Vortex Crystals and Non-Existence of Non-axisymmetric Solitary Waves in the Flierl-Petviashvili Equation

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Abstract

We looked for non-axisymmetric, two-dimensional solitary waves for the Flierl-Petviashvili equation of geophysical fluid dynamics. We found none and conjecture that none exist. However, we did find that the previously known radially symmetric vortices can organize themselves into regular patterns — vortex crystals — when subject to boundary conditions of spatial periodicity. Beginning from non-solitonic initial conditions, the time-dependent flow rapidly organizes itself into such patterns of monopoles plus a small, dispersing transient.

1 Introduction

Flierl and Petviashvili independently derived the following model for the streamfunction $\psi(x, y, t)$ of quasi-two-dimensional large scale flow in the at-

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mosphere and ocean:

$$\frac{\partial}{\partial t}\left(\Delta\psi - \psi\right) + \frac{\partial \psi}{\partial x} + \psi\,\frac{\partial \psi}{\partial x} + J(\psi, \Delta\psi) = 0 \tag{1}$$

where y, x and t are latitude, longtitude and time, respectively, and where subscripts with respect to these coordinates denote differentiation with respect to the subscripted variable, and $J(a,b) \equiv a_x b_y - a_y b_x$ is the usual Jacobian operator acting on arbitrary functions (a,b). We shall dub this the "time-dependent Flierl-Petviashvili" or "FP" equation. The canonical form given above assumes that length and time have been scaled so that the Rossby radius of deformation L_R and the linear long wave speed C_R are both equal to unity; topographic variations are ignored.

The solitary waves of Eq.(1) are of the form

$$\psi(x,y,t) = 2\,c\,\sqrt{1+1/c}\,u(\,\sqrt{1+1/c}\,(x\,-\,c\,t\,),\,y\,) \tag{2}$$

where $u(x,y)$ solves the "Stationary FP" equation:

$$\Delta u - u - u^2 = 0 \tag{3}$$

A full derivation and history of both the time-dependent and stationary forms of the FP equation is given in [13].

Flierl and Petviashvili discovered that (1) has solitary waves ("FP Monopoles") which are *radially symmetric* (in the latitude-longtitude plane) with respect to the center of the disturbance. These are called "monopoles" because they represent anticyclonic vortices in the fluid (clockwise-rotating as viewed from above in the northern hemisphere) . The phase speed c is always less than -1 and the mathematical form is

$$u(x,y) = W(r) \tag{4}$$

where $W(r)$ is the universal function

$$W(r) = \sum_{j=0}^{44} w_j\,\cos\left\{2(j-1)\mathrm{arccot}(r/2)\right\} \tag{5}$$

where the w_j are given in Table 1. The only other steadily-translating solutions that were known before our work are the Korteweg-deVries solitary waves that are functions only of x with $u(x,y) = -(3/2)\mathrm{sech}^2(x/2)$.

The numerical experiments of our previous article showed that after passing over an undersea ridge or valley, an FP Monopole is distorted in shape. Some of these shape-distortions evolve so slowly with time as to suggest that the FP equation might have *non-axisymmetric* two-dimensional solitary waves which would solve Eq.(2). The goal of this follow-up study is to confirm or disprove this hypothesis.

The best strategy would be a rigorous analytic proof of existence or non-existence. Unfortunately, we are not sufficiently clever to offer one. Instead, we have been forced to wage a kind of numerical guerrilla warfare. Our computations make a strong case that non-axisymmetric solutions for the FP equation do not exist. However, we discovered intriguing spatially periodic patterns of monopoles which we shall dub "vortex crystals" [**?**].

We employed three main lines of attack. The first was to solve Eq.(2) in polar coordinates by means of a Fourier-Galerkin method with just two components in the polar angle θ. This reduces the stationary FP equation to a coupled system of two ordinary nonlinear differential equations in r. Although crude, such low order Galerkin-derived systems often have solutions which are satisfactory first guesses for solving the full PDE by Newton's iteration. Although we applied four different methods to the "Two-Mode System", we found no solutions except for the known FP monopole.

The second strategy was to solve the full two-dimensional partial differential equation but subject to boundary conditions of spatial periodicity. The advantage of replacing an infinite plane by a periodic domain is that for small amplitude, the stationary FP equation can be solved *approximately* by *perturbation theory* ("Stokes" or "Poincaré-Lindstedt" expansion). A Fourier pseudospectral method in two dimensions, combined with Newton's iteration, can then continue these solutions to large amplitude where hopefully some peaks will be tall and narrow and good approximations to solitary waves on the infinite plane. This method does reproduce the FP monopole successfully, but fails to generate non-axisymmetric modes. However, the large amplitude continuations of some perturbative approxmations are intriguing patterns of monopoles arranged in regular patterns, the simplest being a hexagonal array of six monopoles on each periodic square.

The third strategy was to solve the time-dependent FP equation from various initial conditions. The good news is that large amplitude, spatially localized solitary waves formed spontaneously from every initial condition we tried. The bad news is that the solitons were always radially symmetric, though often arranged in interesting patterns, again vortex crystals, if the initial condition had sufficient structure.

Table 1
Coefficients w_j of the Rational Chebyshev Series for the FP Monopole $W(r)$

-0.97621121049256	1.25538228185076	-0.21751146985279
-0.06430364505366	-0.00551168103483	0.00339486434098
0.00271007885127	0.00138735466272	0.00058810642786
0.19851729320170E-3	0.03194136755470E-3	-0.02738047042368E-3
-0.03991875025599E-3	-0.34843657309250E-4	-0.25193921061160E-4
-0.16144692269389E-4	-0.09265981588861E-4	-0.04626820363997E-4
-0.17836226154603E-5	-0.02095614080870E-5	0.05448537727938E-5
0.81214246402561E-6	0.81700646366959E-6	0.70036345235849E-6
0.54414285355469E-6	0.39155121176159E-6	0.26204163226590E-6
0.16144776780080E-6	0.08856376240009E-6	0.03905898791359E-6
0.07754334367451E-7	-0.10268877233207E-7	-0.19135234723509E-7
-0.22080608327099E-7	-0.21472121318179E-7	-0.18976849694873E-7
-0.15683651191072E-7	-0.12280177538724E-7	-0.09143951351325E-7
-0.64689635800880E-8	-0.43082144015395E-8	-0.26539087506836E-8
-0.14446578888387E-8	-0.06156591486348E-8	-0.00860468565120E-8

This three-fold numerical strategy may infuriate mathematical purists for whom the theorem-lemma way is the only way. However, two of our three strategies are successful in discovering new patterns of coherent structures. The hexagonal and pentagonal vortex crystals are intriguing additions to the known steadily-translating waves of the FP equation.

We have chosen to describe our failures as well as our successes because the failures, too, are instructive. Much of the numerical analysis and computational science literature is the Quest for the Killer Solver, the magic algorithm that will reliably vanquish all problems in a given class. One of our themes is that for coherent structures, it is better to try multiple lines of attack than a single algorithm. By hiding their mistakes and writing encomiums to a single method, we believe that many authors have done a disservice. For hard problems where existence and convergence proofs are unavailable, one must combine perturbation theory with Newton iterations, spectral methods with Padé approximants, and so on as we have tried to illustrate here.

2 Two-Mode Model: Low Order Galerkin Method

The stationary FP equation is a two-dimensional nonlinear PDE. Since the terms of a Fourier series are a complete basis set, capable of representing any reasonably smooth function, the FP equation can always be solved by a Fourier series in the polar angle θ with radially-varying coefficients:

$$u(r,\theta) = a_0(r) + \sum_{j=1}^{\infty} a_j(r)\cos(j\theta) + \sum_{j=1}^{\infty} b_j(r)\sin(j\theta) \tag{6}$$

The idea of the two-mode model is to truncate this series to the simplest one which gives non-axisymmetric functions:

$$u = A(r) + B(r)\cos(2\theta) \tag{7}$$

We have not include $\cos(\theta)$ or any sine functions because this would be inconsistent with the simplest structure observed in our previous study, which appeared to be symmetric with respect to both the x and y axes. (We also experimented with solutions that lacked symmetry with respect to one axis, but found nothing interesting, so we shall confine our discussion to the Two-Mode Model derived from (7).)

Applying a Galerkin method [5], that is, substituting the expansion into the PDE, multiplying by each of the two basis functions in turn and integrating around a full circle in θ, gives the "Two-Mode Model":

$$\begin{aligned} A_{rr} + \frac{1}{r}A_r - A - A^2 - \frac{1}{2}B^2 &= 0 \\ B_{rr} + \frac{1}{r}B_r - \left\{1 + \frac{4}{r^2}\right\} - 2AB &= 0 \end{aligned} \tag{8}$$

We applied four strategies to solve it:

(1) Pseudospectral/Newton numerical method with arbitrary initial guesses.
(2) Low-order pseudospectral, creating a polynomial equation for the spectral coefficients of such small degree that it can be explicitly solved.
(3) Embedding in a more general problem, plus continuation from a linear eigenvalue problem.
(4) Padé approximant method [6].

The first three methods all employed an expansion of both unknowns $A(r)$ and $B(r)$ as series of rational Chebyshev functions $TB(r;L)$. These functions, which are the images of the usual Chebyshev polynomials under a change of coordinates that maps [-1, 1] into $r \in [-\infty, \infty]$, are a complete orthogonal basis set for an infinite interval. The parameter L is a user-choosable constant which can be varied to improve the rate of convergence of the Chebyshev series. (Usually, $L \sim O(1)$ is optimum when the solution varies on a length scale which is $O(1)$.)

The first method used a lot of basis functions to obtain a set of quadratically nonlinear polynomial equations for the spectral coefficients, which were solved by Newton's iteration. We tried various first guesses, but the numerical method either converged to the previously known FP monopole, which is a radially symmetric function for which $B(r) \equiv 0$, or it never converged at all.

As a fall-back, we went to the second method, which is to reduce the number of unknowns to a sufficiently small number so that the polynomial equations can be solved directly without the need for a first guess or iteration. Truncating to two degrees of freedom gives a system of two quadratic equations which can be reduced to a single quartic equation and solved by radicals. To wring the maximum degree of accuracy from such a small number of unknowns, we did not use the TB functions directly, but instead chose new basis functions which are linear combinations of the TB which vanish at infinity. In addition, for any function in polar coordinates which is well-behaved at $r = 0$, it can be shown that the coefficients of both the radially symmetric part of $u(r, \theta)$ and also that of $\cos(2\theta)$ [and all even cosines] must be symmetric with respect to $r = 0$, and thus can be approximated by the *even degree* rational Chebyshev functions only [5]. Furthermore, $B(r)$ must have a second order root at $r = 0$. Building our basis functions from the TB_{2m} and choosing the coefficients so as to impose the correct behavior at infinity and also, for $B(r)$, the root at the origin gives

$$A(r) = a\frac{1}{1+y^2} \qquad B(r) = b\frac{y^2}{(1+y^2)^2} \tag{9}$$

where $y = r/L$. Collocation at the single point $r = L$ gave two equations which Maple could solve explicitly to give four solutions. One was the trivial solution $a = b = 0$. The second, which turns out to be independent of L, was $a = -2, b = 0$, which is a legitimate approximation to the FP Monopole. (The first method, with a large number of basis functions, converges quite happily and rapidly when given this low order pseudospectral approximation as a first guess.) The other two solutions are

$$a = -\frac{6+L^2}{L^2}, \qquad \leftrightarrow \qquad b = \pm\sqrt{8}\frac{\sqrt{L^4-36}}{L^2} \tag{10}$$

For $L = 3$, these reduce to $a = -1.66667, b = 2.10819$. However, using this as a first guess, we could not obtain solutions from our two mode TB Matlab program. For small L, the coefficients are complex-valued, which is nonsense. Thus, the low order pseudospectral method succeeds only in generating the radially symmetric solitary wave which was discovered by Flierl and Petviashvili.

The third strategy was to embed the Two-Mode Model, which contains no parameters as written, in a larger system that contains a free parameter. For a discrete value of this new, artificial eigenparameter, we obtain non-axisymmetric solutions that bifurcate at infinitesimal amplitude from the FP monopole. For example, if we replace the coefficient of the nonlinear term in the second equation of the two-mode set by a parameter λ and if we set $A(r) = W(r)$, the FP monopole, then the second equation of the Two-Mode Model becomes the linear eigenvalue problem

$$B_{rr} + \frac{1}{r} B_r - \left\{ 1 + \frac{4}{r^2} \right\} B = \lambda W B \tag{11}$$

The smallest eigenvalue is $\lambda = 3.5107$. We can then solve the Embedded Two-Mode set, with the nonlinear term λB^2 instead of $2B^2$, for smaller and smaller values of λ until we reach $\lambda = 2$, at which point we shall have obtained a nonlinear solution to the original parameter-free Two-Mode Set.

Unfortunately, the solution has a limit point at around $\lambda = 3.35$; a graph of the norm of $B(r)$ versus λ rises to an infinite slope at the limit point and then curves back so that there are two solutions for $\lambda > 3.35$, but none for $\lambda < 3.35$. Thus, this continuation strategy failed, too.

All these failures of the pseudospectral method could be attributed to an inadequate knowledge of the expected shape of the non-axisymmetric mode, a common problem in stalking nonlinear solutions. We therefore tried another method, the Padé scheme [6], which does not require a first guess. The first step is to compute the power series expansions of $A(r)$ and $B(r)$ with respect to $r = 0$. Because both functions are symmetric with the respect to the origin, there are only two free parameters in the power series: (a_0, b_2), which are lowest power series coefficients for $A(r)$ and $B(r)$, respectively. (Note that the series for $B(r)$ begins with the quadratic term $b_2 r^2$.) The power series have only finite radii of convergence (probably), so the second step is to convert them into the rational functions known as Padé approximants [1–3]. These will always converge everywhere on the real r-axis if the solution is well-behaved for real r. The third step is to choose the free parameters (a_0, b_2) so that the Padé approximants vanish at infinity. Like the low order pseudospectral method, this gives a system of two polynomial equations in two unknowns. Unfortunately, these equations are of higher order than the pair of quadratics derived by the modified TB series, but the pair can still be reduced to a single equation in a single unknown and all roots computed using a symbolic

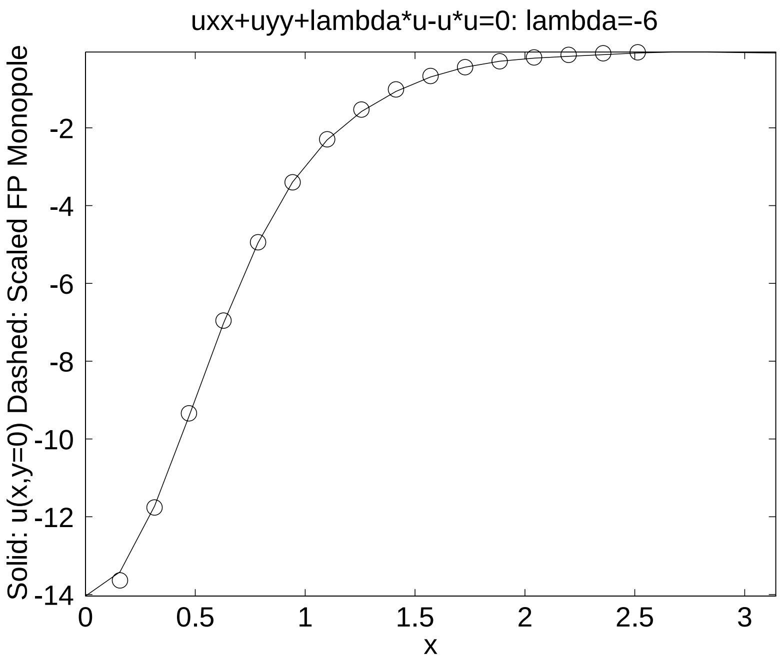

Fig. 1. Solid: Cross-section along the x-axis, that is, $u(x, y = 0)$ for the Fourier pseudospectral solution of $u_{xx} + u_{yy} + \lambda u - u^2 = 0$ with spatial period 2π in both coordinates and $\lambda = -6$. Dashed: $-\lambda\, W((-\lambda)^{1/2} x)$ where $W(r)$ is the FP Monopole. This shows that the solution to the periodic nonlinear eigenvalue problem does indeed converge rapidly to the FP Monopole, even though this is the solution to a (seemingly very different) radially symmetric problem without an eigenparameter on an unbounded domain. The periodic solution used a 5 x 5 spectral basis.

manipulation language like Maple. (The reduction to a single equation and the solution of that equation require the built-in "resultant" and "fsolve" functions, respectively.)

Once again, the only reasonable solution is the FP Monopole. We conclude that if non-axisymmtric FP solitary waves exist, they must be poorly described by the Two-Mode Model.

3 Periodic Solutions in Two Space Dimensions: Vortex Crystals

Another line of attack is to solve the FP equation in *Cartesian* coordinates with *periodic* boundary conditions. If the solution becomes *narrow* compared to the *period* in the limit of large amplitude, then it also becomes a good approximation to an *infinite interval* solution. To obtain a solution of arbitrary amplitude on the periodic domain, it is necessary to insert an eigenparameter λ. This gives

$$u_{xx} + u_{yy} + \lambda u - u^2 = 0 \tag{12}$$

We shall refer to this generalized equation as the "Quadratic-Poisson" equation. It is important to note that its spatially periodic solutions, whether narrow or not, generate legitimate solutions to the time-dependent FP equation of the form

$$\psi(x, y, t) = 2cu([x - ct], y), \qquad \lambda = -\frac{1 + c}{c} \tag{13}$$

with x and y reinterpreted as coordinates in a frame of reference travelling with the wave.

For *small amplitude*, this equation can be approximately solved by an expansion in powers of the amplitude. This series, known variously as a "Stokes" or "Poincaré-Lindstedt" expansion, also provides the value of λ for which (12) has solutions of infinitesimal amplitude. By varying λ is small steps, we can march to larger and larger amplitude to emerge with good approximations to solitary waves (we hope). The perturbation theory gives the first guess for Newton's iteration for small amplitude. As λ is varied through the discrete sequence $\{\lambda_0, \lambda_1, \ldots\}$, the converged numerical solution for λ_j provides the first guess for $\lambda_{j+1} = \lambda_j + \delta$ where δ is small.

We chose the spatial period to be 2π. If $u(x, y; \lambda)$ is a solution to the Quadratic Poisson equation, then so also is $v(x, y; s\lambda) \equiv su(s^{1/2}x, s^{1/2}y; \lambda)$ for arbitrary constant $s > 0$. Consequently, there is no loss of generality in our choice of period since we can obtain solutions for other periods merely by rescaling the spatial coordinates, amplitude, and λ.

Fig. 1 shows the result of continuation from the infinitesimal amplitude solution

$$u = -2 + a\cos(x)\cos(y), \qquad \lambda = -2 \tag{14}$$

to $\lambda = -6$ with boundary conditions of periodicity with period 2π in both coordinates. The "infinitesimal amplitude" solution actually has an $O(1)$ constant, but it still can be calculated by perturbation theory, which is all that matters. The graph shows that troughs for $\lambda = -6$ are indeed approximately accurately by the FP Monopole. One can generate an improved approximation by replicating an infinite number of copies of the monopole $W(r)$, placing one at the center of each periodic box, and summing, that is:

$$u(x, y) \approx \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} -\lambda\, W\left\{(-\lambda)^{1/2}\, \left((x - 2\pi m)^2 \,+\, (y - 2\pi n)^2\right)\right\} \tag{15}$$

This sum of an infinite number of copies of a "pattern function" — in this case, the solitary wave is the "pattern" – is called an "imbricate series". Any

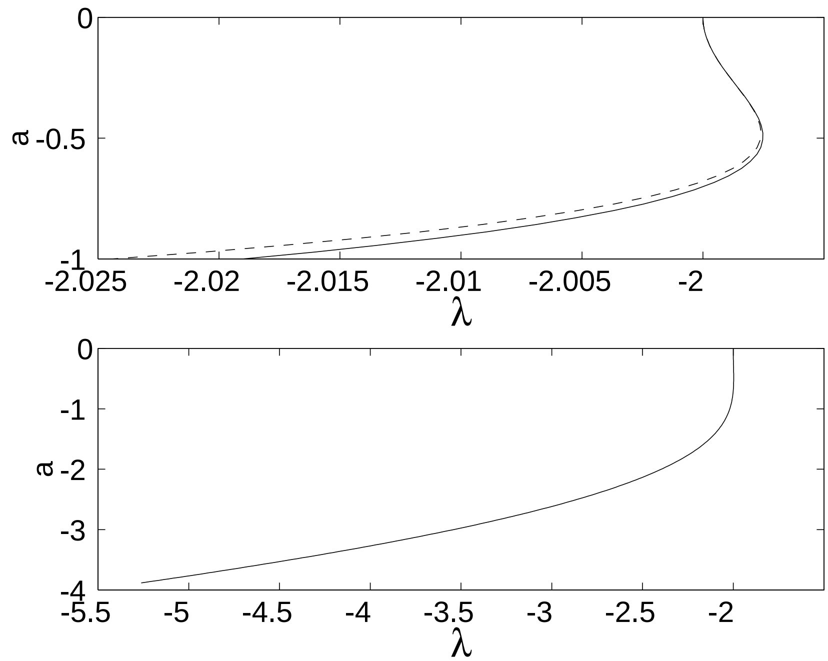

Fig. 2. Both graphs illustrate the amplitude a of the dominant Fourier coefficient, that of $\cos(x)\cos(y)$ versus λ for the Quadratic Poisson equation. In the top panel, the dashed (upper) curve illustrates the two-term perturbative approximation, $\lambda = -2 + (1/48)a^2 - (5003/110592)a^4$. The lower graph shows the exact $a(\lambda)$ on a much wider interval in λ.

solitary wave generates at least an approximate nonlinear, spatially periodic solution by being "imbricated" with a large period.

Thus, although the problem of a solitary wave on an infinite interval would seem to be quite unrelated to that of a nonlinear spatially periodic solution on a periodic domain, the two are closely related. In particular, the strategy of marching from a perturbative solution to a large amplitude solution has generated a good approximation to a solitary wave — but only to the one that we already knew.

Parenthetically, there is a modest technical complication in continuing this solution branch. As the leading Fourier coefficient a decreases from 0, $\lambda(a)$ first increases to a maximum ("limit point" or "fold point") at $\lambda = -1.9975$ for $a = -0.49$, and then decreases monotonically as $a \to -\infty$ as shown in Fig. 2. A Fourier/Newton iteration program which uses λ as the continuation parameter will fail at the limit point. However, by a little trial and error, it was easy to begin the continuation for $\lambda < -2$, thereby bypassing the limit point. To trace the solution branch completely, one could alternatively use pseudoarclength continuation [10,11], extend the perturbation theory to $O(a^4)$ (which gives a good approximation beyond the limit point as illustrated), or take a as the fixed parameter instead of λ, as done in the Stokes' series. We do not understand why, for this branch of solution, λ first increases and then very quickly reverses direction, nor why there is a limit point for such small a that perturbation theory is still accurate.

To look for non-axisymmetric solitons, we first considered the same solution but with the period in y stretched to $2\pi/L$. We do not describe the details of the perturbation theory because these are explained at length, for very similar problems, in texts such as [12] and articles like [4,8,9,7]. Though we prefer to reference our own articles, as being closest in spirit to what would have been given here if we provided details, the method dates back to the nineteenth century and has been used by literally hundreds of investigators. The second order approximation is

$$\begin{aligned}
\lambda_2 &= -\frac{1}{24}\frac{45 - 86L^2 + 45L^4}{(1+L^2)(L^2-3)(3L^2-1)} \\
\lambda &= -1 - L^2 + \lambda_2 a^2 \\
a_{00}^2 &= (1/4)(1+\lambda_2(4L^2+4))/(L^2+1), \qquad a_{02}^2 = -(1/4)/(3L^2-1) \\
a_{20}^2 &= (1/4)/(-3+L^2), \qquad a_{22}^2 = -(1/12)/(1+L^2) \\
u &= -1 - L^2 + a\cos(x)\cos(Ly) \\
&\quad + a^2\left(a_{00}^2 + a_{20}^2\cos(2x) + a_{02}^2\cos(2y) + a_{22}^2\cos(2x)\cos(2Ly)\right) \qquad (16)
\end{aligned}$$

We have given the solution to $O(a^2)$ because the limit $a = 0$ is actually a limit point: $da/d\lambda = \infty$ for zero amplitude, and solutions exist only for $\lambda \leq -1 - L^2$. Thus, the $O(a)$ solution does not specify a unique point. The second order solution for small but nonzero a always was a sufficiently good first guess so that Newton's iteration converged.

The spatial derivatives were discretized by the pseudospectral method, but with a tensor product Fourier basis instead of the rational Chebyshev functions employed for the unbounded, non-periodic domain above. The algebraic equations for the Fourier coefficients were obtained by substituting a truncated Fourier series into the differential equation and demanding that the residual should be equal to zero at a set of evenly spaced grid points equal in number to the number of unknowns [5].

The bad news for $L = 2$ is that we failed, even for large amplitude, to find structures that were the imbrication of non-axisymmetric solitary waves. The good news is that we did find interesting nonlinear solutions. At very small amplitude (relative to the constant in the Fourier series, which is roughly λ at all amplitudes), this mode has a trough at $x = y = 0$. However, there is a topological change at rather small amplitude in which this disappears, and the trough becomes a hexagon of deep troughs as illustrated in Figs. 3 and 4.

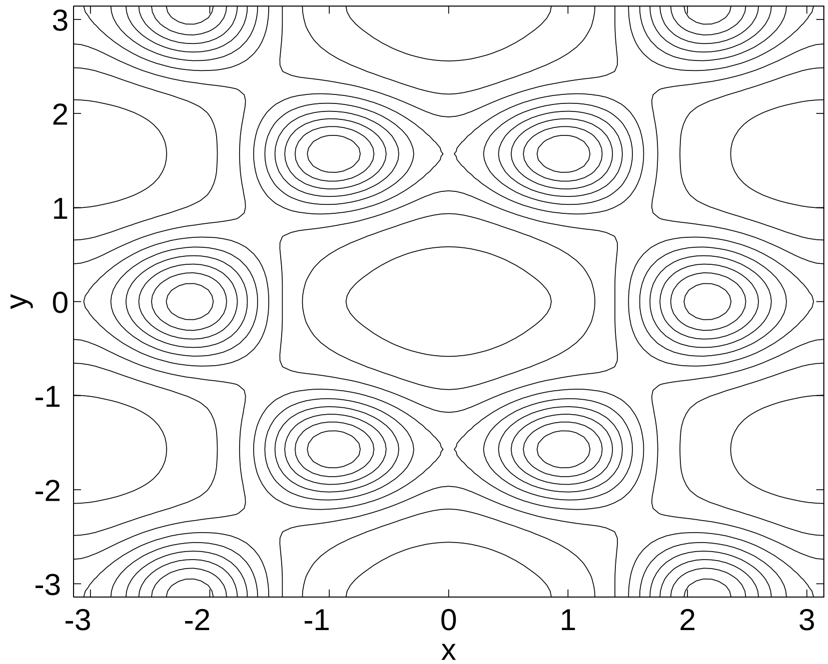

Fig. 3. Solution to the nonlinear eigenvalue problem $u_{xx} + u_{yy} + \lambda u - u^2 = 0$ with $\lambda = -10$ and a spatial period of 2π in x but only π in y so that the parameter $L = 2$ in the perturbative initialization. Contours are from -20 to -2 in intervals of -2 with the deepest troughs in the centers of the six vortices that are at the vertices of the hexagon. The mode has a trough at $x = y = 0$ for very small amplitude, but this becomes long and narrow as the amplitude increases and then disappears so that the deepest troughs occur in a hexagonal pattern.

Similar orderly arrays of vortices have been observed in other systems, as for example [14], and dubbed "vortex crystals". It is far from clear that the turbulent, ever-fluctuating ocean and atmosphere will ever form structures as regular and crystalline as seen in these figures. To use the language of condensed matter physics, turbulent flows are amorphous like glass rather than crystalline like diamond.

Nevertheless, these vortex crystals of the FP equation show that even when forced away from a high degree of symmetry, the patterns are made of radially-symmetric solitary waves only.

The perturbative solution of Eq.(16) can be generalized by adding in other components of the same *total* wavenumber. Thus, the lowest order can be generalized to

$$\lambda = -1 - L^2 \tag{17}$$

$$-1 - L^2 + a \left\{ \cos(x) \cos(Ly) + b \sin(Lx) \cos(y) \right\} \tag{18}$$

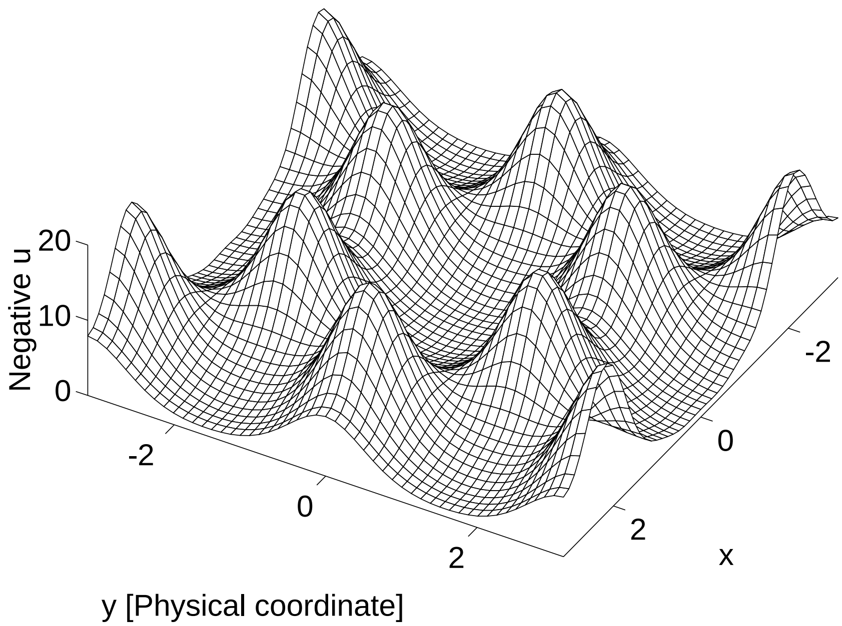

Fig. 4. Same as previous figure, but a surface graph instead of a contour plot. Note that to make this mesh plot easier to visualize, we flipped the sign of u.

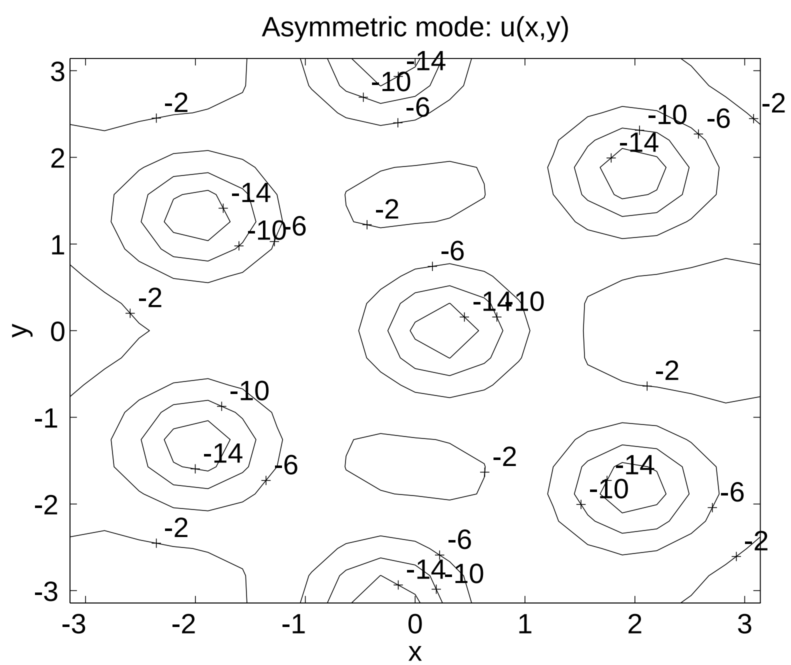

Fig. 5. Unsymmetric mode for $\lambda = -8$ with spatial period of 2π in both x and y. This is the analytic continuation in amplitude a of the mode which at lowest order is $\lambda = -5 + O(a^2), u = -5 + a\{\cos(x)\cos(2y) + \sin(2x)\cos(y)\}$. A basis of 28 sines and cosines in x and 14 cosines in y $[1, \cos(y), \ldots, \cos(13y)]$ were used to compute it.

At lowest order, the new symmetry-breaking parameter b is arbitrary, suggesting there is a one-parameter family of solutions for a given λ, implying that the Jacobian matrix of the Newton iteration will be *singular*. At higher order, however, it turns out that it is possible to remove all the "secular" terms, and so obtain bounded solutions, if and only if

$$b = \pm 1 \tag{19}$$

It suffices to consider the case $b = 1$ only because changing the sign of this parameter creates a phase shift of the patterns without actually altering the shape of $u(x, y)$.

The "lower-symmetry" perturbative solution is

$$\begin{aligned}
\lambda_2 &= -\frac{421}{1320} - b^2 \frac{5}{78} \\
\lambda &= -5 + \lambda_2 a^2 \\
a_{00}^2 &= \lambda_2 + (b^2 + 1)/20 \\
a_{04}^2 &= -1/44, \qquad a_{20}^2 = 1/4, \qquad a_{24}^2 = -1/60 \\
a_{40}^2 &= b^2/44, \qquad a_{11}^2 = b/6, \qquad a_{31}^2 = -b/10 \\
a_{13}^2 &= -b/10, \qquad a_{33}^2 = -b/26, \qquad a_{02}^2 = b^2/4, \qquad a_{42}^2 = b^2/60 \\
u &= -5 + a\left\{\cos(x)\cos(2y) + b\sin(2x)\cos(y)\right\} \\
&\quad + a^2\{a_{00}^2 + a_{20}^2 \cos(2x) + a_{02}^2 \cos(2y) + a_{11}^2 \cos(x)\cos(y) \\
&\quad + a_{40}^2 \cos(4x) + a_{04}^2 \cos(4y) + a_{24}^2 \cos(2x)\cos(4y) \\
&\quad + a_{42}^2 \cos(4x)\sin(2y) + a_{33}^2 \sin(3x)\cos(3y) + a_{31}^2 \cos(3x)\cos(y) \\
&\quad + a_{13}^2 \sin(x)\cos(3y)\}
\end{aligned} \tag{20}$$

Fig. 5 illustrates the large amplitude continuation of this mode for $L = 2$. The vortex crystals do indeed have lower symmetry than the hexagons of previous figures. However, the pattern is still built from the radially symmetry FP vortex.

Single, isolated non-axisymmetric solitary waves on an unbounded plane were not found. However, the vortex crystals have a certain intrinsic interest.

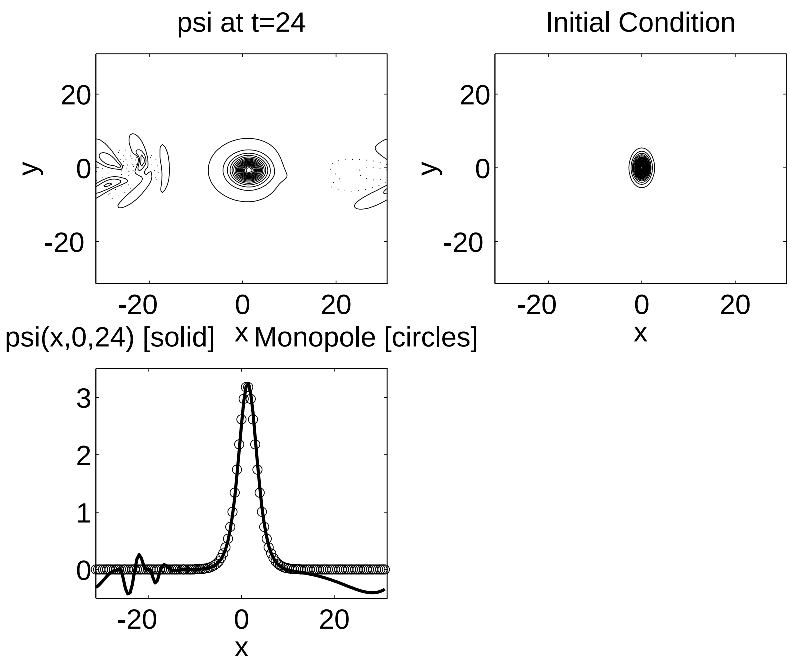

Fig. 6. Solution of the time-dependent FP equation when the initial condition was $\psi(x, y, 0) = -2W(\sqrt{8x^2 + 2y^2})$ where $W(r)$ is the FP monopole function defined in Sec. 1. The solution is shown in a coordinate system moving at the velocity $c = -1.7$ to keep the vortex roughly centered at the origin in the moving reference frame. The two contour plots compare the initial streamfunction with that at $t = 24$. The lower left hand panel, shown also in a zoom plot in the next figure, compares $\psi(x, 0, 24)$ with its best fit by the FP Monopole. This calculation used a 128 x 128 grid with a spatial period of 20π in both coordinates and a time step of 1/27.

4 Time-Dependent Solutions, I: Initial Elliptical Vortices

Our last strategy to search for new solitary waves was to solve the time-dependent FP equation

$$\frac{\partial}{\partial t}\left(\nabla\psi - \psi\right) + \frac{\partial \psi}{\partial x} + \psi\,\frac{\partial \psi}{\partial x} + J(\psi, \nabla\psi) = 0 \tag{21}$$

as an initial-value problem for various initial conditions. We employed the same Fourier pseudospectral time-marching algorithm as described in our previous paper [13] except for third-order Adams-Bashforth time-marching, which was a little less expensive than the fourth order Runge-Kutta used previously. In all cases, the flow rapidly evolved into one or more axisymmetric FP monopoles plus a dispersing wavetrain.

Our first class of experiments employed an initial condition which was chosen to be an axisymmetric soliton, as given by Eq.(4) above, with various phase speeds c and with the contours stretched to be elliptical rather than circular.

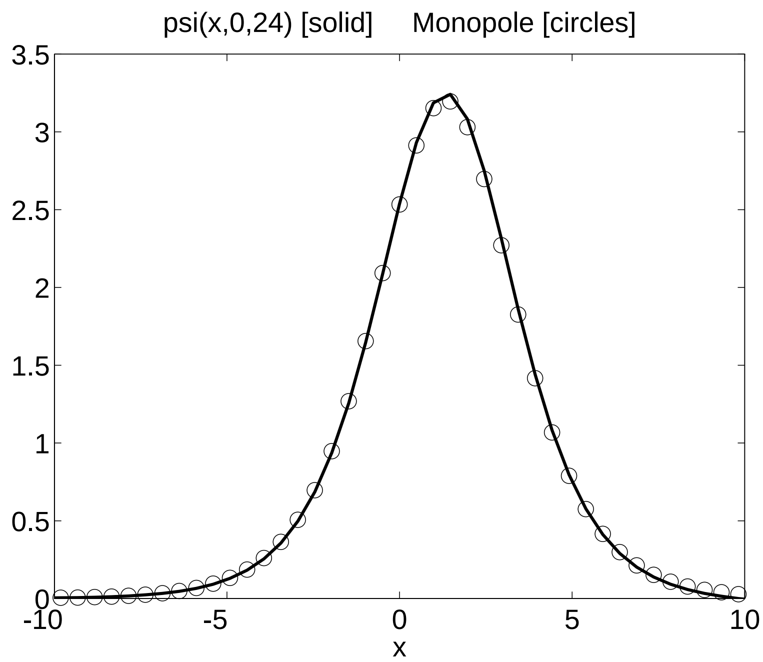

Fig. 7. Same as the lower left graph in the previous figure except that only part of the x-axis is shown. Solid: the numerical solution at $y = 0$ and $t = 24$. Circles: Best fit to this cross-section by a monopole.

We tried elongating the initial condition in both x and y with similar results.

Figs. 6 and 7 illustrate a case where the contours of the initial condition were ellipses with the semimajor axis (along the y-axis) double the semiminor axis.

The initial ellipse quickly evolved to an axisymmetric monopole plus a dispersing transient which is quickly left behind as the solitary wave propagates rapidly westward. Part of the transient wavetrain, which would be only to the right on an infinite interval, has wrapped around and rematerialized on the left of the figure because of the periodic boundary condition in x. Even though the transient is evident both in the contour plot and in the cross-section along the x-axis which is below it in Fig. 6, the central part of the graph is fitted very well by the FP Monopole function. If we zoom in on the central part of the domain, as done in Fig. 7, one can see the fit of the time-dependent solution by the monopole is remarkably good; all the transient is excluded from the smaller x-interval.

The axisymmetrization is much slower when the phase speed is $c = -1.1$, as in [13], rather than $c = -2$ as in the case illustrated. An ellipse which is elongated in the east-west direction, rather than north-south as in the graphs, also evolves more slowly. However, the axisymmetrization from an elliptical initial condition is always monotonic and inevitable.

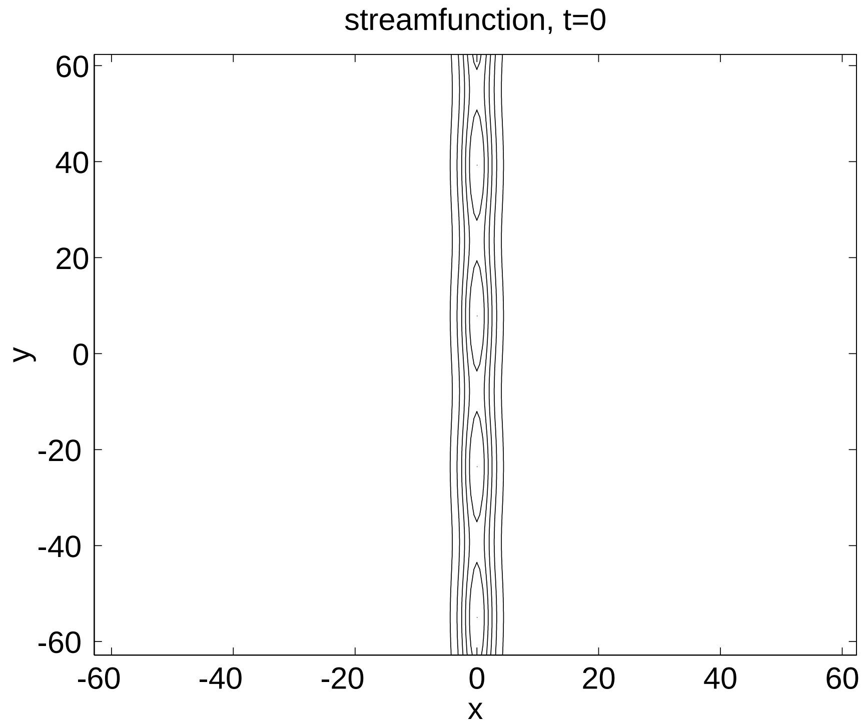

Fig. 8. Streamfunction $\psi(x, y, t = 0)$ when the initial condition was a perturbed KdV soliton: $\psi(x, y, 0) = 3\mathrm{sech}^2\left(8^{-1/2}x\right)(0.9 + 0.1\sin(y/5))$. This calculation used a 256 x 256 grid with a spatial period of 40π in both coordinates.

5 Time-Dependent Solutions, II: Perturbed KdV Solitons

The time-dependent FP equation also has non-axisymmetric solutions which are functions only of x and satisfy the Korteweg-deVries (KdV) equation:

$$\psi(x, y, t) = -3c(1 + 1/c)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{1 + 1/c}\,[x - ct]\right) \qquad (22)$$

where $c < -1$ is the phase speed. Our second set of experiments ran the FP equation from an initial condition which was a y-dependent perturbation of a KdV soliton. With both large and small perturbations, the initial ridge of streamfunction invariably broke up into one or more axisymmetric solitons.

Figs. 8 and 9 illustrate the initial and final streamfunctions for a representative case. The initial perturbation was to multiply the ridge by a sinusoidal perturbation in y with four wavelengths across the period box and an amplitude only $1/10$ that of the unperturbed KdV soliton. Animations, made in Matlab and then converted to Quicktime movies, showed that each of four peaks in the initial condition quickly formed a large axisymmetric vortex ($\max(\psi) \approx 6.43$ versus only 3.0 for the initial condition) and moved rapidly westward (towards negative x). The broken ridge segment left behind then broke up into

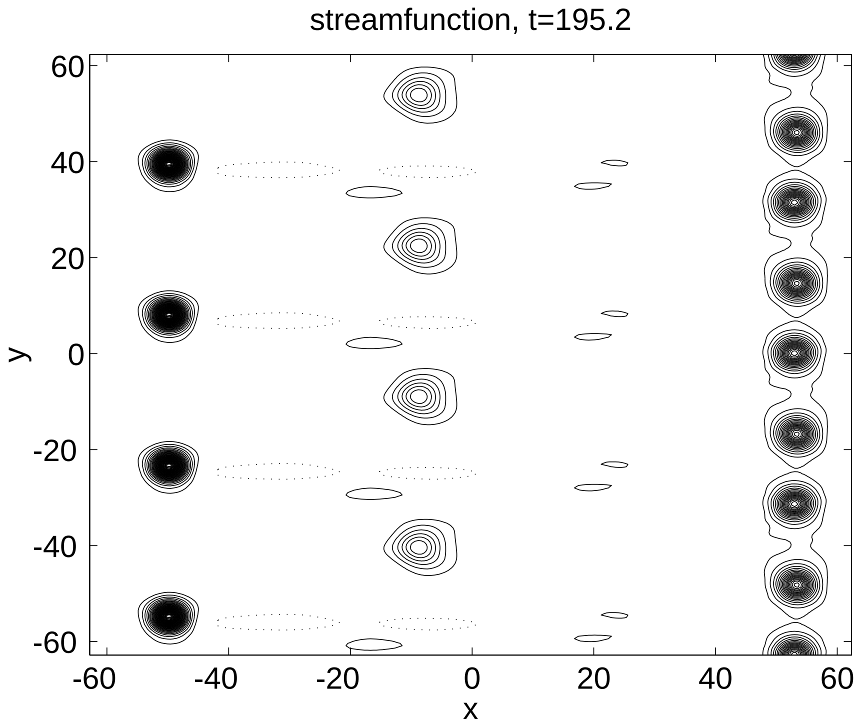

Fig. 9. The streamfunction at $t = 195$ as it has evolved from the perturbed-KdV initial condition shown in the previous figure. Because the perturbation has four wavelengths in y within the square, the axisymmetric monopoles form in groups of four. The tallest monopoles are to the left, the smallest monopoles in the center, and the medium-sized monopoles are at the right (east).

two medium-sized vortices, one at either end, with peaks of about 3.51. The shallowest portion of the ridge, again left behind to the east of the two medium vortices, then axisymmetrized into a small vortex of maximum height about 1.4. Because the initial perturbation had four complete periods in the domain illustrated in these figures, one sees four large, eight medium, and four small vortices in the final contour plot, Fig. 9.

Are these vortices really FP Monopoles? The animations show that these are very persistent and coherent features which evolve little after forming.

Since the animations are not, alas, publishable in the text of a journal, we took cross-sections through each class of vortex and compared them with the FP Monopole which most closely approximates it (Fig. 10). The close agreement shows that all three vortices are very close to the FP Monopole with appropriate rescaling as given by (4), $2c(1 + 1/c)W(\sqrt{1 + 1/c}\, r)$ where $c = -2.344, c = -1.734, c = -1.2924$, respectively.

The quest for a non-axisymmetric two-dimensional FP solitary wave has come up empty once again. However, Fig. 11, which illustrates the smallest of the three sizes of vortices, shows why the search was not entirely foolish. Although

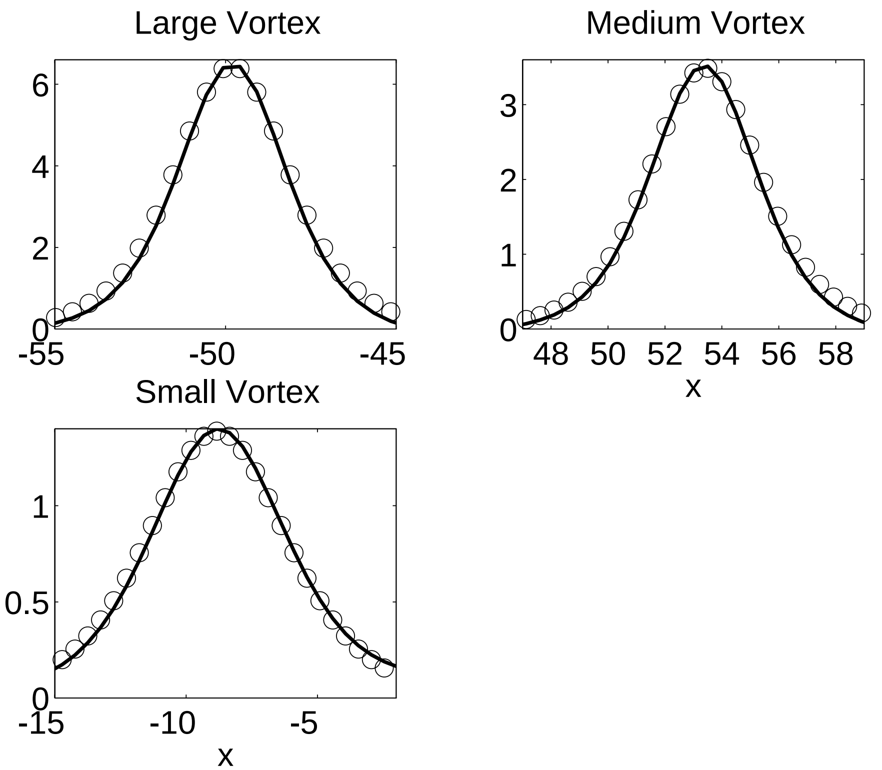

Fig. 10. Same case as previous two figures (perturbed KdV initial condition. Each figure shows a cross-section at fixed y through each of the three sizes of almost axisymmetric monopoles that have formed by $t = 195$.

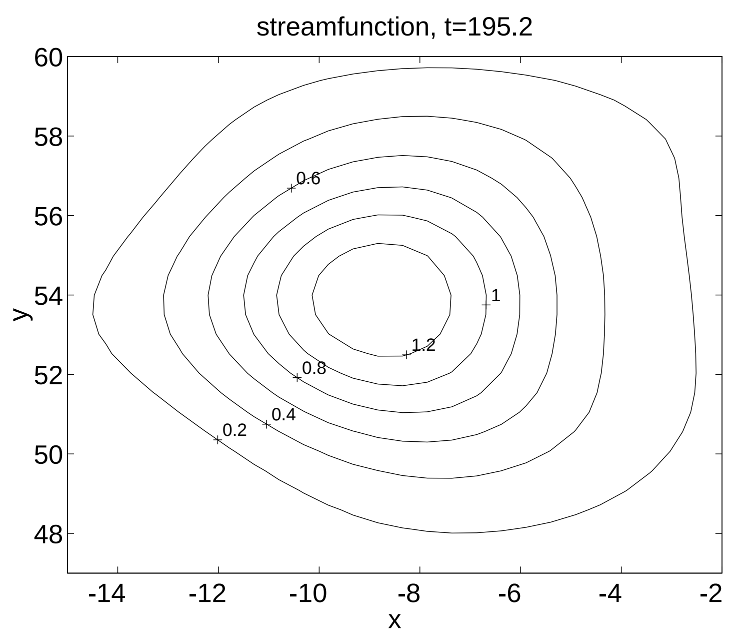

Fig. 11. Same case as previous three figures (perturbed-KdV initial condition). Zoom contour plot of the smallest size of vortex at $t = 195$.

the contours near the center are circular, the outer contours are asymmetric; the outermost is shaped rather like a boat with a pointed "bow" aimed westward and a flat eastern side.

The time evolution of these vortices in the animations leaves no real doubt that they are indeed radially symmetric monopoles. However, the inner contours relax to radial symmetry (for all sizes of vortex) much faster than the outer contours. Furthermore, the smaller and slower a vortex, the slower the relaxation of the outer contours.

Thus, one must expect that, as found in our previous study, weak vortices may *look* non-axisymmetric, at least around the edges of the vortex, for a long time.

In elementary fluids classes, it is common to linearize the flow about some basic state, such as the KdV soliton, and study the eigenfunctions which amplify in time. The rationale for studying the structure of these eigenfunctions is the hope that the finite amplitude structures will be qualitatively similar to those of infinitesimal amplitude. Here, however, the final pattern of vortices bears little resemblance to the instabilities that first develop on the KdV soliton. The basic state itself, which is the KdV soliton, is completely destroyed. The only useful information which could be obtained from a classical, linearized instability study here is a non-existence proof: the KdV ridge soliton is not an observable structure of the time-dependent FP equation because it is unstable to break-up into radially symmetric monopoles.

6 Summary

By employing multiple algorithms, some to solve the nonlinear eigenvalue problem and others to integrate the time-dependent FP equation from various initial conditions, we have searched in vain for two-dimensional non-axisymmetric solitary waves of the Flierl-Petviashvili equation. We conjecture that none exist, and that the only solitary waves of the FP equation are the radially symmetric FP monopoles discovered by Flierl and Petviashvili and the one-dimensional, infinitely long ridges that satisfy the Korteweg-deVries equation. However, a rigorous proof our conjecture is still lacking.

Nevertheless, our search has not yielded only negative results. First, we showed that the FP equation has large amplitude spatially-periodic solutions which have interesting patterns of five or six radial radially symmetric monoples in each periodic box — "vortex crystals". We suspect that the two crystalline patterns illustrated above are only a subset of a much larger, possibly infinite, catalogue of vortex crystal patterns.

In turbulent flows (not studied here), it seems likely that the regularity of the vortex crystals would be disrupted into a slowly-evolving flow that, to continue the analogy with condensed-matter physics, one might dub a“vortex glass”, amorphous rather than crystalline.

Second, our time-integrations have shown that from a variety of non-soliton initial conditions, the FP equation spontaneously evolves to a flow dominated by one or more solitons just like the FP equation. The solitons are always radially symmetric monopoles.

Our third conclusion is that the KdV-solving ridges are unstable. Though we did not attempt an elaborate stability study, the perturbation illustrated above had a maximum amplitude of only 10the KdV soliton rapidly broke up into radially symmetric solitons (plus a small amount of dispersing transients) for all the initial conditions that we tried.

For the FP equation, the radially symmetric vortex solitary wave is the alpha and the omega, the beginning of the story and also the end of the story.

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