ON COMPUTING THE EVENTUAL BEHAVIOR OF A FINITELY PRESENTED FI-MODULE

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Abstract. We provide an easy algorithm for computing the eventual behavior of a finitely presented FI-module in characteristic zero.

In §1, we explain how an FI-matrix gives a presentation for an FI-module. In §2, we recall basic properties of finitely presented FI-modules, including the existence of eventual multiplicities observed by Church-Farb [CF13]. In §3 we give an algorithm that computes these eventual multiplicities, relying on structure theory due to Sam and Snowden [SS16]. The algorithm is adapted from one appearing in this author's thesis.

1. Finitely presented FI-modules

Let FI be the category whose objects are the finite sets $[n] = \{1, 2, \dots, n\}$ and whose morphisms are injections $[n] \hookrightarrow [n']$. An FI**-matrix** consists of two lists of finite sets, $[x_1], [x_2], \dots [x_g]$ and $[y_1], [y_2], \dots [y_r]$, serving as row- and column-labels

$$\begin{array}{c} \\ [x_1] \\ [x_2] \\ \vdots \\ [x_g] \end{array} \begin{array}{c} \begin{array}{cccc} [y_1] & [y_2] & \cdots & [y_r] \end{array} \\ \left[\begin{array}{cccc} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{array} \right] \end{array}$$

where the (i, j)-entry $M_{i,j}$ is required to be a (formal) $\mathbb{Q}$-linear combination of injections $[x_i] \hookrightarrow [y_j]$:

$$M_{i,j} = \sum_{k=1}^{l_{i,j}} m_{i,j}^k \cdot \iota_{i,j}^k.$$

Here, $\iota_{i,j}^k \colon [x_i] \hookrightarrow [y_j]$ is an injection, and $m_{i,j}^k \in \mathbb{Q}$ are rational numbers. Equivalently, $M_{i,j} \in \mathbb{Q}\mathrm{FI}([x_i], [y_j])$ where $\mathrm{FI}([n], [n'])$ is the finite set of injections from $[n]$ to $[n']$.

For any FI-matrix M, a formal construction lets us build an FI**-module** V_M, which is to say, a functor $V_M \colon \mathrm{FI} \to \mathrm{Vect}_{\mathbb{Q}}$. The FI-matrix M **presents** V_M in that V_M is defined as the cokernel of a map that is somehow induced by M:

$$V_M = \operatorname{coker} \left(\bigoplus_{j=1}^{r} P^{y_j} \xrightarrow{\quad M^* \quad} \bigoplus_{i=1}^{g} P^{x_i} \right),$$

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where, for $n \in \mathbb{N}$, $P^n \simeq \mathbb{Q}\mathrm{FI}([n], -)$ is the $\mathbb{Q}$-linearization of the covariant representable functor associated to $[n]$. To understand the induced map M^*, we need Yoneda's lemma, which says

$$\mathrm{Hom}(P^{n'}, P^n) \simeq \mathbb{Q}\mathrm{FI}([n], [n']).$$

2. Representation stability of FI-modules

The category of FI-modules was introduced by Church, Ellenberg, and Farb [CEF15], although the structure theory in characteristic zero is due to Sam and Snowden [SS16] under a different name. We recall two facts about finitely presented FI-modules.

Theorem 2.1 (CEF). *If M is an* FI*-matrix, then the sequence $n \mapsto \dim V_M[n]$ eventually coincides with a polynomial.*

Since the vector space $V_M[n]$ carries an action of a symmetric group S_n, we may ask about its decomposition as an S_n-representation. In characteristic zero, the irreducible representations of S_n are indexed by natural number partitions of the number n as a sum. We write a partition λ as a non-increasing sequence of natural numbers $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ with $n = \sum \lambda_i$. For each partition λ, write W_λ for the corresponding irreducible representation. The action matrices of these representations may be computed in Sage, for example [Dev14]. Instead of asking for the dimension of $V[n]$, we ask for the multiplicity of W_λ as it occurs in an S_n-equivariant decomposition of $V[n]$. We have the following remarkable theorem.

Theorem 2.2 (CEF). *If M is an* FI*-matrix, then for any $\lambda = \lambda_1 \geq \ldots \geq \lambda_k$ the sequence*

$$n \mapsto \textit{multiplicity of } W_{(n+\lambda_1 \geq \lambda_1 \geq \ldots \geq \lambda_k)} \textit{ in } V_M[(n+\lambda_1) + \lambda_1 + \cdots + \lambda_k]$$

is eventually constant. This constant is zero if $\sum \lambda_i$ exceeds the highest row-label of M.

In other words, the end-behavior of the FI-module V_M is extremely organized: once n is large enough, $V_M[n+1]$ can be obtained from $V_M[n]$ by "adding a box in the top row." Church-Farb [CF13] observed this phenomenon in many situations, naming it **representation stability**.

3. Computing the end-behavior of V_M

In light of Theorem 2.2, any FI-matrix M gives rise to a natural number for every choice of λ: the **eventual multiplicity**

$$e_\lambda(M) = \lim_{n\to\infty}(\text{multiplicity of } W_{(n+\lambda_1 \geq \lambda_1 \geq \ldots \geq \lambda_k)} \text{ in } V_M[(n+\lambda_1) + \lambda_1 + \cdots + \lambda_k]).$$

The following construction is key in the computation of $e_\lambda(M)$. It gives a way to replace the injections appearing in an FI-matrix M with usual $\mathbb{Q}$-matrices. This procedure will reduce the computation of $e_\lambda(M)$ to linear algebra over $\mathbb{Q}$.

Construction 3.1. *If λ is a partition of k and $\iota \colon [n] \hookrightarrow [n']$ is an injection, we build an $\binom{n}{k} \times \binom{n'}{k}$ block matrix $I_\lambda(\iota)$. In fact, the rows are indexed by k-subsets of $[n]$ and the columns are indexed by k-subsets of $[n']$. If $s \colon [k] \hookrightarrow [n]$, $s' \colon [k] \hookrightarrow [n']$ are two k-subsets,*

$$I_\lambda(\iota)_{s,s'} = \begin{cases} 0 & (\iota \circ s)([k]) \neq s'([k]) \\ W_\lambda \sigma & \iota \circ s = s' \circ \sigma \end{cases}$$

where $\sigma \in S_k$ is the unique permutation that unscrambles $(\iota \circ s)([k])$, and $W_\lambda \sigma$ is the matrix giving the action of σ on the irreducible S_k-representation W_λ.

Theorem 3.2. *For any* FI*-matrix M, the eventual multiplicity*

$$e_\lambda(M) = \operatorname{corank} I_\lambda(M)$$

where $I_\lambda(M)$ is the rational block matrix obtained by applying Construction 3.1 to each entry of the FI*-matrix M:*

$$I_\lambda(M)_{i,j} = \sum_{k=1}^{l_{i,j}} m_{i,j}^k \cdot I_\lambda(\iota_{i,j}^k).$$

Corollary 3.3. *If M is an* FI*-matrix, the eventual multiplicity of the trivial representation in $V_M[n]$ is given by*

$$\lim_{n\to\infty} \dim \left(V_M[n]\right)^{S_n} = \operatorname{corank} M',$$

where M' is the rational matrix obtained from M given by replacing every injection with the rational number 1*.*

Proof. Construction 3.1 degenerates completely: $I_\emptyset \iota$ is the 1×1 identity matrix for any injection ι. □

Remark 3.4. Since the I_λ are fixed for all time, it is possible to precompute $I_\lambda \iota$ for small λ and ι and injection between relatively small sets. This makes the computation of e_λ exactly as fast as a rank computation whenever the presentation matrix M only uses injections that have already been computed. We also mention that $e_\lambda(M) = 0$ whenever λ is a partition of a number exceeding every row-label of M. This observation makes Theorem 3.2 algorithmic. Moreover, by work of Ramos [Ram], a finitely presented FI-module V_M matches its eventual behavior once $n \geq \min\{X, Y\} + Y$, where $X = \max x_i$ and $Y = \max y_j$; this gives an algorithm computing $V_M[n]$ for all $n \in \mathbb{N}$.

Question 3.5. *If we knew the indecomposable injective* FI*-modules over $\mathbb{F}_p$, a similar algorithm would become available in that setting. What are the indecomposable injective* FI*-modules in characteristic p?*

Proof of Theorem 3.2. In [SS16], Sam-Snowden prove that the free FI-modules known as $M(\lambda)$ are injectives in the category of finitely generated FI-modules over the rationals. We have described $M(\lambda)$ in Construction 3.1, naming them I_λ to emphasize their injectivity. Exactness of the functor $\operatorname{Hom}(-, I_\lambda)$ guarantees that $V_M \mapsto \operatorname{corank} I_\lambda(M)$ is additive in short exact sequences of finitely presented FI-modules. Sam-Snowden also show that the Grothendieck group of the Serre quotient

$$\{\text{finitely presented FI-modules}\}/\{\text{FI-modules that are zero on large enough sets}\}$$

has a basis given by the classes $[M(\lambda)]$. It is easy to check that the theorem holds for these representations, and so it also holds for all finitely presented representations by additivity. □

References

[CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.*, 164(9):1833–1910, 2015.

[CF13] Thomas Church and Benson Farb. Representation theory and homological stability. *Adv. Math.*, 245:250–314, 2013. `doi:10.1016/j.aim.2013.06.016`.

[Dev14] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 6.6)*, 2014. `http://www.sagemath.org`.

[Ram] Eric Ramos. Homological invariants of FI-modules and FI_G-modules. `arXiv:1511.03964`.

[SS16] Steven V. Sam and Andrew Snowden. GL-equivariant modules over polynomial rings in infinitely many variables. *Trans. Amer. Math. Soc.*, 368(2):1097–1158, 2016. `doi:10.1090/tran/6355`.