# Representation Theory of Combinatorial Categories

John Wiltshire-Gordon

(Portions are joint with Jordan Ellenberg)

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Thanks to Ben Salisbury and Peter Tingley for the invitation, and also to Loyola and the AMS for hosting this event.

1. Motivation
2. Definitions
3. Examples
4. Theorem
5. Applications

Combinatorists of old considered assignments

$$\sigma \longmapsto M_\sigma$$

rules for converting permutations to square matrices

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$$M_\sigma \circ M_\tau = M_{\sigma\circ\tau}$$

$$M_e = I$$

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$$M_f \circ M_g = M_{f \circ g}$$

$$M_{\mathbb{1}} = I$$

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if for every function

$$f : [m] \longrightarrow [n]$$

we have an induced map

$$Mf : M[m] \longrightarrow M[n]$$

so that $(Mf)\circ(Mg) = M(f\circ g)$

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$\mathcal{F}$ is the primordial combinatorial category.

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The same formula works for any function $f$

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and so the entire category $\mathcal{F}$ acts, not just the symmetric group.

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where $i, j, k, \ell, m \in [n]$.

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where $i, j, k, \ell, m \in [n]$. In fact $V[n] \simeq H^1(\overline{\mathcal{M}_{0,n}}(\mathbb{R});\mathbb{Q})$

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where $i, j, k, l, m \in [n]$. In fact $V[n] \simeq H^1(\overline{\mathcal{M}_{0,n}}(\mathbb{R}); \mathbb{Q})$

This is a result of Etingof, Henriques, Kamnitzer, and Rains.

They give a similar presentation for every $H^p$.

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You could argue that the notion of "transformation" is more fundamental than the notion of "symmetry."

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abstract transformations $\rightsquigarrow$ concrete linear transformations

1. MOTIVATION
2. DEFINITIONS
3. EXAMPLES
4. THEOREM
5. APPLICATIONS

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intertwining the structure maps

$$\begin{tikzcd} Vx \arrow[r, "\varphi_x"] \arrow[d, "Vf"'] & Wx \arrow[d, "Wf"] \\ Vy \arrow[r, "\varphi_y"] & Wy \end{tikzcd}$$

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Representations of $\mathcal{D}$ form an abelian category where ker/im/coker are formed pointwise at the objects of $\mathcal{D}$

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If $\mathcal{D}$ has infinitely many arrows, a single vector typically generates an infinite-dimensional representation.

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If $\mathcal{D}$ is enriched in $\mathrm{Vect}_{\mathbb{F}}$, we recover representations of an $\mathbb{F}$-algebra in the same way.

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$\longrightarrow$ Most categories aren't Noetherian or Artinian, and that makes computations too hard.

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Artinian

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$$d \longmapsto \dim V_d$$

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| --- | --- |
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**Thm** Any f.g. representation $V : \Delta \longrightarrow \mathrm{Vect}_{\mathbb{F}}$ has
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$$\begin{aligned} \mathcal{FI} &\longrightarrow \mathrm{Top}^{op} \\ [n] &\longmapsto \mathrm{Injections}([n], X) \end{aligned}$$

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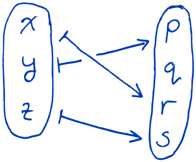
Precomposing with another injection $[m] \hookrightarrow [n]$ gives a representation

$$\mathcal{FI} \longrightarrow \mathrm{Top}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{F}}$$

$$[n] \longmapsto \mathrm{Injections}([n], X) \longmapsto H^k(\mathrm{Conf}_n X ; \mathbb{F})$$

Here is an injection $f: [3] \longrightarrow [4]$



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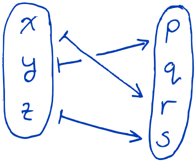


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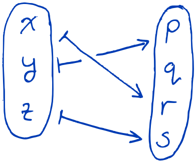
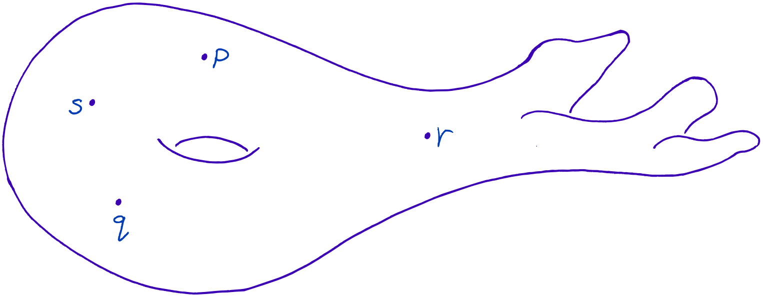



Start with a point in $\mathrm{Conf}_4 X$.

How does $f$ produce a point in $\mathrm{Conf}_3 X$ ?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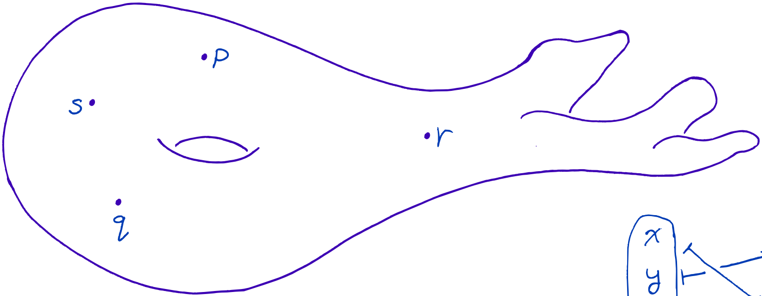
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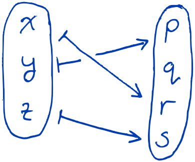
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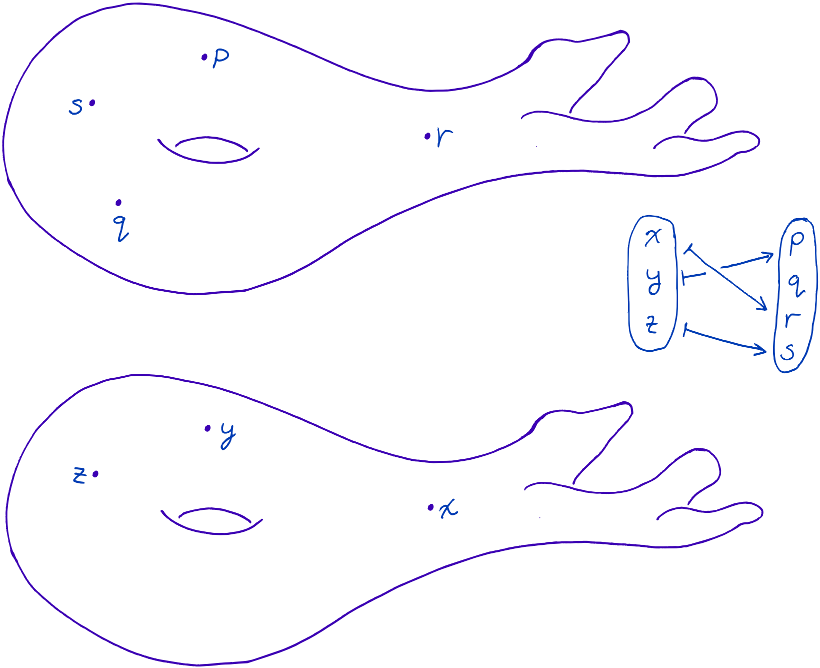
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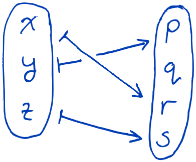
Here is an injection $f : [3] \longrightarrow [4]$

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So $\mathcal{F}\mathcal{I}$ acts by forgetting points and relabeling the ones that remain.

<u>Thm</u> (CEF) The $\mathcal{FI}$ representation

$$[n] \longmapsto H^k \operatorname{Conf}_n X$$

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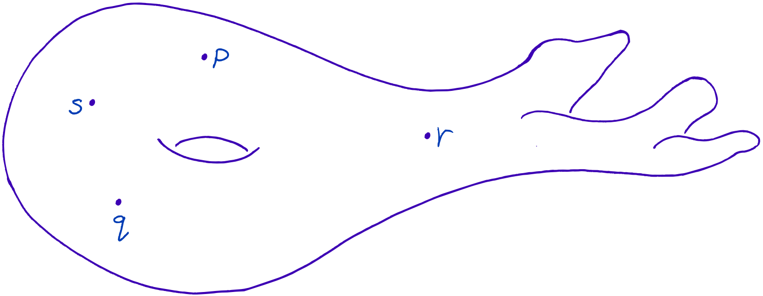
Let's try the construction with $\Delta$ instead of $\mathcal{FI}$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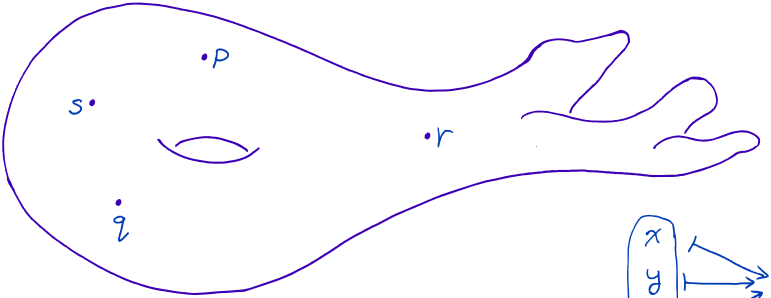
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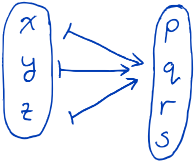
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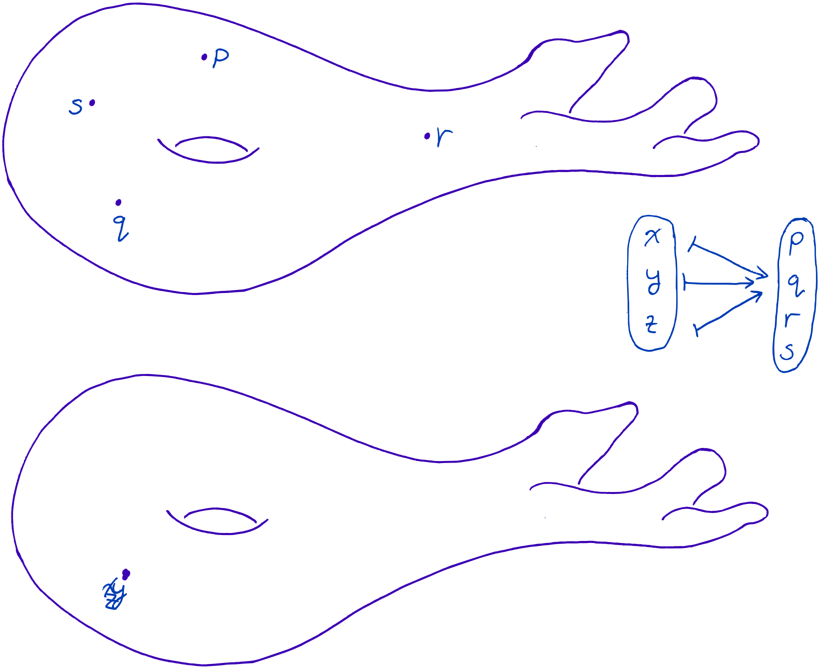

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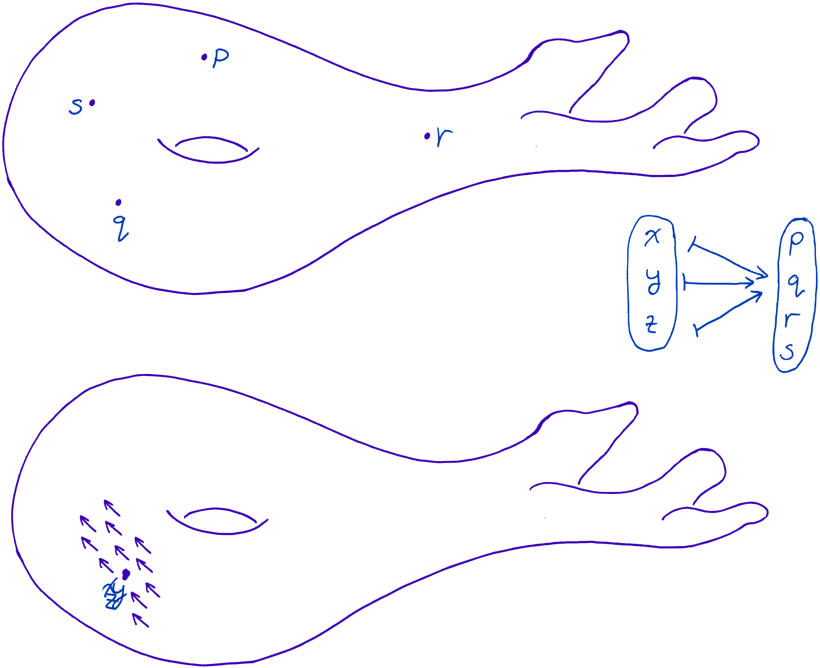
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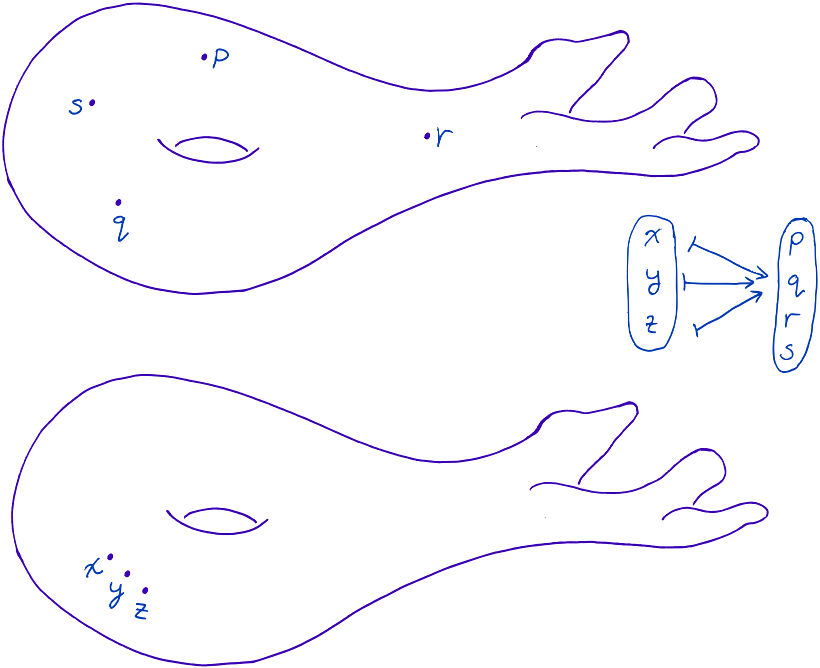
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Thm (Joint with JSE) If $X$ has an everywhere nonvanishing vector field, then the $\Delta$-representation

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Table of values for $\dim H^k(\mathrm{Conf}_n \; ; \mathbb{Q})$

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computed from a spectral sequence due to Totaro

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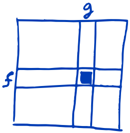
$M_s =$ 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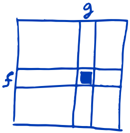
$M_s =$ 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$$M_s = \begin{array}{cc} & \begin{array}{ccc} \phantom{0} & g & \phantom{0} \end{array} \\ f & \left[\begin{array}{ccc} \; & \; & \; \\ \; & \blacksquare & \; \\ \; & \; & \; \end{array}\right] \end{array} \qquad (f,g)\text{-entry } \blacksquare = \begin{cases} 1 & s \circ f = g \\ 0 & s \circ f \neq g \end{cases}$$

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It turns out that $\underset{d}{\leq}$ is reflexive and transitive.

<u>Defn</u> A homological modulus for a category is a function $\mu : \{\text{objects of } \mathcal{D}\} \longrightarrow \left\{\begin{array}{l}\text{finite subsets of} \\ \text{objects of } \mathcal{D}\end{array}\right\}$

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Category

$\mathcal{F}$

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Thank You!

Thm can be found in arXiv: 1508.04107