HOMOTOPICAL ALGEBRA SEMINAR: COMPUTATIONS I

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1. Basic Toolkit

Algorithm 1.1. Compute a one-sided inverse of a full-rank matrix.

Algorithm 1.2. Compute the nullspace of a matrix.

Algorithm 1.3. Starting with a commuting square in $\mathbb{Q}$-Vect

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "t"] & Q \arrow[d, "u"] \\ R \arrow[r, "g"] & S, \end{tikzcd}$$

compute the corresponding map of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \ker f \arrow[r] \arrow[d] & P \arrow[r, "f"] \arrow[d, "t"] & Q \arrow[r] \arrow[d, "u"] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \ker g \arrow[r] & R \arrow[r, "g"] & S \arrow[r] & \operatorname{coker} g \arrow[r] & 0. \end{tikzcd}$$

Algorithm 1.4 (Equalizers and Coequalizers)**.** Use the previous algorithm to compute $\mathbb{Q}$-Vect limit and colimit functors of shape

$$\bullet \rightrightarrows \bullet \, .$$

Say what to do to objects, say what to do to maps, and provide the "∃!" maps coming from the universal property.

Algorithm 1.5. Given a finite poset P, compute the 0-, 1-, and 2-simplices of its nerve NP along with the face and degeneracy maps. Do the same for a finite group G and its nerve NG.

Algorithm 1.6 (Finite Limits and Colimits)**.** Let $\mathcal{I}$ be a finite category. A functor F from $\mathcal{I}$ to $\mathbb{Q}$-Vect can be encoded as a simplicial map NF from $N\mathcal{I}$ to $N\mathbb{Q}$-Vect. Given NF_0, NF_1, and NF_2, compute the limit of F. Say what to do to individual functors F, say what to do given a natural transformation $F \longrightarrow F'$, and provide the "∃!" maps coming from the universal property.

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2. Problems

Problem 2.1. Let G be the symmetric group on 3 objects. Define a map of sets

$$\begin{aligned} f\colon \quad G^2 &\longrightarrow G \\ (x,y) &\longmapsto xyx^{-1}y^{-1}. \end{aligned}$$

$\mathbb{Q}$-valued functions pull back along f to produce a linear map f^*. Compute the exact sequence

$$0 \longrightarrow \ker f^* \longrightarrow \mathbb{Q}^6 \longrightarrow \mathbb{Q}^{36} \longrightarrow \operatorname{coker} f^* \longrightarrow 0.$$

Compute the invariants with respect to the action of G by conjugation.

Problem 2.2. Let F be the field $\mathbb{Q}(i)$. Consider the map of varieties

$$g : M^{4\times 4}(F) \longrightarrow M^{6\times 6}(F)$$

sending a matrix to its matrix of 2×2 minors. Let g_1^* be the linear map pulling back degree 1 homogeneous polynomials to degree 2 homogeneous polynomials. Find the trace of complex conjugation acting on the kernel, image, and cokernel of g_1^*.