Computing with Finitely Presented Representations of a Category

John D. Wiltshire-Gordon

Presented at Banff International Research Station April 2016

COMPUTING WITH FINITELY PRESENTED REPRESENTATIONS OF A CATEGORY

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Thanks to DANIEL ERMAN and GREGORY G. SMITH for the invitation to participate and speak. Thanks also to NASSIF GHOUSSOUB and the staff here at the Banff Center

1. Matrices over a Category
2. Matrices as Presentations
3. Interpreting Presentations
4. Computing with Finitely Presented Modules
5. Dimension Zero Categories

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A matrix over a category $\mathcal{C}$ with coefficients in a commotative ring R has

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→ entries that are linear combinations of $\mathcal{C}$-morphisms from the row label to the column label.

Formally, define

$$\mathrm{Mat}_R^{\mathcal{C}}\left(x_1 \oplus \cdots \oplus x_k \longrightarrow y_1 \oplus \cdots \oplus y_l\right) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} R \cdot \mathrm{Hom}_{\mathcal{C}}(x_i, y_j)$$

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$$\mathrm{Mat}_R^{\mathcal{C}}(x^{\oplus} \to y^{\oplus}) \otimes_R \mathrm{Mat}_R^{\mathcal{C}}(y^{\oplus} \to z^{\oplus}) \xrightarrow{-\cdot-} \mathrm{Mat}_R^{\mathcal{C}}(x^{\oplus} \to z^{\oplus})$$

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extending the composition law of $\mathcal{C}$ bilinearly.

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So define a category Poly

$$\mathrm{Ob}(\mathrm{Poly}) = \mathbb{Z}$$

$$\mathrm{Hom}_{\mathrm{Poly}}(m, n) = \left\{ \begin{array}{c} \text{monomials of} \\ \text{degree } n-m \end{array} \right\}$$

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Composition is given by multiplication

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Let's take a look at a (usual) matrix over $\mathbb{R}$

$$M = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad M \in \mathrm{Mat}_{\mathbb{R}}(3 \times 2)$$

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This matrix presents an R-module $R^3 \big/ \text{Column Span of } M$

Which can also be written

$$\frac{Mat_R(3 \times \underline{1})}{M \cdot Mat_R(2 \times \underline{1})}$$

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$$V_M(c) = \frac{\mathrm{Mat}_R^{\mathcal{C}}(x^\oplus \longrightarrow c)}{M \cdot \mathrm{Mat}_R^{\mathcal{C}}(y^\oplus \longrightarrow c)}$$

Let's try out the definition on the matrix

$$M = \begin{bmatrix} xz - y^2 & yw - z^2 & xw - yz \end{bmatrix} \in \mathrm{Mat}_{R}^{\mathrm{Poly}}(0 \longrightarrow 2 \oplus 2 \oplus 2)$$

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$$V_M(c) = \frac{\mathrm{Mat}(0 \longrightarrow c)}{M \cdot \mathrm{Mat}(2 \oplus 2 \oplus 2 \to c)}$$

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$$= \frac{R \cdot \{\text{monomials of degree } c\}}{\text{polynomial combinations of the columns of } M}$$

In other words, $V_M(c)$ is the degree c part of the graded R-module presented by M.

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it is a graded module for the polynomial ring $R[x, y, z, w]$.

But what is a (right) module for a category $\mathcal{C}$?

That's easy. It's a functor $\mathcal{C} \longrightarrow \mathrm{Mod}_R$.

And indeed, $M \in \mathrm{Mat}_R^{\mathcal{C}}(x^{\oplus} \to y^{\oplus})$ gives rise to a right $\mathcal{C}$-module

$$V_M : \mathcal{C} \longrightarrow \mathrm{Mod}_R$$

where a morphism $f : c \longrightarrow c'$ induces a linear map

$$V_M(c) \longrightarrow V_M(c')$$

by postmultiplication with the little matrix $[f]$.

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We say V_M is a **finitely presented $\mathcal{C}$-module with presentation matrix M.**

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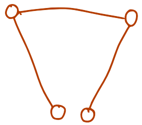
Suppose we have a finite graph



and on each node we have a stack of indistinguishable tokens (or a pile of sand, your choice)

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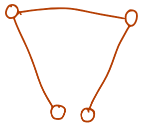

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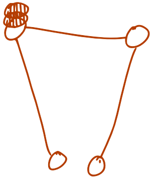
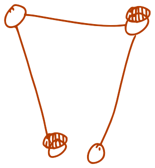
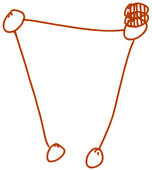
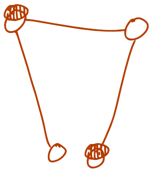
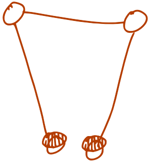
Introduce an equivalence relation on configurations of tokens generated by moves of the form

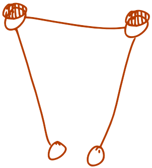




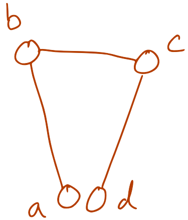








As a shorthand for



write $x^a y^b z^c w^d$

a monomial in the polynomial algebra $\mathbb{Q}[x,y,z,w]$

Then the relations we expressed read

$$y^2 = xz \qquad z^2 = yw \qquad yz = xw$$

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Let's give this matrix to M2, see what we get

$$\frac{1 - 3t^2 + 2t^3}{(1-t)^4} =$$

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$$1 + 4t + 7t^2 + 10t^3 + 13t^4 + \ldots$$

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For example $10t^3$ means there are 10 equivalence classes of positions for 3 tokens.

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For example $10t^3$ means there are 10 equivalence classes of positions for 3 tokens. (There are!)

0 0 3 0	1 0 2 0	0 1 2 0	1 1 1 0	1 0 1 1
0 0 0 3	0 1 0 2	1 0 0 2	1 1 0 1	0 1 1 1

In this interpretation, variables are locations and exponents give the number of indistinguishable tokens at each location

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"Unordered configurations of points on the space parametrizing the variables" if you like.

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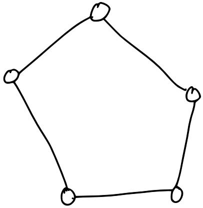
(Would it be fun to study $\mathbb{C}[x_m]/x_m^2$ where $m \in$  ?)

Next example:



how many ways are there to color this graph so that adjacent vertices receive distinct colors?

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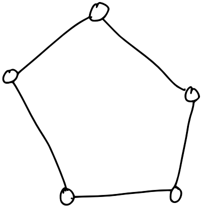
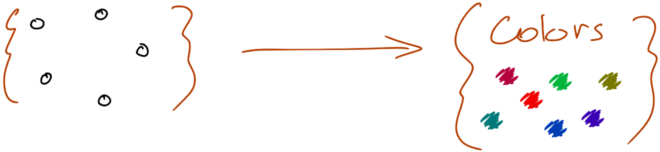


how many ways are there to color this graph so that adjacent vertices receive distinct colors?

Well, a coloring is just a certain sort of function



So it occurs to us to try $\mathcal{C}$ = category of finite sets with functions

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We write $acbbd$ for the function

$$\{1, 2, 3, 4, 5\} \xrightarrow{\begin{array}{c} 1 \mapsto a \\ 2 \mapsto c \\ 3 \mapsto b \\ 4 \mapsto b \\ 5 \mapsto d \end{array}} \{a, b, c, d\}$$

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We write acbbd for the function

$$\{1, 2, 3, 5, 4\} \longrightarrow \{a, b, c, d\}$$

$$\begin{aligned} 1 &\longmapsto a \\ 2 &\longmapsto c \\ 3 &\longmapsto b \\ 4 &\longmapsto b \\ 5 &\longmapsto d \end{aligned}$$

for example. Here is a matrix M

$$\begin{array}{cc} & \begin{array}{ccccc} 4 & 4 & 4 & 4 & 4 \end{array} \\ 5 & \begin{bmatrix} \text{aabcd} & \text{abbcd} & \text{abccd} & \text{abcdd} & \text{abcda} \end{bmatrix} \end{array}$$

as before, form the fraction

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$$\frac{\mathrm{Mat}_{\mathbb{Q}}^{\mathrm{Fin}}\left([5] \longrightarrow [n]\right)}{[\text{aabcd abbcd abccd abcdd abcda}] \cdot \mathrm{Mat}_{\mathbb{Q}}^{\mathrm{Fin}}\left([4]\oplus[4]\oplus[4]\oplus[4]\oplus[4] \longrightarrow [n]\right)}$$

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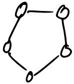
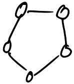
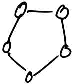
$$\frac{\mathbb{Q}\cdot\{\text{functions } [5] \to [n]\}}{\mathbb{Q}\cdot\{\text{functions that factor through one of those five entries}\}}$$

So this has a basis indexed by

valid n-colorings of 

So this has a basis indexed by valid n-colorings of 

and M gives a presentation for a **Fin**-module V_M with the property

$V_M[n] = \mathbb{Q} \cdot \{$ valid n-colorings of  $\}$

What would the presentation matrix

$$\begin{bmatrix} aabcd & abbcd & abccd & abcdd & abcda & abcde-bcdea \end{bmatrix}$$

mean?

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[aabcd abbcd abccd abcdd abcda abcde - bcdea]

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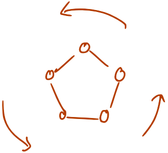
Colorings up to rotation



Configurations of n distinct distinguishable points in $\mathbb{C}$

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By a result of Arnol'd, we have a presentation

$$H^*\left(PConf^n(\mathbb{C});\mathbb{C}\right)$$

$$\simeq \frac{\bigwedge^* \{\omega_{ij}\}}{\left(\omega_{ij}-\omega_{ji},\ \omega_{ii},\ \omega_{ij}\wedge\omega_{jk}+\omega_{jk}\wedge\omega_{ki}+\omega_{ki}\wedge\omega_{ij}\right)}$$

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where i, j, k range over the finite set $[n]$

In particular,

$$H^2(PConf^n(\mathbb{C})) \simeq \frac{\langle w_{ab} \otimes w_{cd} \rangle}{\langle w_{aa} \otimes w_{bc},\ w_{ab} \otimes w_{cd} - w_{ba} \otimes w_{cd},\ w_{ab} \otimes w_{cd} + w_{cd} \otimes w_{ab},\ w_{ab} \otimes w_{bc} + w_{bc} \otimes w_{ca} + w_{ca} \otimes w_{ab} \rangle}$$

Removing the cruft, we have a matrix / Fin

$$M =$$

$$\begin{array}{cc} & \begin{array}{cccc} 3 & 4 & 4 & 3 \end{array} \\ 4 & \begin{bmatrix} aabc & abcd - bacd & abcd + cdab & abbc + bcca + caab \end{bmatrix} \end{array}$$

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So that

$$H^2(\mathrm{PConf}^n(\mathbb{C})) \simeq V_M[n]$$

where V_M is the Fin-module presented by M.

$$M = 4\left[\overset{4}{abcd + bacd} \qquad \overset{4}{abcd + bcda} \qquad \overset{5}{abcd + bcde + cdea + deab + eabc}\right]$$

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$$H^1(\overline{\mathcal{M}_{0,n}}(\mathbb{R});\mathbb{Q}) \simeq V_M[n]$$

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By a theorem of PAVEL ETINGOF, ANDRE HENRIQUES, JOEL KAMNITZER, ERIC RAINS

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$$a_{iii}^2 \cdot a_{jjj}^2 + a_{iij}^2 \cdot a_{jji}^2 + a_{iji}^2 \cdot a_{jij}^2 + a_{ijj}^2 \cdot a_{jii}^2$$

$$-2\big(a_{iii} \cdot a_{iij} \cdot a_{jji} \cdot a_{jjj} + a_{iii} \cdot a_{iji} \cdot a_{jij} \cdot a_{jjj} + a_{iii} \cdot a_{ijj} \cdot a_{jii} \cdot a_{jjj}$$

$$+ a_{iij} \cdot a_{iji} \cdot a_{jij} \cdot a_{jji} + a_{iij} \cdot a_{ijj} \cdot a_{jji} \cdot a_{jii} + a_{iji} \cdot a_{ijj} \cdot a_{jij} \cdot a_{jii} \big)$$

$$+4\big(a_{iii} \cdot a_{ijj} \cdot a_{jij} \cdot a_{jji} + a_{iij} \cdot a_{iji} \cdot a_{jii} \cdot a_{jjj} \big)$$

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$$+ a_{iij} \cdot a_{iji} \cdot a_{jij} \cdot a_{jji} + a_{iij} \cdot a_{ijj} \cdot a_{jji} \cdot a_{jii} + a_{iji} \cdot a_{ijj} \cdot a_{jij} \cdot a_{jii}\big)$$

$$+4\big(a_{iii} \cdot a_{ijj} \cdot a_{jij} \cdot a_{jji} + a_{iij} \cdot a_{iji} \cdot a_{jii} \cdot a_{jjj}\big)$$

$\{i,j\} \times \{i,j\} \times \{i,j\}$ hyperminors

$i,j \in [n]$.

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Presentation matrix?

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Let's try the category $\mathbf{Rel}$ = finite sets with relations

What will a presentation look like?

$$\frac{\langle x_S \cdot x_T \rangle}{\langle x_{S \cup T} \cdot x_{T \cup U} + x_{T \cup U} \cdot x_{U \cup S} + x_{U \cup S} \cdot x_{S \cup T} \rangle}$$

where S, T, U range over subsets of $[n]$

What about matrices / $\mathrm{Vect}_{\mathbb{F}_q}$?

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Hard to say what this means.

We have variables indexed by matrices over $\mathbb{F}_q$

1. Matrices over a Category
2. Matrices as Presentations
3. Interpreting Presentations
4. Computing with Finitely Presented Modules
5. Dimension Zero Categories

How will we compute with presentation matrices?

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In a classical setting, there are essentially two styles

Column Reduction and Gröbner Bases

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works for: Finite-dimensional algebras | Polynomial algebras

Warm-up Let A be a finite dimensional algebra over a field.

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Suppose we have a presentation matrix $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ filled with elements of A

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which is a usual matrix over the ground field.

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→ FI should be reasonably fast, especially if we use recent results of THOMAS CHURCH and JORDAN ELLENBERG

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finite dimensional algebras	:	dim = 0
FI	:	dim = 1
Polynomial algebra in k variables	:	dim = k

So let's keep it easy and study
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A category $\mathcal{C}$ has <u>dimension zero</u> over a field $\mathbb{F}$ if every finitely generated $\mathcal{C}$-module V has a finite filtration

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\ell = V$$

So that each successive quotient V_i / V_{i-1} is simple.

Fin is dimension zero over any field!

So Fin is more computationally analogous to a finite dimensional algebra than a polynomial algebra or FI.

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I have computed the simples and their injective hulls. The same substitution trick works on presentation matrices / Fin!

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<u>Defn</u> Write $x \leq_c y$ if the identity matrix is in the span of the matrices Q_s.

Crucial fact about $\leq_{c}$, $c \in \mathcal{C}$

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So that for all $c, x \in \mathcal{C}$ there exists $y \in \mu(c)$ such that $x \underset{c}{\leq} y$.

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Claim: $[3] \underset{[1]}{\leq} [2]$

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Proof: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

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In fact, $[2]$ is a maximum for $\underset{[1]}{\leq}$

and so we may take $\mu([1]) = \{[2]\}$

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So to check if $\mathcal{C}$ is dimension zero, you play around with the various preorders $\leq_c$ until you find elements that form a joint upper bound. After collecting some data, you conjecture and prove some general rule that works for any $c \in \mathcal{C}$

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So to check if $\mathcal{C}$ is dimension zero, you play around with the various preorders $\leq_c$ until you find elements that form a joint upper bound. After collecting some data, you conjecture and prove some general rule that works for any $c \in \mathcal{C}$ at which point you have μ

$$\mu([k]) = \{[k], [k+1]\}$$

serves as a homological modulus for Fin

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Other dimension zero categories will give new styles of presentation.

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Computing Hilbert series becomes algorithmic as soon as we know the simples and their injective hulls.

Thank You!