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Notation, Mathematical (*see*: Notation, Logical)

Notation is a conventional written system for encoding a formal axiomatic system . Notation governs:

- the rules for assignment of written symbols to elements of the axiomatic system
- the writing and interpretation rules for well-formed formulae in the axiomatic system
- the derived writing and interpretation rules for representing transformations of formulae, in accordance with the rules of deduction in the axiomatic system.

All formal systems impose notational conventions on the forms. Just as in natural language, to some extent such conventions are matters of style and politics, even defining group affiliation. Thus notational conventions display sociolinguistic variation ; alternate conventions are often in competing use, though there is usually substantial agreement on a 'classical' core notation taught to neophytes.

Excluding logic (see *Notation, Logical*), mathematics as a whole is divided into three parts, of which one is **algebra** (roughly, the realm of the discrete, including number theory and combinatorics), another **geometry** (roughly, the realm of the visibly-codable, including topology), and the third what ordinary people call *calculus*, but which is spoken of by mathematicians as **analysis** (roughly, the realm of the continuous, including statistics, differential and integral calculus, theory of functions, and complex variables). All these areas overlap and grow into one another fractally; in particular, they crucially involve the concept of number.

Numbers and Arithmetic Symbols. Though there appear to be some cultures where counting and numbers are unknown (see Everett 2004), almost all languages contain some numbers, and almost all cultures practice counting and other mathematical activities. The written representation of quantities and numbers is as old as, and may even predate and prefigure (see Schmandt-Besserat 1992 and Zimansky 1993), writing itself. After alphabetic writing was developed, it was normal to use individual letters to represent particular numbers in an additive system like Roman numerals, where, for instance, *MDCCVII* represents 1000+500+100+100+5+1+1, or 1707. Similar systems existed for ancient Greek and Hebrew; even after the widespread use of Hindu/Arabic numerals, the Hebrew system of alphabetic number representation survived as a Kabbalistic numerological system known as *Gematriya*, in which the number associated with any Hebrew word has gnostic significance. See Ifrah 1991, Ch. 2 in Barrow 1992, and Pettersson 1996 for history.

The Hindu/Arabic numerals, with their **positional notation**, were originally developed in India during the Gupta empire and spread to the Muslim world, whence they made their way to Europe after the Crusades. In positional notation, the position of each numeral in a series indicates the power of the base that it enumerates; thus *1707* in base ten represents $(1x10^3)+(7x10^2)+(0x10^1)+(7x10^0)$. Such a notation (including *zero*, which is necessary for it to work) represents an enormous advance in numeration over additive systems, because it makes possible algorithmic calculations like long division.

Hindu/Arabic numerals in base ten are all but universal in the writing systems of the modern world. While the *decimal* (base ten) system is the norm, in principle any numeric base can be used, and several have been: the Babylonian numeric system was *sexagesimal* (base 60, from which we inherit the 360º circle and the 60-minute hour); the Mayan system was *vigesimal* (base 20); vestiges of a duodecimal (base 12) system remain in English vocabulary (*dozen, gross*); and several systems based on powers of 2 – *binary* (base 2, digits *0* and *1*), *octal* (base 2^3, digits *0-7*), and *hexadecimal* (base 2^{2^2}, digits *0-9* plus *A-F*) – are in common use in computing contexts.

By contrast with the numerals, all of the usual symbols for arithmetic operations (+, –, ×, ÷) are of European origin, originating among a congeries of symbols used between the 15th and 17th centuries. + is simply the stylized letter † of *et*, an early ampersand. The symbol for division (÷), now seen as an obvious mnemonic of the fraction notation with a horizontal bar introduced by the Arabs, was in fact originally used in Europe for subtraction, and it is from this sign that the modern – sign comes. Finally, × is only one of several symbols for multiplication, introduced and promoted by William Oughtred (1574-1660) in the *Clavis Mathematicae* (1631). Other multiplication notation in use today includes the raised dot (*a•b*), which Leibniz preferred, and simple juxtaposition (*ab*), which appears to have been the original notation in India, dating from the 10th century and appearing sporadically in Arabic and European sources afterward. By the 18th century, these symbols had become general in European mathematical notation.

Other symbols. Mathematics is a gigantic intellectual area, famously unapproachable by the uninitiated, replete with multiple notational conventions and the materials and will to invent new ones. Considering the size of the field and the elaborate profusion of its notation, this article can do no more within its space constraints than touch on the notations of several areas (besides logic) that impinge on natural language, and remark on general notational tendencies in mathematics.

One example is the notation used in modern integral and differential calculus, which descends from Leibniz's original invention in the seventeenth century; Newton famously invented calculus independently before Leibniz, but published later. More importantly for our purpose, he used a different notation which, while patriotically employed in English mathematics for the next century, proved awkward and was eventually abandoned for Leibnizian differentials.

The basic concepts in calculus are the **derivative** of a function $y = f(x)$ and its **integral**. The derivative is another function f' whose value at any point x is the instantaneous rate of change with respect to x of the value y of $f(x)$. This function in turn has its own derivative f'', which is thus the **second derivative** of f, and so on. Newton used overdot $\dot{y}$ for the derivative. The integral of a function is another function whose value represents the area between the curve and an axis. Newton used overbar $\bar{y}$ to represent the integral.

While first and second derivatives are still quite commonly indicated with prime marks, replication of dots and primes becomes difficult when speaking of higher derivatives, or when using a variable. Leibniz's notation, unlike Newton's, used a ratio $\frac{dy}{dx}$ between the **differentials** dy and dx (infinitesimal increments in y and x) to represent the derivative. Second and higher derivatives are indicated by exponents on the differentials; thus

$f'(x) = \frac{dy}{dx}$, and $f''(x) = \frac{d^2y}{dx^2}$, producing a natural **operator** notation $\frac{d}{dx}$ for differentiation, in contrast to Newton's notation, which did not lend itself to such generalization.

For integrals Leibniz used a special operator, the long-S **integral** symbol $\int f(x)\,dx$. The differential *dx* after the function in this notation is a mnemonic for **multiplication**, just as the ratio between differentials symbolizes **division** in the derivative notation. Leibniz's convention recalls the Fundamental Theorem of Calculus (i.e, derivative and integral are inverses) by use of the inverse operations multiplication and division to represent them; by contrast, Newton's notation was arbitrary and had no mnemonic value.

Following the invention of the calculus, mathematics endured for several centuries a disconnect between axiomatic theory and actual practice. Calculus worked, for reasons which mathematicians could not explain, and about which engineers did not care. The result was that practice got very far ahead of axiomatic theory, until at the beginning of the 19^{th}-century mathematicians resolved to put such axiomatization at the front of their goals.

By 'axiomatization', mathematicians mean the establishment of a system of undefined elements, axioms, and rules of inference for deriving theorems from the axioms, based on the model of Euclidean geometry, which for millennia has stood as the exemplar of pure mathematical thought. This is what is meant in mathematics as a 'formal system'; in linguistics this term has a different sense, since linguists do not establish axioms or prove theorems.

The most significant area of mathematics in this axiomatization effort has been set theory, which now, as a result of the 19^{th}-century work, is taken to underlie all areas of mathematics and logic. Set theory has only three undefined terms: set, element of a set, and inclusion in a set. These are universally symbolized with capital letters to name sets, lower-case letters to name elements in them, and the relation symbol ϵ for set inclusion, thus $a \epsilon A$ indicates that *a* is an element of the set *A*. Sets may also be specified by using curly brackets, e.g. the set of English vowel letters is $\{a, e, i, o, u\}$, and the set *N* of natural numbers is $\{1, 2, 3, \ldots\}$. Since most interesting sets are infinite, or at least very large, simple enumeration is rarely sufficient; thus one finds qualified set descriptions, e.g. the set of perfect squares can be described as $\{x^2\mathbf{:}\ x \epsilon N\}$, pronounced 'the set of all *x* squared, such that *x* is in *N*'.

From this set convention come all the various uses of parentheses and brackets for inclusion and specification, like the **ordered pair** *(0, 1)* in algebra, which is simply a special set consisting of two elements **in order**. Exactly the same notation *(0, 1)* in topology is used to denote an **open set**, i.e, the set of all points on a line lying between *0* and *1*, but **not** including the endpoints; if an endpoint is included, the notation changes to a square bracket, so that the **closed set** *[0, 1]* has a largest and a smallest member, while the open set *(0, 1)* has neither. In analysis, *(0, 1)* following a function symbol like *f* implies that *f* is a function of two arguments to be applied to *0* and *1*.

Fonts play a more important role in mathematical notation than one might expect, given the arbitrariness of font choice. The problem in mathematics is that there is an infinite number of concepts that one might wish to represent with a simple symbol, but there is a limited number of simple symbols that can be printed easily. Hence it is common to mix and match Latin, Greek, and Fraktur alphabets (among others), in boldface, italic, and plain variants, each indicating some systematic feature of the concept under discussion. Such use is quite idiosyncratic, with considerable individual variation and specific traditions in each sub-area. For instance, in relativity theory, tensor indices use Greek letters when they vary from *0* to *3* and Latin letters when they vary from *1* to *3*.

General tendencies. Mathematical notation, like all mathematics, is an exemplification of the grounding metaphors that structure the field (see Lakoff & Nuñez 2000). These, in turn, are mostly based on the basic human cognitive activity of predication, instantiated by concepts like **function**, **operator**, or **relation**, which name various symbols and types of sentence-like symbolic syntax in all areas of mathematics and logic: a function $f(x)$ is a clause with a predicate and argument(s); an equation $a = b$ is a topic-comment clause. All mathematical notation is ultimately pronounceable, a subset of written language, though like Chinese characters it is not language-specific.

References

Barrow, John D. 1992. *Pi in the Sky: Counting, Thinking, and Being*. Little, Brown.

Cajori, Florian. 1928-29. *A History of Mathematical Notation.* (2 vol.) Lasalle, Illinois: The Open Court Publishing Co.

Everett, Daniel L. 2005. 'Cultural Constraints on Grammar and Cognition in Pirahã: Another Look at the Design Features of Human Language', *Current Anthropology* 46.4.

Ifrah, Georges. 2000. *The Universal History of Numbers: From Prehistory to the Invention of the Computer*. John Wiley (orig publ as *Histoire Universelle de Chiffres*, 1981).

Lakoff, George, and Rafael Nuñez. 2000. *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being*. Basic Books.

Pettersson, John S. 1996. 'Numerical Notation', Section 69 (pp 795-806) in Daniels & Bright (eds), *The World's Writing Systems*. Oxford U Press.

Schmandt-Besserat, Denise. 1992. *How Writing Came About*. University of Texas Press.

Zimansky, Paul. 1993. Review of D. Schmandt-Besserat (1992), *Journal of Field Archaeology* 20:513-17.