

Notation, Logical (*see*: Notation, Mathematical)

**Notation** is a conventional written system for encoding a formal axiomatic system .

Notation governs:

- the rules for assignment of written symbols to elements of the axiomatic system
- the writing and interpretation rules for well-formed formulae in the axiomatic system
- the derived writing and interpretation rules for representing transformations of formulae, in accordance with the rules of deduction in the axiomatic system.

All formal systems impose notational conventions on the forms. Just as in natural language, to some extent such conventions are matters of style and politics, even defining group affiliation. Thus notational conventions display sociolinguistic variation ; alternate conventions are often in competing use, though there is usually substantial agreement on a ‘classical’ core notation taught to neophytes.

This article is about notational conventions in formal logic, which is (in the view of most mathematicians) that branch of mathematics (logicians, by contrast, tend to think of mathematics as a branch of logic; both metaphors are correct, in the appropriate formal axiomatic system) most concerned with many questions that arise in natural language, e.g, questions of meaning, syntax, predication, well-formedness, and – for our purposes, the most important such – detailed, precise specification. Specification is the purpose of notation, both in mathematics and in science, but such precise conventions are unavoidably context-sensitive. Thus the use of logical notation is different in logic and in linguistics. Bochenski 1948 (English translation 1960) is still the best short introduction to logical notation.

Almost all logical notation is modern, dating from the last century and a half. However, there is some prior work that deserves comment here, since logic, alone of all mathematical fields, was widely studied and significantly developed in the European Middle Ages.

The principal concern of Medieval logicians was the syllogism. By the time of the Renaissance, there was an extensive and thorough account of syllogistic. One of its major achievements was the development of systematic names for the modes of the syllogism. These names (as conventionally grouped, *Barbara, Celarent, Darii, Ferio, Barbarix, Feraxo; Cesare, Festino, Camestres, Baroco, Camestrop, Cesarox; Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison; Bramantip, Camenes, Dimaris, Fesapo, Fresison, Camenop*) constitute the first real notational convention in logic.

The names are mnemonic, designed to be chanted, like Paṇini’s rules. The three vowels in each name are the letters *A, E, I, O*, which mark the vertices of the Square of Opposition, indicating the proposition type (respectively, Universal Affirmative, Universal Negative, Existential Affirmative, and Existential Negative) of each of the three propositions of the syllogism. The letters *s, p, m*, and *c* also have specific meanings in these mnemonics, summarizing relevant logical properties of each type, thus serving the notational goal of detailed, precise specification. Syllogistic is largely of historical interest in modern logic, but its concerns, terminology, and notation continued to be used and understood until well into the development of modern logic.

In modern logic and mathematics, notation is a necessary part of a *calculus*, one of a number of special sets of formalized concepts and techniques for manipulating them. The metaphor refers to the origins of classical *calculation*, which was performed with pebbles (Lat *calculus*) on a counting-table or *abacus*. This metaphor licenses notational practice with formal systems, which is to

- **Encode**: represent parts (hopefully, natural parts) of a quantity, concept, or truth with symbols, then
- **Calculate**: push those symbols around in conventionally-accepted fashions, hoping thereby to
- **Decode**: find in the changed symbolic patterns representations of previously unknown quantities, concepts, or truths.

Calculi are an invention of the Seventeenth Century; the best known are Leibniz's integral calculus and Newton's differential calculus, which (together) are understood as the default meaning of *calculus* in modern English.

There are many calculi in modern mathematics, some of which exist in name only, like Leibniz's putative *calculus ratiocinator*. In symbolic logic, which is the closest thing to what Leibniz called for, the two most important calculi, each with its own notational conventions, are Propositional Calculus and Predicate Calculus, both of which were originally intended straightforwardly (as the titles of their original publications show) as tools for the representation of human thought. Language did not enter into the picture at first, except as a transparent expression of thought.

**Propositional Calculus.** Symbolic logic, and its notation, originated in the works of George Boole (1815-1864), of which Boole (1854) is the best known. Boole's intention was to produce an algebraic account of propositions as combined via what we have come to call **Boolean** connectors, principally (logical) *and*, *or*, *not*, *equivalent*, and *implies*, which then achieved the dual status of English words that are not only prominent in logical discussion, but are also mathematically defined functors. Boole first made explicit the alternation between logical *and* and *or* enshrined in DeMorgan's Laws, comparing them (respectively) to multiplication and addition in algebra; thus he represented *x and y* as  $xy$ , while *x or y* was  $x+y$ ; he used numbers throughout, using  $1-x$  to represent *not x*, for instance. This notational convention, and the system based on it, is often called **Boolean Algebra** (though technically it is a *complemented distributive lattice*, not an algebra). Boole's logic did not use quantifiers *per se*; instead he dealt with the quantification inherent in syllogistic by using the traditional letters A, E, I, O.

Propositional calculus, the calculus of arbitrary whole propositions without regard to their predicates or arguments, uses two major notations. One, usually called 'Classical' or 'Standard', exists in numerous individual variations and is usually the one taught to students; the other, called 'Polish', 'Łukasiewicz', or 'Prefix', is standardized and in widespread technical use.

**Classical notation** for propositional calculus uses lower-case letters for propositions (traditionally  $p, q, r, s$ ) and special symbols for their connectives. The two truth values of a proposition are usually either **T** and **F**, or **1** and **0**. In ternary logics, **T / F / #** is more common than numeric codes, since arithmetic systems like **1 / 0 / -1** or **1 / ½ / 0** make implicit algebraic claims.

The Classical special symbols for functors include:

|                               |  |
|-------------------------------|--|
| <i>not p</i> :                | $\neg p, \bar{p}, \sim p, \bar{p}$                     |
| <i>p and q</i> :              | $p \wedge q, p \cdot q, p \& q, pq$                    |
| <i>p or q</i> :               | $p \vee q$   |
| <i>p implies q</i> :          | $p \supset q, p \rightarrow q, p \Rightarrow q$        |
| <i>p is equivalent to q</i> : | $p \equiv q, p \leftrightarrow q, p \Leftrightarrow q$ |

In each case, the first symbol is the most widely accepted. In addition to functors, propositional logic also contains symbols for pragmatic connectives used in proofs, such as entailment, which usually uses a single arrow ( $\rightarrow$ ), and assertion, which uses a variety of symbols, including  $\vdash$ .

There are also parentheses, since grouping of formulae can introduce significant ambiguity, which is anathema in logic. In extended use, parentheses were found to be burdensome, since balancing them was a frequent source of avoidable error. To combat this, Whitehead and Russell in their monumental *Principia Mathematica* (1910-13), developed a special parenthesis-free notation to augment their Classical formulae, based on using groups of 1, 2, 3, ... dots to separate propositions. This version is rarely seen today.

**Polish notation** was developed and popularized by Jan Łukasiewicz (1878-1956) in the early 1920s as a byproduct of his development of ternary logic, for which he also invented the truth table. In this notation, propositions are again represented by lower-case letters, but functors are upper-case letters placed immediately before their argument(s): *not p* is **Np**, *p and q* is **Kpq**, *p or q* is **Apq**, *p implies q* is **Cpq**, and *p is equivalent to q* is **Eppq**. Since functors form valid propositions, these can be nested indefinitely without recourse to parentheses; for instance, De Morgan's Laws, which are stated in Classical notation as  $\neg(p \wedge q) \equiv \neg p \vee \neg q$  and  $\neg(p \vee q) \equiv \neg p \wedge \neg q$ , are stated in Polish notation respectively as **EKNpqANpNq** and **EANpqKNpNq**.

Since the prefixal position of the Polish functors is arbitrary, a postfixal variant, called **Reverse Polish Notation**, or RPN (linguists always note that it should be called 'Japanese Notation', because it acts exactly like an SOV language), is equally valid, and is widely used in computing circles, since it turns out to be ideally adapted to performing calculations using a pushdown stack. In RPN, De Morgan's laws are stated as **pNqKpNqNAE** and **pNqApNqNKE**.

Modal Logic, an extension of propositional calculus into modality, introduces two more common notational symbols,  $\diamond p$  for *p is possibly true* (in Polish notation **Mp**, for **Möglich**), and  $\square p$  for *p is necessarily true* (Polish **Lp**, for **Logisch**). De Morgan's Laws for modal logic (where  $\square$  is associated with  $\wedge$  and  $\diamond$  with  $\vee$  – see McCawley 1993 for details) can thus be stated

$$\neg \square p \equiv \diamond \neg p \quad (\text{Polish ENLpMNp})$$

and

$$\neg \diamond p \equiv \square \neg p \quad (\text{Polish ENMpLNp}).$$

**Predicate Calculus.** Quantified Predicate Calculus (both First- and Second-Order) was first axiomatized and used notationally by Gottlob Frege (1848-1925) in 1879, a quarter-century after Boole. In predicate calculus, the atomic proposition of propositional calculus is split into predicate and argument(s), allowing far more representation of actual natural language phenomena. To represent predication, Frege introduced the now-standard functional notation, widely used in mathematics. In this notation, an atomic proposition  $p$  could now be seen to consist of a **predicate** (typically using upper-case letters) operating on **arguments** expressed by following parenthesized variables, in the same way as a mathematical function like  $f(x,y) = (x^2 + y^2)^{1/2}$ , e.g. **TALL (x) = X is tall**, **SEE (x, y) = X sees Y**, and **GIVE (x, y, z) = X gives Y to Z**.

In particular, quantifiers were separated by Frege for the first time from their traditional Aristotelian *A, E, I, O* notation. Quantifiers in natural language are specialized words that often involve special syntax; normally they appear in construction with some noun, which they are said to **bind**. However, their syntax varies widely, and quantifier ambiguities are frequent.

Modern logic admits what McCawley 1993 calls “the logicians’ favorite quantifiers”:

- the **existential quantifier**,  $\exists x$ , pronounced ‘for some  $x$ ’ or ‘there exists an  $x$ ’,
- and
- the **universal quantifier**,  $\forall x$ , pronounced ‘for all/every/each  $x$ ’.

The  $x$ ’s in each case are *dummy variables*; they do no more than indicate which variable in the proposition following is to be considered bound by the quantifier.

Quantifiers are rigidly controlled in the formulae in order to avoid ambiguity (and indeed to allow natural language ambiguities to be explicated). They are placed before the formula containing the variable they bind, and their relative placement serves to denote the concept of scope, which is highly relevant to the three natural language elements represented in logic by operators, i.e. quantification, negation, and modality, all of which govern scope phenomena like Negative Polarity. Thus the two ambiguous readings of *A boy beat every girl at tennis* are represented by  $(\exists x) (\forall y) \text{BEAT}(x, y)$  and  $(\forall y) (\exists x) \text{BEAT}(x, y)$ .

Naturally, there are variations in quantifier notation as well: a formula like De Morgan’s Laws for quantifiers, which can be written  $\text{ENPx}\phi\text{xSxN}\phi\text{x}$  and  $\text{ENSx}\phi\text{xPxN}\phi\text{x}$  [ $\text{Px}$  is  $(\forall x)$  and  $\text{Sx}$  is  $(\exists x)$ ] in Polish notation, comes out as  $\neg(\forall x) \phi(x) \equiv (\exists x) \neg\phi(x)$  and  $\neg(\exists x) \phi(x) \equiv (\forall x) \neg\phi(x)$  in Classical notation, which also optionally admits a simple parenthesized variable  $(x)$  instead of  $(\forall x)$ , and also one with a circumflex ‘hat’ ( $\hat{y}$ ) instead of  $(\exists y)$ , in the appropriate position. The use of parentheses, colons, brackets, and other punctuation with quantifiers is inconsistent and follows individual style, which is usually oriented towards scope delimitation.

### References

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