Foundations: The 4 Schools

Set Theory
Improving on Cantor
Zermelo-Fraenkel Axioms for Set Theory (1908 & 1922):
- Two sets are identical if they have the same members
- The empty set exists
- Infinite sets exist
- If $x$ and $y$ are sets, then the unordered pair $\{x,y\}$ is a set
- The union of a set of sets is a set
- Any property that can be formalized in the language of the theory can be used to define a set
- The set of all subsets of any given set is a set
- $x$ does not belong to $x$
- The axiom of choice

Formalism
Founded by David Hilbert
Completely axiomatic
Used formal system of theorem statement and proof.
Used a restricted metalanguage for the logic used to prove theorems.
Devastated by Gödel’s Proofs, but for many still the “official” theory.

Logicism
Precursors: de Morgan and Boole
1910 Russell & Whitehead, Principia Mathematica
Problems: The Axioms of Reducibility, Infinity, and Choice
Avoid antinomies by resort to Theory of Types

Intuitionism
Precursors:
Kronecker ‘God made the natural numbers; all the rest is the work of man’
Poincaré on logistic attempts to define number:
‘ “Zero is the number of objects that satisfy a condition that is never satisfied.” But as never means in no case I do not see that any great progress has been made.’
‘ “1 is the number of a class in which any 2 elements are identical.” This seems to be a definition admirably suited to give an idea of the number 1 to people who have never heard of it before; but I am afraid if we asked what 2 is, they would be obliged to use the word one.’
1907 Brouwer’s dissertation
Accepted:
- “Intuitively obvious” set of natural numbers.
- Addition and Multiplication
- Rational numbers, and certain irrationals that can be constructed.
Rejected:
- Infinite sets and cardinal numbers
- Law of Excluded Middle in logic
- Proof by Mathematical Induction
- Existence Proofs that are not constructive
- Dependence on logic or axiomatic method