Euler's Formula

Many functions can be calculated by the sum of a series of decreasing terms ("Taylor Series"). Since each term gets smaller, a point is reached (the point of "diminishing returns") after which further summation is unnecessary to give a certain degree of accuracy. Therefore, in practice, such series are often used to calculate the values of the functions.

The **exponential function** (e^x) and the trigonometric functions **sine** (sin x) and **cosine** (cos x) have the following Taylor Series:

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots
\]

\[
sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots
\]

\[
cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots
\]

Note that these series resemble each other very much, and if one could somehow generate exactly the right pattern of plus and minus signs, it seems like it would be possible to add the series for sine and cosine to get the series for the exponential function. As it happens, if we put an i in each x in the exponential series we wind up with exactly the right pattern of signs. There are no i's in the even power terms (i.e., the cos x series), since the even powers of i are real numbers (i^2 = -1, i^4 = +1), but there will be i's in the odd power terms (the sin x series, since i^3 = -i, i^5 = +i).

The result is **Euler's Formula**, which states that:

\[e^{ix} = \cos x + i \sin x\]

When we substitute π for x in Euler's Formula, we get:

\[e^{i\pi} = \cos \pi + i \sin \pi\]

Now cos \(\pi = -1\), and sin \(\pi = 0\), facts which can be looked up anywhere or verified on any calculator costing more than $10. So the expression above simplifies thus:

\[e^{i\pi} = -1 + (0 \times i) = -1\]

Or, put in standard polynomial form:

\[e^{i\pi} + 1 = 0\]

Q. E. D.