Entry Conditions and the Market Value of Capital

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Abstract

This paper presents a model of aggregative investment. In the model, the business sector expands by adding new establishments. Costs of investment are linear but, in the case of net investment, depend upon the risk of entry failure. We can measure the risk, and time-series variations in it, from micro data. We show that average entry-failure rates are sufficiently high that entry risk alone can explain about one-half of the intangible capital stock of the US. Despite linear costs, we show that our model has potentially interesting dynamic implications.

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1 Introduction. We propose a new model of the cost of investment. In the model, the business sector expands by adding new establishments, and a part of the cost of net investment arises from entry-failure risk. The new cost component enters business profit calculations linearly, as seems consistent with several recent findings (Hall [2004], Shapiro [1986])[1]. An advantage of the new approach is that we can calibrate entry-failure rates directly from micro data. We show that entry-failure rates in the data are quite high, and that they can explain a value of Tobin's q noticeably above 1. Second, we show that time series variations in risk are a possible explanation for the positive correlation between investment spending and Tobin's q described in the original literature (e.g., Hayashi [1982], Summers et al [1981]). Third, although our data series on entry is too short, at this point, to reveal trends, it hints that temporary changes in entry-failure probabilities tend to accompany macro shocks. Analysis of the model shows that short-lived changes in risk may have large impacts on both interest rates and investment.

In our model, the physical capital of existing business establishments is subject to wear and tear depreciation, and establishments themselves are subject to obsolescence and death according to an exogenous Poisson process with parameter φ. New establishments arise from new investment to take the place of those that terminate.[2,3] The new establishments have new characteristics, which determine their appeal. We assume that the only sure way to verify the workability of a new investment-project design is to have an entrepreneur undertake the project, opening production at normal operating scale.[4] Successful entry,

[1] See also Abel and Blanchard [1986] and others

[2] A "firm" can include many establishments (i.e., plants). In our model, an existing firm expands by adding new establishments.

[3] An earlier draft of this paper allowed a fraction of successful entrants the opportunity to duplicate themselves a given number of times, while avoiding startup costs. That opportunity corresponds to a favorable cost-function parameter in, for example, Jovanovic [1982] or Hopenhayn [1992]. This draft drops such features for simplicity.

[4] As in Jovanovic [1982] (Hopenhayn [1992] and Hopenhayn and Rogerson [1993]), prospective investment projects know only the overall probability of successful entry. In our model, once they attempt entry, they learn their potential quickly.

having probability θ, creates a new business establishment. Failure leads to immediate scrapping of a project's physical capital.

Roughly speaking, a dollar's worth of new investment must have a market value that exceeds 1 in the event of success. Otherwise, attempted entry would cease. This creates an equilibrium value, in most time periods, of Tobin's average q higher than 1. Recent work (e.g., Laitner and Stolyarov [2003], Hall [2001], McGrattan and Prescott [2000]) suggests a value of q in the range of 1.50, and this paper shows that the substantial entry-failure rate evident in US data can, by itself, explain about one-half of the total. (The model attributes the balance of q to entry-planning costs.)

Despite linear costs, our formulation's general equilibrium structure makes dynamic analysis possible. We show that variations in entry risk can lead to a positive correlation over time between the aggregative investment-to-capital ratio and Tobin's average q — as the literature has tended to find.

Our last section shows that temporary increases in the riskiness of investment can have large elasticities. We argue that macro disruptions, such as the advent of a new general purpose technology (Laitner and Stolyarov [2003, 2004]), an oil price shock (Baily et al [1981]), or a financial crisis, can raise entry risks for short intervals. The possibility of infrequent, yet sometimes sharp, short-term, simultaneous declines in interest and net investment rates is one of the most interesting implications of the model.

The organization of this paper is as follows. Section 2 presents the model. Section 3 focuses on the model's implications for intangible capital. Section 4 calibrates parameters, using Census data on establishments to determine θ and φ. Section 5 considers permanent changes in θ, and Section 6 examines changes in entry risk that are temporary. Section 7 concludes.

2 Model. The production sector consists of a continuum of business "establishments," each with its own "location" in the space of characteristics.[5] Each establishment produces units of output that is useful either for consumption or investment. Output from one location is a perfect substitute for that of any other.[6] Suppose the capital stock of establishment i, started at time t, and currently age s is $X(i\,,\,t\,,\,s)$. We assume that each location supports $X(i\,,\,t\,,\,s) \leq 1$. A prospective establishment constructs a "plan of operations," including a scale $X(i\,,\,t\,,\,0) \in [0\,,\,1]$. Constructing a plan costs $\xi \geq 0$. After developing a plan of operations, a prospective establishment invests and begins production. With probability θ, the project survives entry; with probability $1-\theta$, it fails. At failure, the scrap value of the project's investment is $\eta \cdot X(i\,,\,t\,,\,0)\,,\ \eta \in [0\,,\,1)$. With success, the

[5] A "firm" could consist of one "establishment," or many. Our data, see below, as well as our analysis, is based upon establishments.

[6] If the output of establishment i is $Y(i\,,\,t\,,\,s)$ and aggregate output at time t is Y_t, Dixit-Stigliz aggregation, with one establishment per niche $(i\,,\,t-s\,,\,s)$, would yield

$$Y_t = \left[\int \int [Y(i\,,\,t-s\,,\,s)]^{\omega}\, di\, ds \right]^{1/\omega}, \quad \omega \in (0\,,\,1)\,.$$

One could think of our model as an approximation for $\omega \to 1$.

project can continue with $X(i\,,\,t\,,\,s) = X(i\,,\,t\,,\,0)$ until its "death." Establishment death follows a Poisson process, with death rate φ. The Poisson processes are independent across establishments.[7]

A prospective entrant cannot know whether it will succeed or fail until it implements its plan of operations and begins production. Implementing a plan of operations requires testing at the full scale of eventual operation. Entrepreneurs can diversify their holdings of new projects. There is no reason for $X(i\,,\,t\,,\,0) < 1$; hence, we assume $X(i\,,\,t\,,\,0) = 1$.

A successful entrant subsequently endures wear and tear depreciation at rate $\delta \in (0\,,\,1)$. It must invest to cover this depreciation in order to stay in business. Such investment is not subject to entry risk.

Let aggregate physical investment in new projects be $J_t \geq 0$. If the climate for production becomes very adverse, some establishments may voluntarily close and scrap their capital. Let $J_t^S \leq 0$ be the aggregate flow of voluntarily scrapped capital.

We normalize the price of output to 1. Then the unit price of investment goods for prospective new projects is

$$P_{0t} = 1 \quad \text{all} \quad t\,. \tag{1}$$

Let P_{1t} be the unit value of capital in successful projects. Let

$$P^* \equiv \frac{1-\eta}{\theta} + \eta + \frac{\xi}{\theta}\,. \tag{2}$$

Assuming an elastic supply of prospective new projects at all times and free entry, we have

$$P_{0t} + \xi = \theta \cdot P_{1t} + (1-\theta)\cdot\eta \Longleftrightarrow P_{1t} = \frac{1+\xi-(1-\theta)\cdot\eta}{\theta} = P^* \quad \text{for} \quad J_t > 0\,, \tag{2$'$}$$

$$\eta = \theta \cdot P_{1t} + (1-\theta)\cdot\eta \Longleftrightarrow P_{1t} = \eta \quad \text{for} \quad J_t^S < 0\,, \tag{2$''$}$$

$$\theta \cdot P_{1t} + (1-\theta)\cdot\eta \in [\eta\,,\,1+\xi] \Longleftrightarrow P_{1t} \in [\eta\,,\,P^*] \quad \text{for} \quad J_t = 0 = J_t^S\,. \tag{2$'''$}$$

Letting X_t be the active physical stock at time t, Y_t be national output, and C_t aggregate consumption, we have

$$Y_t - C_t + \eta\cdot\varphi\cdot X_t + \eta\cdot(1-\theta)\cdot J_t - \eta\cdot J_t^S = J_t + \xi\cdot J_t + \delta\cdot X_t\,. \tag{3}$$

The left-hand side sums sources of physical capital for investment: $Y_t - C_t$ is the flow of new units of investment good; $\eta\cdot\varphi\cdot X_t$ is the flow of scrap from dying establishments; $\eta\cdot(1-\theta)\cdot X_t$ is the flow of scrap from failed entrants; and, $-\eta\cdot J_t^S$ (recall $J_t^S \leq 0$) is the flow of scrap from voluntary disinvestment. On the right-hand side, J_t is the flow of investment for new entry, $\xi\cdot J_t$ is the product of the planning cost per entry attempt and the number of entry attempts (recall that the scale of each attempt is 1), and $\delta\cdot X_t$ is the

[7] See Laitner/Stolyarov [2009] for a specification with endogenous life spans.

flow of capital to cover wear and tear depreciation. Since $\eta < 1$, we should not observe voluntary scrapping and positive net investment simultaneously. In other words, there will be one, and only one, of the following:

$$J_t > 0 \text{ and } J_t^S = 0\,; \quad J_t = 0 = J_t^S\,; \quad \text{or}\,, \quad J_t \text{ and } J_t^S < 0\,. \tag{4}$$

We assume that the national income and product accounts do not (properly) treat entrant planning costs as a final good, and as a part of gross investment I_t. Thus, if Y_t^{NIPA} and I_t^{NIPA} are NIPA GDP and gross investment, respectively, we have

$$Y_t = Y_t^{NIPA} + \xi \cdot J_t \quad \text{and} \quad I_t = I_t^{NIPA} + \xi \cdot J_t\,. \tag{5}$$

Consider the evolution of X_t. The flow of investment in new projects is J_t, of which $\theta \cdot J_t$ leads to increments in X_t. Voluntary scrapping depletes X_t at flow rate J_t^S. Similarly, the year-by-year death of active projects at Poisson rate φ leads to an exit flow $\varphi \cdot X_t$. Hence,

$$\dot{X}_t = \theta \cdot J_t + J_t^S - \varphi \cdot X_t\,. \tag{6}$$

It will be convenient to index active establishments at time t with $j \in \mathcal{J}_t$. We constantly renumber the indices as necessary to keep $\mathcal{J}_t$ an interval. Since $X_{jt} = 1$ in our model, we use, in fact,

$$\mathcal{J}_t = [0\,,\, X_t]\,. \tag{7}$$

If the time-t output of establishment j is Y_{jt} and its labor input is N_{jt}, and if the common TFP level is $e^{g \cdot t}$, $g > 0$, we assume

$$Y_{jt} = A \cdot [X_{jt}]^{\alpha} \cdot [E_{jt}]^{1-\alpha}\,, \quad A > 0\,, \quad \alpha \in (0\,,\,1)\,, \tag{8}$$

where

$$E_{jt} \equiv N_{jt} \cdot e^{g \cdot t}\,.$$

We have

$$Y_t = \int_{\mathcal{J}_t} Y_{jt}\, dj\,, \tag{9}$$

$$X_t = \int_{\mathcal{J}_t} X_{jt}\, dj\,, \tag{10}$$

$$E_t = \int_{\mathcal{J}_t} E_{jt}\, dj\,. \tag{11}$$

Our model has a representative household. The household inelastically supplies

$$N_t = N_0 \cdot e^{n \cdot t}\,, \quad n > 0\,, \tag{12}$$

units of labor every t, and it seeks to maximize

$$\int_0^\infty e^{-\rho \cdot t} \cdot N_t \cdot u(C_t/N_t)\, dt\,, \tag{13}$$

with

$$\rho > 0\,, \quad u(c) \equiv \frac{c^\beta}{\beta}\,, \quad \beta < 1\,, \quad \beta \neq 0\,.$$

<u>Equilibrium</u>. This section defines equilibrium and then proves existence and uniqueness.

We have

Definition: An "equilibrium" is a time path

$$\{N_t;\, X_t;\, J_t;\, J_t^S;\, P_{0t};\, P_{1t};\, (N_{jt}, X_{jt}, Y_{jt})\ \textit{all}\ j \in \mathcal{J}_t;\, Y_t;\, C_t\} \quad \textit{all} \quad t \geq 0\,,$$

with X_0 given and

(i) Free entry and P_{1t} is as in (2);

(ii) Profit maximizing choices for N_{jts} on the part of establishments, taking the wage as given; $X_{jt} = 1$ all $j \in \mathcal{J}_t$; and, establishment output from (8);

(iii) Representative household maximization of (13), subject to the household's income constraint;

(iv) Labor supply (12); and, full employment.

We also have

Definition: A "steady-state equilibrium" (SSE) is an equilibrium in which all endogenous variables grow geometrically and r_t is constant.

Turning to analysis, we have

Lemma 1. *If all establishments hire labor to maximize their profit, taking the wage, W_t, as given, we have an aggregate production function*

$$Y_t = F(X_t\,,\, E_t) \equiv A \cdot [X_t]^\alpha \cdot [E_t]^{1-\alpha}\,. \tag{14}$$

We have

$$\frac{\partial F(X_t\,,\, E_t)}{\partial E_t} = W_t\,, \tag{15}$$

and, if MPX_{jt} is the marginal product of X for active establishment j, we have

$$\frac{\partial F(X_t\,,\,E_t)}{\partial X_t} = MPX_{jt} \quad \textit{all} \quad j\ t\,. \tag{16}$$

Proof: See Appendix.

Then

Proposition 1. *Our model has a unique equilibrium for any $X_0 > 0$. The equilibrium converges to a SSE.*

Proof: See Appendix.

Letting

$$x_t \equiv \frac{X_t}{E_t}\,, \quad c_t \equiv \frac{C_t}{X_t}\,, \quad j_t \equiv \frac{J_t}{E_t}\,, \quad j_t^S \equiv \frac{J_t^S}{E_t}\,,$$

Proposition 1 yields a phase diagram as in Figure 1 in the Appendix. There is a stationary point at a. The equations of motion from analysis of the Hamiltonian (see the Appendix) make a a saddlepoint.

There are three "regimes" implicit in Figure 1: regime 1 occurs when $x_t \in (0\,,\,x_b)$; regime 2 when $x_t \in (x_b\,,\,x_c)$; and, regime 3 when $x_t \in (x_c\,,\,x_d)$.

<u>Regime 1</u>. For any $x_t \in (0\,,\,x_b)$, the unique equilibrium time path is coincident with the saddle's stable arm. In this range, $j_t > 0$, with $j_t = j^* > 0$ at a; $j_t^S = 0$; and,

$$P_{1t} = P^*\,. \tag{17}$$

The last follows from (2'). If r_t is the real interest rate, free entry implies

$$P_{1t} = \int_t^\infty \varphi \cdot e^{-\varphi\cdot(s-t)} \cdot \Big(\int_t^s R(t\,,\,u) \cdot [\frac{\partial F(X_u\,,\,E_u)}{\partial X_u} - \delta]\, du + \eta \cdot R(t\,,\,s)\Big)\, ds\,, \tag{18}$$

where

$$R(t\,,\,u) \equiv e^{-\int_t^u r_z\, dz}\,.$$

For regime 1, the derivative of (18) yields

$$\frac{\partial F(X_u\,,\,E_u)}{\partial X_u} = P_{1t} \cdot r_t + [\delta + \varphi \cdot (P_{1t} - \eta)]\,. \tag{19}$$

Expression (19) is the model's version of Jorgenson's familiar cost of capital formula. The left-hand side is the marginal product of capital at any active establishment (see

Lemma 1). Units of capital in active establishments are worth $P_{1t} = P^*$. The opportunity cost of financial investments in the latter is $P_{1t} \cdot r_t$. Wear and tear depreciation, at rate δ, can be covered with new investment goods, for which the cost is $P_{0t} = 1$. Establishment death leads to a financial loss per unit of X of $P_{1t} - \eta$.

Regime 2. Returning to Figure 1, if the stable arm never reaches the dashed consumption curve, $x_b = \infty$.[8] In our calibrated examples, however, the stable arm cuts the consumption curve at $x_b < \infty$. Then for $x_t \in (x_b\,,\, x_c)$, the equilibrium growth trajectory is coincident with the consumption curve. Proposition 1 shows that $j_t = 0 = j_t^S$ in regime 2. Likewise, condition (2") shows that $P_{1t} \in [\eta\,,\, P^*]$. We can pin P_{1t} down precisely from the proof of Proposition 1. The costate variable μ_t is the value, in units of utility, of the marginal unit of X_t; hence,

$$P_{1t} = \frac{\mu_t}{u'(c_t)}\,. \tag{20}$$

We can derive r_t from (18).

Regime 3. The proof of Proposition 1 shows that equilibrium growth may follow arc $\overline{cd}$ after x_c. In our calibrations, a finite x_c always, in fact, emerges, and the equilibrium growth path never intersects the dashed consumption curve thereafter. For $x_t > x_c$, we have $j_t = 0$; $j_t^S < 0$; and, $P_{1t} = \eta$ (see (2‴)). This is the only regime in which some establishments voluntarily scrap their capital.

Discussion. Aggregative data implies that the US economy has spent most of the post WWII era in regime 1. We can see that as follows.

In our stylized model, $Y_t - C_t$ corresponds to gross investment, say, I_t.[9] Equation (3) shows

$$I_t = \big(1 - \eta \cdot (1-\theta)\big) \cdot J_t + \xi \cdot J_t + \eta \cdot J_t^S + (\delta - \eta \cdot \varphi) \cdot X_t\,. \tag{21}$$

Let D_t be NIPA depreciation. We assume

$$D_t = (1-\eta) \cdot \varphi \cdot X_t + \delta \cdot X_t\,. \tag{22}$$

In other words, NIPA measured depreciation includes the write-down to scrap of the plant and equipment at establishments reaching the end of their lives, plus replacement needed to offset wear and tear.[10]

NIPA net investment would be

[8] Parameter values could yield $\eta \cdot \varphi - \delta > 0$, for example, with the dashed consumption curve rising for all x_t.

[9] We would include government spending on goods and services in C_t, and ignore government investment.

[10] An alternative would add losses from entry failure, including lost planning investments. Then

$$I^{NIPA} - D_t = \big(1 - \eta \cdot (1-\theta)\big) \cdot J_t + \eta \cdot J_t^S - \varphi \cdot X_t \,. \tag{23}$$

US NIPA data shows positive aggregate gross investment for all $t \geq 1929$. Net investment has sometimes been negative. However, most of the latter instances occurred before the post WWII era: net private domestic fixed investment was negative 1931-34 and 1943-45, net private nonresidential investment was negative 1931-35, 38-39, and 42-44; and, net private domestic business investment (only presented since 1960) was negative in 2009. In our model, $J_t^S < 0$ implies $J_t = 0$, so that (23) implies $I_t^{NIPA} - D_t < 0$. Even with $J_t > 0$ (so that $J_t^S = 0$), on the other hand, $I_t^{NIPA} - D_t \leq 0$ is possible. If $I_t^{NIPA} - D_t > 0$, (23) implies $J_t > 0$.

$J_t > 0$ puts the economy in regime 1, and Section 4 calibrates on the basis of that case.

3 Intangible Capital. Section 2 argues that the US economy has spent virtually the full post WWII period in regime 1. The present section examines the implied aggregate stock of intangible capital and the value of Tobin's q.

In regime 1, new entry replenishes the economy's stock of active capital, replacing dying establishments and, in a SSE, enabling the aggregate capital stock X_t to grow in-step with E_t. A prospective entrant buys a unit of new investment good, for price 1, and starts production, implementing its plan of operations. If the plan of operation succeeds, the economy gains a new business establishment. Otherwise, the prospective entrant must scrap its capital. Successful entry reveals information, namely, it shows a particular "plan of operations" to be a good one. The amount by which the value of the new establishment exceeds the book value of its capital, $(P_{1t} - 1) \cdot X_{jt} = P_{1t} - 1$, is a measure of the value of the information. The economy's corresponding aggregate intangible capital stock is

$$(P_{1t} - 1) \cdot \int_{\mathcal{J}_t} X_{jt} \, dj = (P_{1t} - 1) \cdot X_t \,. \tag{24}$$

Similarly, Tobin's average q equals the ratio of the market value of active capital, say, $K_t = P_{1t} \cdot X_t$, to its book value, X_t:

$$q_t = \frac{K_t}{X_t} = \frac{P_{1t} \cdot X_t}{X_t} = P_{1t} \,. \tag{25}$$

In regime 1, Proposition 1 shows

$$P_{1t} = P^* = \frac{1-\eta}{\theta} + \eta + \frac{\xi}{\theta} > 1 \,. \tag{26}$$

Formally, we have then established

$$D_t = (1-\eta) \cdot (1-\theta) \cdot J_t + (1-\theta) \cdot \xi \cdot J_t + (1-\eta) \cdot \varphi \cdot X_t + \delta \cdot X_t \,.$$

This alternative does not seem as plausible in terms of the description of NIPA depreciation — and it tends to yield unsatisfactory calibration values for δ and η.

Proposition 2: *Equilibrium growth eventually leads to regime 1, in which*

$$P_{1t} = P^* > 1\,.$$

In regime 1, the aggregate value of intangible capital is

$$(P^* - 1) \cdot X_t > 0\,.$$

And, Tobin's average q is

$$q_t = P_{1t} = P^* > 1\,.$$

Proposition 2 implies that Tobin's average q should have exceeded 1 for the US economy in most post WWII years. The next section attempts to gauge the part of Tobin's average q that risk of entry-failure can explain.

4 Calibration. This section calibrates the model's vector of parameters

$$(\theta\,,\,\varphi\,,\,\eta\,,\,\xi\,,\,\delta\,,\,\alpha\,,\,A\,,\,n\,,\,g\,,\,\rho\,,\,\beta)\,. \tag{27}$$

We try values $\beta = 0\,,\,-1\,,\,...\,,\,-5$. We use standard values for α, A, n, and g; microeconomic data sources to calibrate the key new parameters θ and φ; and, macroeconomic data to determine η, ξ, δ, and ρ.

Standard values. Assume $\alpha = 0.30$, $n = 0.01$, and $g = 0.02$. Without loss of generality, normalize $A = 1$.

Microeconomic data. We use microeconomic data shown in the diagram on the next page to calibrate θ and φ. Our preliminary estimates are $\theta = 0.80$ and $\varphi = 0.11$.

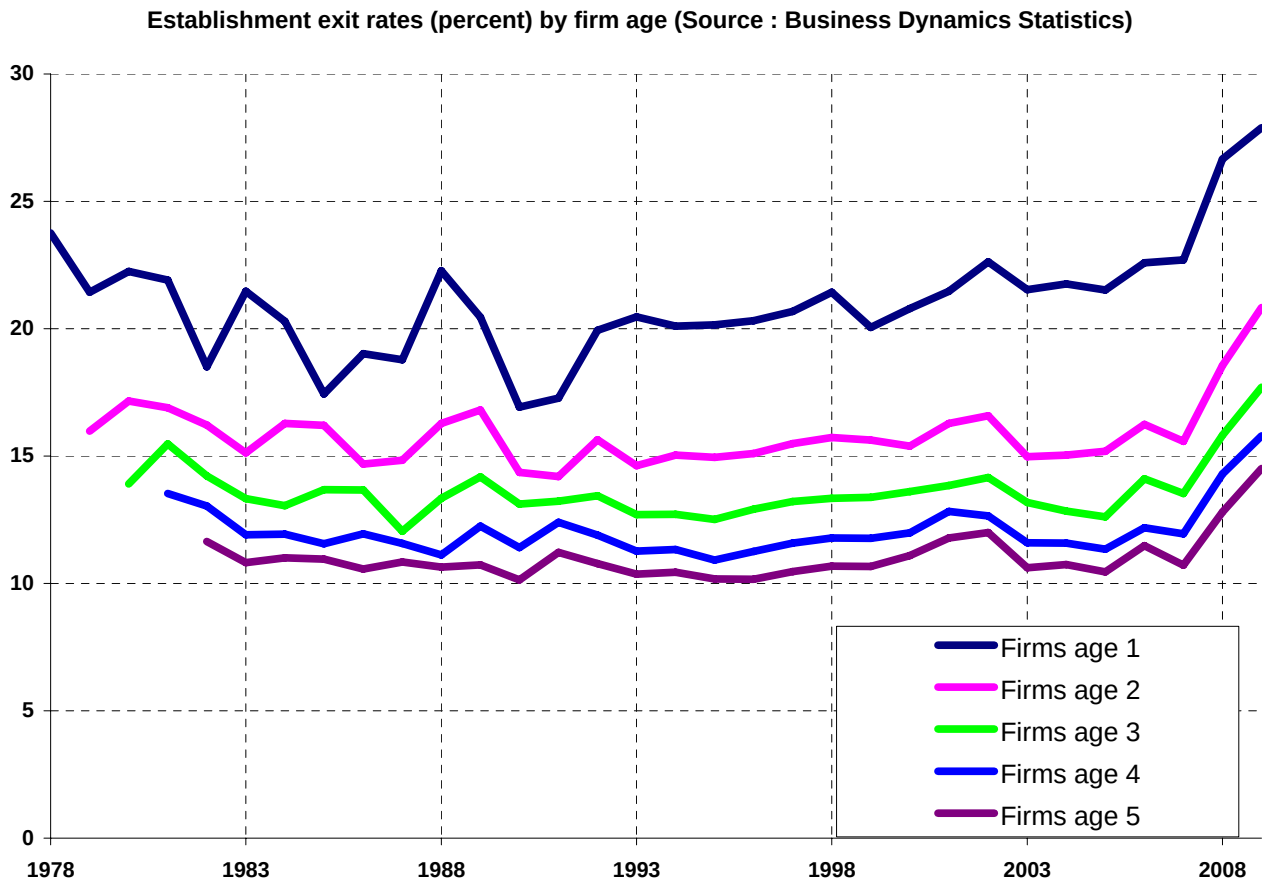
Macroeconomic data. We assume $r^* = 0.10$;[11] set the SSE ratio D_t/Y_t^{NIPA} to $d^* = 0.11$ using NIPA data 1950-2010; and, set the SSE ratio I_t^{NIPA}/Y_t^{NIPA} to $i^* = 0.17$ using NIPA data over the same years.[12]

Section 3 shows that in regime 1, Tobin's average q equals P^*. Laitner and Stolyarov [2003, p.1258] estimate $q = 1.48$ for US nonresidential capital. Hall [2001, fig.13] reports similar numbers for US nonfinancial corporations.[13] McGrattan and Prescott [2000, tab 1] estimate 1.62 for the corporate sector. Nonresidential private fixed assets are slightly

[11] See Laitner and Stolyarov [2003].

[12] Note that we combine aggregate private consumption and government spending on goods and services in our C_t.

[13] Hall's data shows peak values for q of 1.5-1.7 around 1970 and again for 1995. Laitner and Stolyarov's [2003] model implies that the peaks yield the best estimates of q.


Establishment exit rates (percent) by firm age (Source : Business Dynamics Statistics)
30
25
20
15
10
5
0
1978
1983
1988
1993
1998
2003
2008
Firms age 1
Firms age 2
Firms age 3
Firms age 4
Firms age 5

less than half of private fixed assets, on average, in the NIPA "All Fixed Asset Table 1.1." Assuming that intangible capital is zero for residential capital, we set[14]

$$P^* = (0.47) \cdot (1.48) + (0.53) \cdot (1.00) \approx 1.23\,. \tag{28}$$

Parenthetically, we think that measuring Tobin's average q on a year-to-year basis is very difficult. The numerator is a market value, potentially observable in the Flow of Funds, for example. The BEA constructs a version of X_t, a potential denominator, from a perpetual inventory equation. Laitner/Stolyarov [2003, Figure 1], presents ratios from these sources. The period 1974-85 seems inconsistent with the present paper's model because $J_t > 0$ yet apparently $P_{1t} < 1$. Laitner and Stolyarov [2003] argue, however, that standard figures for X_t can be misleading. They suggest that the steep decline in asset prices about 1970 occurred because of the advent of a new general purpose technology (namely, microprocessor chips) and that it caused significant, abrupt obsolescence. Old capital was not immediately scrapped, but its resale value precipitously declined. New capital gradually replaced it, at the rate of net investment. In the interim, the BEA's X_t, constructed using a constant depreciate rate, was too high. In other words, if technological progress is vintage specific, average q can be a poor proxy for marginal q in some time periods.

We develop estimates of η, ξ, and δ as follows. Let $y_t \equiv Y_t/E_t$. From Lemma 1,

$$y_t = A \cdot [x_t]^\alpha\,.$$

At the SSE, (6) implies

$$\theta \cdot j^* = (\varphi + n + g) \cdot x^*\,. \tag{29}$$

Then at the SSE, (22) implies

$$d^* \cdot \left[A \cdot [x^*]^{\alpha-1} - \frac{\xi}{\theta} \cdot (\varphi + n + g)\right] = (1-\eta) \cdot \varphi + \delta\,; \tag{30}$$

(2) implies

$$P^* = \frac{1-\eta}{\theta} + \eta + \frac{\xi}{\theta}\,; \tag{31}$$

(19) implies

$$\alpha \cdot A \cdot [x^*]^{\alpha-1} = P^* \cdot r^* + \delta + \varphi \cdot (P^* - \eta)\,; \tag{32}$$

and, (3), after substituting from (31) and dividing by x^*, implies

[14] An alternative approach (i.e., Laitner and Stolyarov [2003]) would remove residential capital and investment (and corresponding consumption service flows) from the model. The present approach has the advantage of maintaining a close connection to the aggregative data.

$$\frac{\xi}{\theta}\cdot(\varphi+n+g)+i^*\cdot\left[A\cdot[x^*]^{\alpha-1}-\frac{\xi}{\theta}\cdot(\varphi+n+g)\right]=(P^*-\eta)\cdot\varphi+\delta+P^*\cdot(n+g)$$

$$\Longleftrightarrow (1-i^*)\cdot\frac{\xi}{\theta}\cdot(\varphi+n+g)+i^*\cdot A\cdot[x^*]^{\alpha-1}=(P^*-\eta)\cdot\varphi+\delta+P^*\cdot(n+g)\,. \quad (33)$$

Expressions (30)-(33) constitute 4 linear equations in $(\eta\,,\,\xi\,,\,\delta\,,\,A\cdot[x^*]^{\alpha-1})$. Given our A and α, we can extract $(\eta\,,\,\xi\,,\,\delta\,,\,x^*)$.

Let

$$\zeta\equiv\rho-n-\beta\cdot g\,. \quad (34)$$

The proof of Proposition 1 shows that at the stationary point $(x^*\,,\,c^*)$, the first-order condition for the costate variable μ_t yields

$$P^*\cdot(\zeta+\varphi+n+g)=\alpha\cdot A\cdot[x^*]^{\alpha-1}-\delta+\eta\cdot\varphi\,.$$

The latter expression, in combination with (32), yields

$$r^*=\rho+(1-\beta)\cdot g\,, \quad (35)$$

from which we determine ρ. Finally, (3) and (29) yield c^*:

$$c^*=A\cdot[x^*]^{\alpha}-P^*\cdot(\varphi+n+g)\cdot x^*+(\eta\cdot\varphi-\delta)\cdot x^*\,. \quad (36)$$

Table 1. Calibrations of $(\eta\,,\,\xi\,,\,\delta\,,\,\rho)$

given $\theta=0.80$, $\varphi=0.11$, and various values of β

β	P^*	η	ξ	δ	ρ	x^*	c^*
0.000	1.230	0.565	0.097	0.035	0.080	1.451	0.908
-1.000	1.230	0.565	0.097	0.035	0.060	1.451	0.908
-2.000	1.230	0.565	0.097	0.035	0.040	1.451	0.908
-3.000	1.230	0.565	0.097	0.035	0.020	1.451	0.908
-4.000	1.230	0.565	0.097	0.035	0.000	1.451	0.908
-5.000	1.230	0.565	0.097	0.035	-0.020	1.451	0.908

Source: see text.

Table 1 presents results. Expression (2) shows if $\xi=0$, the risk of entry failure alone explains almost half of the economy's intangible capital:

$$\frac{1-\eta}{\theta}+\eta=1.11\,.$$

We can use (5) to compute the implied difference between the SSE $y_t \equiv Y_t/E_t$ and $y_t^{NIPA} \equiv Y_t^{NIPA}/E_t$ and between $i_t \equiv I_t/E_t$ and $i_t^{NIPA} \equiv I_t^{NIA}$:

$$\frac{y^{NIPA*}}{y^*} = 0.98 \quad \text{and} \quad \frac{i^{NIPA*}}{i^*} = 0.88\,.$$

5 Permanent Changes in the Entry-failure Rate. The original idea of Tobin's q was that a high marginal q signaled a favorable moment for investment (Tobin [1969], Hayashi [1982], Summers et al [1981], Abel and Blanchard [1986], and many others). This section shows that the connection between q and investment will tend to be opposite to this when changes in q result from variation in θ. Nevertheless, we show that a positive correlation between I/K and q, which was the specific focus of the existing literature, is also consistent with our analysis.

<u>Investment and q</u>. Suppose that the economy resides in a SSE for $t \leq 0$ but that the entry-success rate θ is subject to an exogenous, permanent decline at time $t = 0+$.

Taking a general equilibrium perspective, although the decline in θ causes $P_{1t} = P^*$ to rise, it ultimately makes physical capital accumulation more arduous, as waste from entry failures is greater. The increase in P_{1t} reflects the fact that entry is more difficult, rather than signaling that prospects for future profits are brighter. In the end, we find that x^* should decline. Formally,

Proposition 3: *An increase in P^* stemming from a permanent decline in θ causes x^* to fall.*

Proof: See Appendix.

Consider the perspective of agents within the economy. First, think about entrepreneurs. Although in Proposition 3 a higher P_{1t} indicates a higher reward for successful entry, the corresponding lower θ means a greater risk of entry failure. The expected value of the reward for attempting entry does not, on balance, change. In other words, a decline in θ need neither damp nor abet entrepreneurs' enthusiasm for new investment.

In the case of creditors, the story is different. In our model, the SSE real interest rate, say, r^* does not change when θ does: we have $r^* = \rho + (1-\beta)\cdot g = \bar{r}^*$. As θ declines, the price of a unit of investment in active capital rises to, say, $\bar{P}^* > P^*$. In other words, the financial investment needed to purchase a given stream of interest payments now costs more. In response, creditors reduce the supply of financing for new investment projects. This leads to the lower x^* of Proposition 3.

Although Proposition 3 suggests that SSE investment will decline, the result refers to x^* rather than i^*. Our simulations below, however, show that a permanent decline in θ does lower i^*.

More subtly, while we detrend variables with E_t, other normalizations can yield different outcomes. For example, to circumvent aggregation problems stemming from convex adjustment costs (Hayashi [1982]), the existing literature focuses on the relationship of $I_t/X_t = i_t/x_t$ and P_{1t}. Surprisingly,

Proposition 4: *An increase in P^* stemming from a permanent decline in θ causes i^*/x^* to rise.*

Proof: See Appendix.

The intuition is as follows. Suppose a decline in θ causes P^* to rise. If i^* and x^* decline equally, equilibrium will not be restored. When the entry-failure rate increases, x^* must decline more than i^*. As the latter occurs, the ratio i^*/x^* rises.

Conceivably the impact effect of a permanent change matters more, in practice, than the long-term consequences. Figures 2a-b show that the impact effect on consumption of a permanent drop in θ can be either positive or negative. At time $t = 0+$, c_t jumps to the stable arm of the saddle at $\bar{a}$. Let the jump be dc^+. On the other hand, x_t moves at the finite rate of investment, so that $x_{0+} = x_0 = x^*$. Geometrically, dc^+ depends on the slope of the phase diagram's stable arm at point $\bar{a}$ relative to the slope of the line segment connecting the old and new stationary points, dc^*/dx^* .

The proof of Proposition 1 yields

$$\dot{x}_t = \frac{A \cdot [x_t]^\alpha + (\eta \cdot \varphi - \delta) \cdot x_t - c_t}{P^*} - (\varphi + n + g) \cdot x_t \,,$$

$$\dot{c}_t = \frac{c_t}{1-\beta} \cdot [\frac{\alpha \cdot A \cdot [x_t]^{\alpha-1} + (\eta \cdot \varphi - \delta)}{P^*} - \zeta - \varphi - n - g] \,.$$

Linearizing the right-hand sides with respect to x_t and c_t about $(x^*\,,\,c^*)$, form a matrix M. One eigenvalue will be positive, and the other negative. Call the negative eigenvalue ν, and the corresponding eigenvector $(v^x\,,\,v^c)$.[15] Figures 2a-b show

$$dc^+ = -(\frac{dx^*}{dP^*}dP^*) \cdot [\frac{v^c}{v^x} - \frac{dc^*}{dx^*}] \Longleftrightarrow \frac{dc^+}{dP^*} = -\frac{dx^*}{dP^*} \cdot [\frac{v^c}{v^x} - \frac{dc^*}{dx^*}] \,.$$

We have $y_t = c_t + i_t$ and $dy^+/dP^* = 0$; so, $di^+/dP^* = -dc^+/dP^*$. The half-life for convergence to the new SSE is $\ln(0.50)/\nu$.

<u>Simulations</u>. Our numerical analysis utilizes the calibrations from Table 1. P^* varies monotonically with θ, and we present results in the form di^*/dP^*.

The first-order conditions for regime 1 in the proof of Proposition 1 yield (36) and

$$\frac{\alpha \cdot A \cdot [x^*]^{\alpha-1} + (\eta \cdot \varphi - \delta)}{P^*} = \zeta + \varphi + n + g \,. \tag{37}$$

We can solve for $(x^*\,,\,c^*)$. Table 1 shows the solution. We can also differentiate to find $\partial x^*/\partial P^*$ and $\partial c^*/\partial P^*$. Table 2 presents outcomes.

The first column of Table 2 bears out our conjecture based upon Proposition 3: we find that the lower x^* accompanying a decrease in θ leads to a lower i^* as well, given our

[15] We are working with infinitesimal changes $dx^* = (\partial x^*/\partial P^*)\,dP^*$ and dc^*. In this case, the stable arm at a has the same slope as the stable arm at $\bar{a}$.

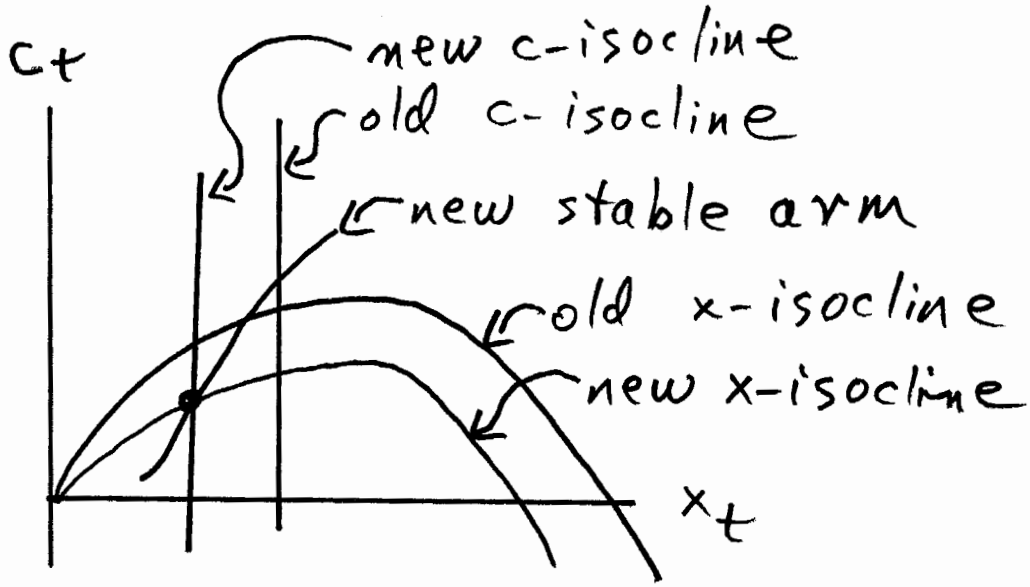


Figure 2a

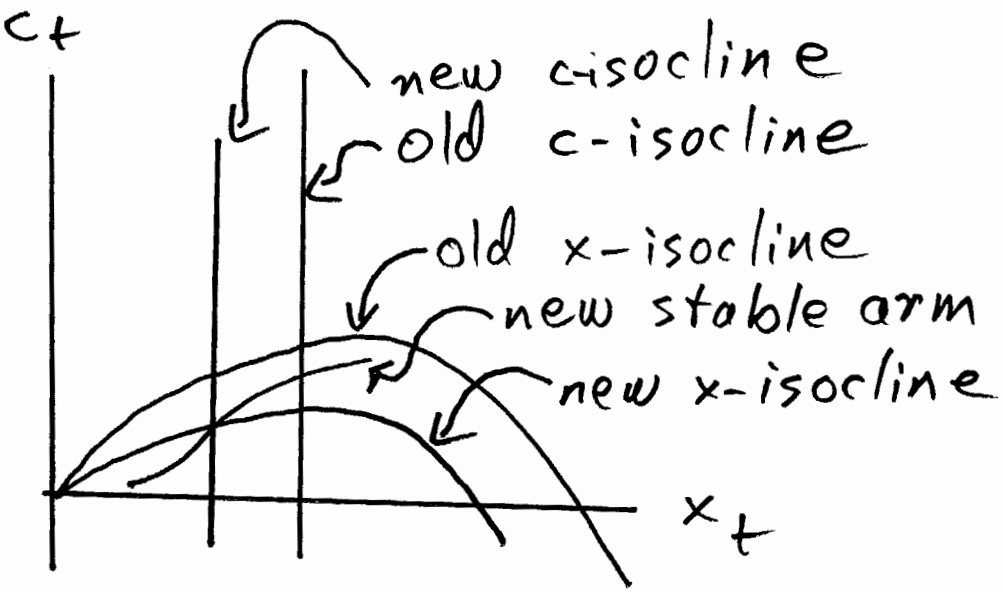


Figure 2b

Table 2. Long-run Consequences for Investment of a Change in P^* Stemming from a Change in θ (Table 1 Parameters Values)

β	$\frac{dx^*/x^*}{dP^*/P^*}$	$\frac{dc^*/c^*}{dP^*/P^*}$	$\frac{dy^*/y^*}{dP^*/P^*}$	$\frac{di^*/i^*}{dP^*/P^*}$	$\frac{di^{NIPA*}/i^{NIPA*}}{dP^*/P^*}$
all values	-1.596	-0.495	-0.479	-0.409	-0.497

Source: see text.

Table 3. Impact Consequences for Investment of a Change in P^* Stemming from a Change in θ (Table 1 Parameters Values)

β	ν	Half-life	$\frac{v^c}{v^x}$	$\frac{dc^*}{dx^*}$	$\frac{dc^+}{dP^*}$	Elasticities	
						$\frac{di^+/i^*}{dP^*/P^*}$	$\frac{di^{NIPA+}/i^{NIPA*}}{dP^*/P^*}$
0.000	-0.226	3.066	0.364	0.194	0.320	-1.871	-2.023
-1.000	-0.151	4.583	0.272	0.194	0.147	-0.859	-0.990
-2.000	-0.118	5.854	0.232	0.194	0.071	-0.415	-0.536
-3.000	-0.099	7.001	0.208	0.194	0.026	-0.153	-0.268
-4.000	-0.086	8.072	0.192	0.194	-0.004	0.025	-0.087
-5.000	-0.076	9.089	0.180	0.194	-0.027	0.155	0.046

Source: see text.

calibrations. We can see that the elasticity $d\ln(i^*/x^*)/d\ln(P^*)$ will be slightly larger than 1.

Table 3 presents impact outcomes.

We can see that the slope of the stable arm depends upon β. A low β implies a low intertemporal elasticity of substitution. If the economy starts at $x_0 < x^*$, a low intertemporal elasticity makes the representative agent desire a relatively flat consumption trajectory. Hence, a low β tends to lead to a low v^c/v^x.

With $\beta = 0$ (i.e., $u(c) = \ln(c)$), Table 3 shows $di^+/dP^* < 0$. For $\beta \leq -5$, on the other hand, $di^{NIPA+}/dP^* > 0$. Since x_t is a non-jump variable, the elasticities of i^+/x_0 and i^+ are the same.

A comparison of Tables 2-3 shows that the impact elasticity on investment can be larger or smaller than the long-run elasticity, or it can have the opposite sign. Since Table 3 shows half-lives for convergence to the permanent new solution of 3-9 years, the impact effects are potentially important. For $\beta \leq -5$, they would tend to generate a positive correlation between i_t^{NIPA}/x_t and q_t.

Discussion. Our model is not antithetical to the traditional literature (e.g., Hayashi [1982], Summers et al [1981]). It is rather a special case, with a linear cost function

for investment. The earlier papers looked for an empirical correlation generated by a convex cost-of-adjustment specification. Possibly shocks that shifted the cost function — as with θ changing — were playing an important role in the data studied. In that case, Proposition 4 and Table 3 suggest that linear costs might account for the positive correlation that the literature found.

6 Temporary Changes in the Entry-failure Rate. This section considers changes in θ that are temporary. We find that elasticities can be much larger than those for permanent changes. Section 4's data suggests that a substantial, presumably temporary, decline in θ occurred during the recent recession. We turn briefly to empirical evidence below.

Model. We model a temporary change in θ as follows. For $t \leq 0$, the economy rests at (x^*, c^*). At $t = 0+$, θ drops to $\bar{\theta} \equiv \theta + d\theta$. P^*, therefore, rises to $\bar{P}^* = P^* + dP^*$. Agents know that the underlying change in θ is temporary, yet they are unsure of exactly how long it will persist. There is a Poisson process with parameter λ such that the first Poisson event is a (permanent) restoration of the original θ. If λ is large, the period with $d\theta < 0$ is likely to be brief — the period's expected duration is $1/\lambda$. If $\lambda = 0$, the change lasts forever — reinstating the analysis of Section 5.

Intuitively, the equilibrium response of i_t and r_t may be larger when $1/\lambda$ is smaller. As in Section 5, entrepreneurs are indifferent to changes in θ: the entry-failure risk rises at $t = 0+$, but a corresponding upward adjustment in P_{1t} fully compensates potential entrants. Creditors, on the other hand, will dislike increases in the failure rate, and they may be especially wary of temporary increases. At $t = 0+$, (19) shows that r_t must fall: the left-hand side is unchanged, but P_{1t} rises on the right. Equation (18) shows the situation may be even worse than for a permanent change. P_{1t} will fall after the temporary change ends, giving creditors a capital loss. The rate of the loss will be greater the sooner it transpires — i.e., the sooner the Poisson event occurs. Prior to restoration of the status quo, r_t must be low enough to compensate entrepreneurs making new investments for the coming loss. Creditors end up paying more (i.e., $\bar{P}^* > P^*$) for a stream of interest payments that is lower — perhaps, much lower.

Let $V(x_t)$ be the value, in time-0 units of utility, of the maximized problem of Proposition 1. Then the representative agent's criterion is

$$\begin{aligned}
&\int_0^\infty \lambda \cdot e^{-\lambda \cdot s} \cdot [\int_0^s e^{-\zeta \cdot t} \cdot u(c_t)\, dt + V(x_s) \cdot e^{-\zeta \cdot s}]\, ds \\
&= \int_0^\infty \int_t^\infty \lambda \cdot e^{-\lambda \cdot s} \cdot e^{-\zeta \cdot t}\, u(c_t)\, ds\, dt + \int_0^\infty \lambda \cdot e^{-\lambda \cdot s} \cdot e^{-\zeta \cdot s} \cdot V(x_s)\, ds \\
&= \int_0^\infty e^{-(\lambda+\zeta) \cdot t} \cdot [u(c_t) + \lambda \cdot V(x_t)]\, dt\,.
\end{aligned}$$

Assume we stay in regime 1. Put a bar over the variables of the new model to distinguish them from the model of Proposition 1. Start the analysis at $t = 0+$.

Write $(\bar{x}_t\,, \bar{c}_t) = (\bar{x}_t(P)\,, \bar{c}_t(P))$ to show the dependence of the new variables on P. The event that we wish to study has $P = \bar{P}^*$. Notice that since x is a non-jump variable,

we have $\bar{x}_0(P) = x^*$. If $P = P^*$, the new problem is identical to the one of Proposition 1, for which $(x^*\,,\,c^*)$ is the stationary solution. Hence, $\bar{x}_t(P^*) = x^*$ all t.

Following a temporary change in θ, the representative agent solves

$$\max_{\bar{c}_t}(P) \int_0^\infty e^{-(\lambda+\zeta)\cdot t} \cdot [u(\bar{c}_t(P)) + \lambda \cdot V(\bar{x}_t(P))]\, dt\,, \tag{38}$$

$$\text{subject to:} \quad \dot{\bar{x}}_t(P) = \frac{A \cdot [\bar{x}_t(P)]^\alpha + (\eta \cdot \varphi - \delta) \cdot \bar{x}_t(P) - \bar{c}_t(P)}{P} - (\varphi + n + g) \cdot \bar{x}_t(P)\,, \tag{39}$$

$$\bar{x}_0(P) = x^*\,, \tag{40}$$

where $P = \bar{P}^*$.

Suppose the Poisson event (governed by λ) occurs at T. When the duration uncertainty is resolved at T, we jump to the stable arm of the model of Proposition 1. In other words, at the moment T of resolution, consumption jumps from $\bar{c}_T$ to c_t^a, with $(x_T^a\,,\,c_T^a)$ on the equilibrium path of Figure 1. Since x is a non-jump variable, we have $x_T^a = \bar{x}_T(\bar{P}^*)$. This behavior is implicit in the definition of $V(.)$.

As in Section 5, if we linearize Proposition 1's first-order conditions at $(x^*\,,\,c^*)$, i.e., at point a in Figure 1, one eigenvalue, ν, will be negative and the other positive. The corresponding eigenvector is $(v^x\,,\,v^c)$. For infinitesimal changes, the stable arm at point a is locally determined by the $(v^x\,,\,v^c)$. Thus, for $t \geq T$, we have

$$\frac{c_t^a(P) - c^*}{x_t^a(P) - x^*} = \frac{v^c}{v^x} \Longleftrightarrow c_t^a(P) = c^* + \frac{v^c}{v^x} \cdot (x_t^a(P) - x^*)\,. \tag{41}$$

Defining a Hamiltonian

$$\begin{aligned}\bar{\mathcal{H}} \equiv\; & u(\bar{c}_t(P)) + \lambda \cdot V(\bar{x}_t(P)) + \\ & \bar{\mu}_t(P) \cdot [\frac{A \cdot [\bar{x}_t(P)]^\alpha + (\eta \cdot \varphi - \delta) \cdot \bar{x}_t(P) - \bar{c}_t(P)}{P} - (\varphi + n + g) \cdot \bar{x}_t(P)]\,,\end{aligned} \tag{42}$$

we have first-order conditions

$$u'(\bar{c}_t(P)) = \frac{\bar{\mu}_t(P)}{P}\,, \tag{43}$$

$$\begin{aligned}\dot{\bar{\mu}}_t(P) = \; & (\lambda + \zeta) \cdot \bar{\mu}_t(P) - \lambda \cdot V'(\bar{x}_t(P)) - \\ & \bar{\mu}_t(P) \cdot [\frac{\alpha \cdot A \cdot [\bar{x}_t(P)]^{\alpha-1} + (\eta \cdot \varphi - \delta)}{P}] + \bar{\mu}_t(P) \cdot (\varphi + n + g)\,.\end{aligned}$$

Using the notation $\widehat{z}_t \equiv \dot{z}_t / z_t$, and noting that $V'(\bar{x}_t) = \mu_t$, where μ_t is the costate from the model of Proposition 1, we can rewrite the latter as

$$\widehat{\bar{\mu}}_t(P) = \lambda \cdot (1 - \frac{\mu_t}{\bar{\mu}_t(P)}) + \zeta - \frac{\alpha \cdot A \cdot [\bar{x}_t(P)]^{\alpha-1} + (\eta \cdot \varphi - \delta)}{P} + (\varphi + n + g)\,. \tag{44}$$

Equations (39) and (43) are exactly as those in the proof of Proposition 1. Equation (44) differs only in the addition of a new term

$$\lambda \cdot (1 - \frac{\mu_t}{\bar{\mu}_t(P)})\,. \tag{45}$$

If $\lambda = 0$, so that the change in θ is permanent, this term disappears — and the analysis of Section 5 emerges. Likewise, if $d\theta = 0$, the Poisson event does not announce a change — so, the analysis of Proposition 1 re-emerges.

We expect $\mu_t < \bar{\mu}_t(\bar{P}^*)$ because the costate is the value (in units of utility) of one more unit of active capital, and active capital is harder to obtain in the new problem. Thus, the stationary point in the phase diagram of the new problem should lie to the left of point a from Figure 1: $\bar{x}^*(\bar{P}^*) \equiv \bar{x}^* < x^*$. Because $\bar{P}^* > P^*$, the isocline from (39) will be lower; hence, we also expect $\bar{c}^*(\bar{P}^*) \equiv \bar{c}^* < c^*$. Since (45) involves multiplication by λ, we would, in general, expect more dramatic results for a larger λ — that is to say, for a shorter duration change in θ — as we reasoned above.

In manipulating (44), the first-order condition for c shows

$$\lambda \cdot (1 - \frac{\mu_t}{\bar{\mu}_t(P)}) = \lambda \cdot (1 - \frac{P^* \cdot u'(c_t^a)}{P \cdot u'(\bar{c}_t(P))})\,. \tag{46}$$

We can then compute $d\bar{c}^*(P)/d\bar{x}^*(P)$ and the eigenvector $\bar{v}(P) \equiv (\bar{v}^x(P)\,,\, \bar{v}^c(P))$ locally characterizing the stable arm for problem (38).

The temporary increase in P begins at $t = 0+$. At that moment, we must jump to the stable arm for point $\bar{a}$, determined by $(\bar{x}^*\,,\, \bar{c}^*)$. The jump, illustrated in Figure 3, must be vertical — since c, but not x, is a jump variable. The analysis resembles that for impact effects in Section 5. We can compute $di^+(P)/dP$ and $di^{NIPA+}(P)/dP$ as in Section 5.

Let $\bar{r}_t(P)$ be the interest rate corresponding to $(\bar{x}_t(P)\,,\, \bar{c}_t(P))$. Over the time interval $(t\,,\, t + dt]$, an establishment has probability $\varphi\, dt$ of death, $\lambda\, dt$ of seeing an end to the temporary period with $d\theta < 0$, and $1 - \varphi\, dt - \lambda\, dt$ of continuing with P. Hence,

$$\begin{aligned} P = {} & (\alpha \cdot A \cdot [\bar{x}_t(P)]^{\alpha-1} - \delta)\, dt + \eta \cdot (1 - \bar{r}_t(P)\, dt) \cdot \varphi\, dt + P^* \cdot (1 - \bar{r}_t(P)\, dt) \cdot \lambda\, dt \\ & + P \cdot (1 - \bar{r}_t(P)\, dt) \cdot (1 - \varphi\, dt - \lambda\, dt)\,. \end{aligned} \tag{47}$$

Dropping terms of order $[dt]^2$,

$$\begin{aligned} P = {} & (\alpha \cdot A \cdot [\bar{x}_t(P)]^{\alpha-1} - \delta)\, dt + \eta \cdot \varphi\, dt + P^* \cdot \lambda\, dt + P - P \cdot (\varphi + \lambda)\, dt - P \cdot \bar{r}_t(P)\, dt \\ \Longleftrightarrow {} & \alpha \cdot A \cdot [\bar{x}_t(P)]^{\alpha-1} - \delta = P \cdot \bar{r}_t(P) + P \cdot (\varphi + \lambda) - \varphi \cdot \eta - P^* \cdot \lambda\,. \end{aligned} \tag{48}$$

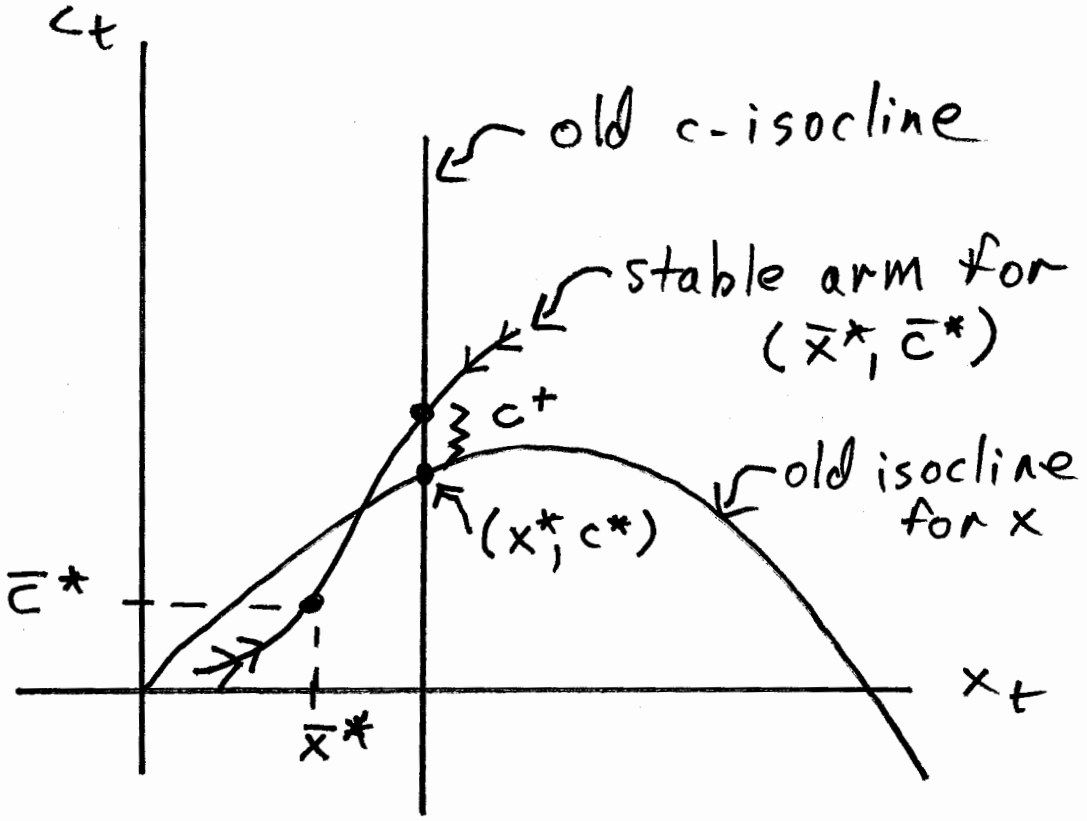

c_t
old c-isocline
stable arm for
$(\bar{x}^*, \bar{c}^*)$
c^+
old isocline
for x
(x^*, c^*)
$\bar{c}^*$
x_t
$\bar{x}^*$

Figure 3

Let r^* be the interest rate at stationary point a in Figure 1. Because $\bar{x}_t(P^*) = x^*$ all t, we have

$$\bar{r}_t(P^*) = r^* \quad \text{all} \quad t \geq 0\,.$$

We want the differential of $\bar{r}_t(P)$ with respect to P to be

$$d\bar{r}_t(P) = \bar{r}_t(P^*) + \frac{\partial \bar{r}_t(P^*)}{\partial P}\, dP = r^* + \frac{\partial \bar{r}_t(P^*)}{\partial P}\, dP\,. \tag{49}$$

Differentiating (48) with respect to P,

$$\alpha \cdot (\alpha - 1) \cdot A \cdot [\bar{x}_t(P)]^{\alpha-2} \cdot \frac{\partial \bar{x}_t(P)}{\partial P} = \bar{r}_t(P) + P \cdot \frac{\partial \bar{r}_t(P)}{\partial P} + (\varphi + \lambda)$$
$$\Longleftrightarrow \frac{\partial \bar{r}_t(P^*)}{\partial P} = \frac{\alpha \cdot (\alpha - 1) \cdot A \cdot [\bar{x}_t(P^*)]^{\alpha-2}}{P^*} \cdot \frac{\partial \bar{x}_t(P^*)}{\partial P} - \frac{r^* + \varphi + \lambda}{P}\,.$$

To find the impact change in $\bar{r}_t(P)$, recall that x is a non-jump variable. Hence,

$$\frac{dr^+}{dP} = \frac{\partial \bar{r}_0(P^*)}{\partial P} = -\frac{r^* + \varphi + \lambda}{P^*}\,. \tag{50}$$

Table 4 presents impact outcomes. The top section examines impact effects on investment. The first column characterizes a permanent change in P. Columns 2-6 investigate changes of average duration of 5 years, 2 years, 1 year, 2 quarters, and 1 quarter, respectively. The elasticity magnitudes rise as we move to temporary changes of shorter duration. Not surprisingly, the elasticities for net investment are appreciably larger.

The bottom part of Table 4 considers the impact on the (real) interest rate. The elasticities are very large. A 5 percent drop in θ, which raises P^* from 1.23 to 1.29, drives $r^* = 0.100$ below 0 for $\lambda \geq 2.0$, and more than halfway to 0 for $\lambda = 1.0$.

Empirical record. Laitner and Stolyarov [2003, 2004] argue that improvements in the micro processor created a new general purpose technology in the early 1970s, leading to abrupt obsolescence for existing capital, and excellent opportunities for new investment.[16] In the context of the present paper, coordination problems for the new technology — i.e., need for a common operating system, for service and support, for software, etc. — might have lowered θ in the short run.

Rising oil prices at the same time, may have contributed to entry risk (Baily et al [1981]). There was uncertainty about the permanence of the change. Beyond that, there were multiple possible national responses (i.e., conservation, biofuels, hybrid automobiles, green energy sources, off-shore oil drilling, nuclear power, etc.). Given network externalities, the number of credible options would itself, as in the preceding paragraph, tend to raise entry risk.

[16] See the discussion of general purpose technologies in Helpman, ed., *General Purpose Technologies and Economic Growth.* Cambridge, MA: MIT Press, 1998.

Table 4. Impact Effect of a Temporary Change in P^* Stemming from a Change in θ (Table 1 Parameters Values)

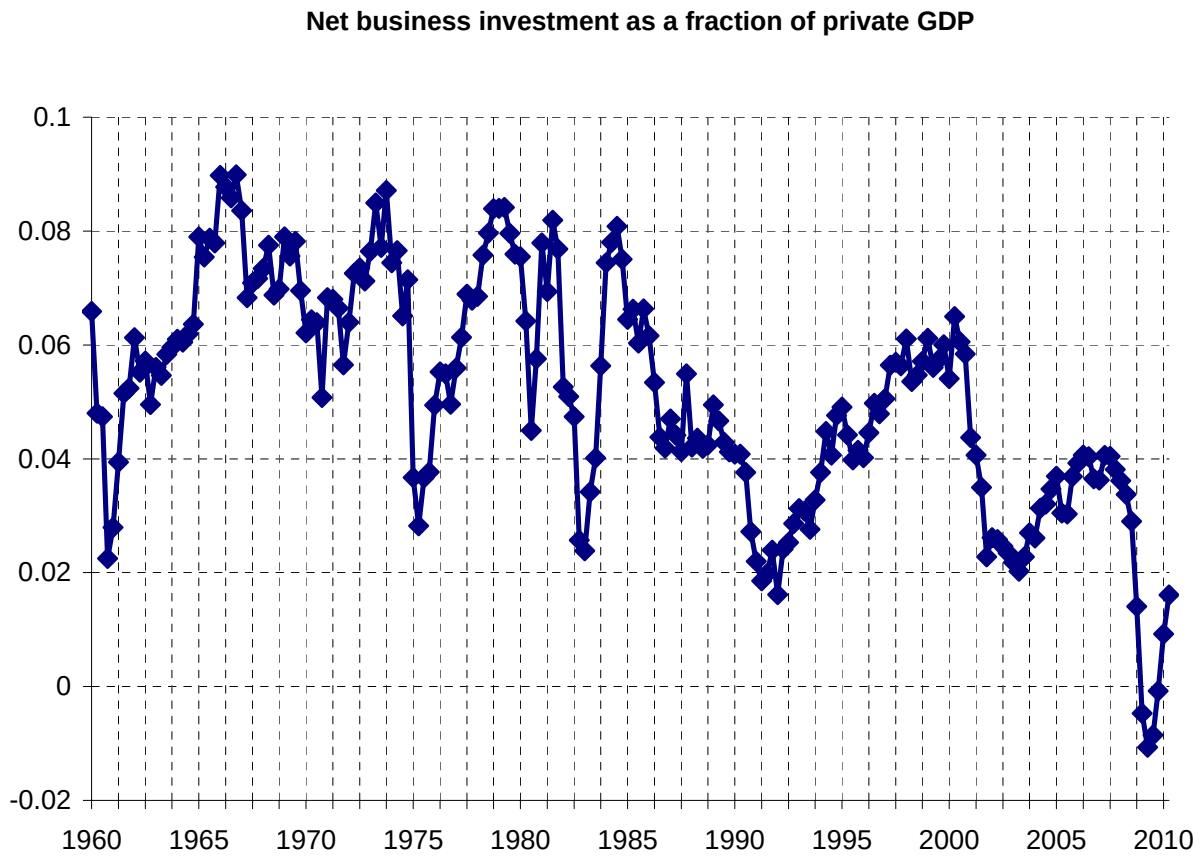
β	$\lambda =$					
	0.00	0.20	0.50	1.00	2.00	4.00
Elasticity: $\frac{di^{NIPA+}/i^{NIPA*}}{dP^*/P^*}$						
0.000	-2.023	-3.027	-3.587	-3.945	-4.192	-4.342
-1.000	-0.990	-1.618	-1.907	-2.073	-2.181	-2.244
-2.000	-0.536	-1.074	-1.294	-1.414	-1.490	-1.533
-3.000	-0.268	-0.780	-0.974	-1.076	-1.139	-1.175
-4.000	-0.087	-0.596	-0.778	-0.871	-0.927	-0.959
-5.000	0.046	-0.469	-0.644	-0.732	-0.785	-0.815
Elasticity: $\frac{di^{net\ NIPA+}/i^{net\ NIPA*}}{dP^*/P^*}$						
0.000	-5.732	-8.577	-10.164	-11.177	-11.879	-12.302
-1.000	-2.805	-4.585	-5.404	-5.875	-6.180	-6.357
-2.000	-1.520	-3.042	-3.667	-4.007	-4.221	-4.343
-3.000	-0.760	-2.211	-2.760	-3.050	-3.228	-3.328
-4.000	-0.247	-1.689	-2.203	-2.467	-2.627	-2.717
-5.000	0.129	-1.330	-1.825	-2.075	-2.225	-2.308
Elasticity: $\frac{dr^+/r^*}{dP^*/P^*}$						
all values	-2.100	-4.100	-7.100	-12.100	-22.100	-42.100

Source: see text.

The next page presents a figure showing net investment as a fraction of GDP. The period of the 1970s as a whole yielded low real returns to financial investors, whereas the reverse followed for the next 20 years (e.g., Smithers and Wright [2000, chart 3.1]).

Both the investment and real-return evidence for the 1970-95 period seem broadly consistent with Table 4. Our argument is that creative destruction in the early 1970s led to the 1972-74 stockmarket crash, and, in turn, created potential new investment opportunities. Coordination problems, however, may have temporarily increased P_{1t} — i.e., marginal q. (Average q, on the other hand, would have been low, due to measurement problems — see below.) Creditors preferred to delay new investment until the problems were resolved. In the interim, the rate of return on financial investments was low. By the early 1980s, the temporary episode was over. In ensuing years, investment was high. While P_{1t} would have actually been somewhat below its temporary high during the 1970s, the marginal product of (new) capital was great enough to deliver large financial returns to creditors. Measured average q rose steadily as new capital replaced plant and equipment made obsolete in the early 1970s.

In 2008, financial-market turmoil from the subprime mortgage crisis made almost all


Net business investment as a fraction of private GDP
0.1
0.08
0.06
0.04
0.02
0
-0.02
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010

businesses struggle. Our micro data was available by this time, and Section 4 shows that entry risk rose. Again, our model predicts a reduction in interest rates and in net investment. The net business investment figure on the preceding page confirms that investment dropped very sharply.[17]

Information on Treasury Inflation Protected Securities (TIPS), available after 1997, provides a general indication of the time path of real interest rates. The US Treasury provides data on 5, 10, and 20 year TIPS yield to maturities.[18] Graphs of January data show a flat yield curve with $r \approx 0.02$ in 2006-07. By 2008, however, the yield on 5-year TIPS was 0.01, whereas the 20-year yield remained at the 2006 level. By 2011, the 5-year yield was just 0.0025, while the 20-year remained almost as before, at about 0.016.[19] The data is, therefore, consistent with a temporary reduction in real interest rates.

Again, investment and interest rates both follow patterns roughly consistent with the model's implications for a temporary increase in entry risk.

Discussion of the earlier literature. Section 5 noted the literature's finding of a positive correlation in aggregative time series data of average q and I/K. Proposition 4 and/or Table 3 may be part of the answer. Another part may be as follows.

We have suggested that after a macro shock, coordination problems sometimes lead to a temporary spell in which marginal q is high. Table 4 implies that during the spell, short-term interest and net investment rates may be quite low. Section 4 argues that measured average q may be severely biased downward at the same time, due to unusually great obsolescence of older capital that the denominator of measured average q will tend to overlook. If investment and measured average q are low at the same time, a spurious positive correlation between the two may emerge.

7 Conclusion. This paper presents an equilibrium theory of investment that stresses the role of risk as a determinant of aggregate investment expenditure. In the model, net investment, for new and old firms, takes the form of adding new establishments. We measure risk from establishment failure rates. We do not invoke a convex cost-of-adjustment function.

We show that risk of failure is empirically large and can explain about one-half of the intangible capital in the US. (Planning costs account for the remainder in our model.)

We suggest several ways in which our model can explain the positive correlation of investment and average q characterized in the existing literature.

Increases in risk that are temporary have the largest elasticities in our model, potentially leading to sharp downturns in investment and short-term interest rates. We argue

[17] In our model, part of NIPA "depreciation" covers replacement of establishments that die from obsolescence. Replacement investment for this component of "depreciation" must breakeven. In other words, having NIPA net investment fall to 0 is insufficient, by a considerable margin, to thrust the economy out of what Section 2 calls "regime 1."

[18] See $http://www.treasury.gov/resource-center/$.

[19] Note that the 2010 yield curve is very similar to 2011. The January 2, 2009, curve, however, is erratic.

that the most recent recession provides evidence consistent with this analysis, and that our model may help to interpret aggregative-data patterns from the earlier periods as well.

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Appendix

Proof of Lemma 1. Firm maximization requires

$$W_t = \frac{\partial Y_{jt}}{\partial N_{jt}}, \tag{A1}$$

which implies

$$\frac{X_{jt}}{E_{jt}} = \frac{X_{0t}}{E_{0t}} \quad \text{all} \quad j \in \mathcal{J}_t\,. \tag{A2}$$

Then

$$X_t = [\text{from (9)}] = \int_{\mathcal{J}_t} \frac{X_{jts}}{E_{jt}} E_{jt} dj = [\text{from (A2)}] = \frac{X_{0t}}{E_{0t}} \int_{\mathcal{J}_t} E_{jt} dj = [\text{from (10)}] = E_t \cdot \frac{X_{0t}}{E_{0t}}.$$

So,

$$\frac{X_{0t}}{E_{0t}} = \frac{X_t}{E_t} \tag{A3}$$

Hence,

$$Y_t = [\text{from (8)}] = \int_{\mathcal{J}_t} Y_{jt} dj = [\text{from (7)}] = \int_{\mathcal{J}_t} A \left(\frac{X_{jt}}{E_{jt}}\right)^{\alpha} E_{jt} dj = [\text{from (A2)}] =$$

$$A \left(\frac{X_{0t}}{E_{0t}}\right)^{\alpha} \int_{\mathcal{J}_t} E_{jt} dj = [\text{from (10)}] = A \left(\frac{X_{0t}}{E_{0t}}\right)^{\alpha} E_t = [\text{from (A3)}] = A X_t^{\alpha} E_t^{1-\alpha}.$$

The envelope theorem establishes (14). Euler's formula then establishes (15). ■

Proof of Proposition 1: Let $j_t \equiv J_t/E_t$ and $j_t^S \equiv J_t^S/E_t$. We want to solve

$$\max_{c_t, j_t, j_t^S} \int_0^{\infty} e^{-(\rho-n-\beta\cdot g)\cdot t} \cdot u(c_t)\, dt \tag{A4}$$

$$\dot{x}_t = \theta \cdot j_t + j_t^S - (\varphi + n + g) \cdot x_t\,,\ x_0 \text{ is given,}$$

$$j_t = \begin{cases} \frac{A\cdot[x_t]^{\alpha} - c_t + (\eta\cdot\varphi-\delta)\cdot x_t}{1-\eta\cdot(1-\theta)}, & \text{if } A\cdot[x_t]^{\alpha} - c_t + (\eta\cdot\varphi-\delta)\cdot x_t \geq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{A5}$$

$$j_t^S = \begin{cases} \frac{A\cdot[x_t]^{\alpha} - c_t + (\eta\cdot\varphi-\delta)\cdot x_t}{\eta}, & \text{if } A\cdot[x_t]^{\alpha} - c_t + (\eta\cdot\varphi-\delta)\cdot x_t < 0 \\ 0, & \text{otherwise} \end{cases}. \tag{A6}$$

The variable j_t registers the flow of new investment projects. In a severe downturn, the economy might actually scrap some establishments, and, in that case, $j_t^S < 0$ measures the flow of scrap investment usable for replacement investment.

(A6) yields a standard set of constraints:

$$\eta \cdot j_t^S \leq A \cdot [x_t]^{\alpha} - c_t + (\eta\cdot\varphi-\delta)\cdot x_t\,,$$

$$j_t^S \leq 0\,.$$

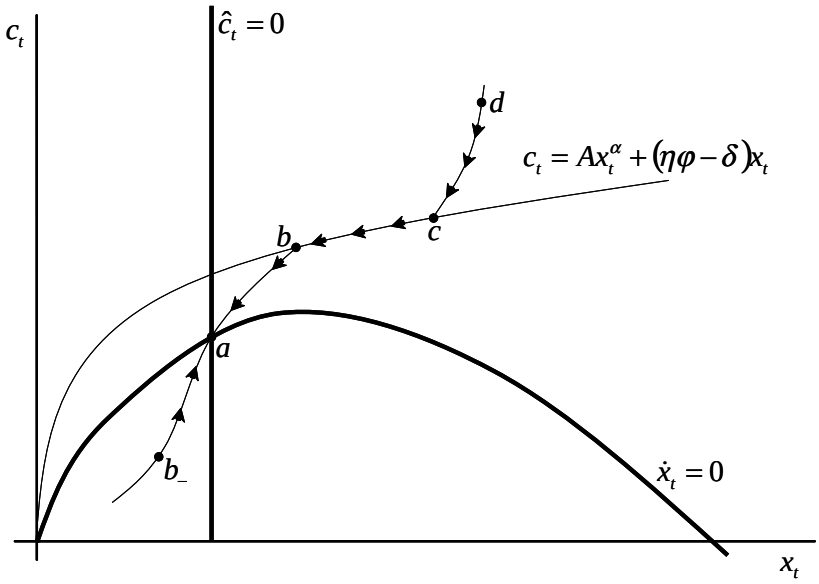

Figure 1: Phase diagram for the solution.

However, (A5) is conditional on $A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t \geq 0$. We can remove the conditionality as follows. Note that either $j_t^S = 0$ and $[1 - \eta \cdot (1-\theta)] \cdot j_t = A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t \geq 0$ or $j_t = 0$ and $\eta \cdot j_t^S = A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t$. Thus, we can substitute

$$[1 - \eta \cdot (1-\theta)] \cdot j_t = A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t - \eta \cdot j_t^S \,,$$

$$j_t \geq 0 \,,$$

for (A5). The right-hand side of the first of these inequalities will always be nonnegative.

We form a current-value Hamiltonian,

$$\begin{aligned} \mathcal{H} \;\equiv\; & u(c_t) + \mu_t \cdot [\theta \cdot j_t + j_t^S - (\varphi + n + g) \cdot x_t] \\ & + \Omega_t^1 \cdot [A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t - \eta \cdot j_t^S - [1 - \eta \cdot (1-\theta)] \cdot j_t] + \omega_t^1 \cdot j_t \\ & + \Omega_t^2 \cdot [A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t - \eta \cdot j_t^S] - \omega_t^2 \cdot j_t^S . \end{aligned}$$

The Lagrange multipliers Ω_t^i and ω_t^i should be nonnegative and follow continuous paths; the costate μ_t should be positive and continuous; the controls are j_t, j_t^S, and c_t; and, the state variable is x_t. The Hamiltonian is concave in $(x_t \,,\, c_t \,,\, j_t \,,\, j_t^S)$.

Our solution follows Figure 1. The stationary point in the phase diagram is a. Initial conditions x_0 consistent with path $\overline{ab}$ or $\overline{ab'}$ allow us to follow the solution of Proposition 1 in the text. Larger initial conditions x_0 require that we start on the arc $\overline{bc}$ where $j_t = 0 = j_t^S$, so that there is neither new investment nor forced-closure disinvestment. Still larger initial conditions require forced closures of existing establishments for optimality, along arc $\overline{cd}$.

First-order conditions for the controls are

$$u'(c_t) = \Omega_t^1 + \Omega_t^2 \,, \tag{A7}$$

$$\theta \cdot \mu_t - [1 - \eta \cdot (1-\theta)] \cdot \Omega_t^1 + \omega_t^1 = 0 \,, \tag{A8}$$

$$\mu_t - \eta \cdot \Omega_t^1 - \eta \cdot \Omega_t^2 - \omega_t^2 = 0 \,. \tag{A9}$$

The costate equation is

$$\begin{aligned}\mu_t &= (\rho - n - \beta \cdot g) \cdot \mu_t + (\varphi + n + g) \cdot \mu_t \\ &\quad -(\Omega_t^1 + \Omega_t^2) \cdot (\alpha \cdot A \cdot [x_t]^{\alpha-1} + \eta \cdot \varphi - \delta).\end{aligned} \tag{A10}$$

When $j_t = 0 = j_t^S$, we have

$$c_t = A \cdot [x_t]^\alpha + (\eta \cdot \varphi - \delta) \cdot x_t \tag{A11}$$

Step 1. Consider initial conditions that start us along $\overline{ab}$.

As stated, this is the case of Proposition 1. We have a conventional saddlepoint phase diagram. We have $j_t^S = 0$, $\omega_t^1 = 0$, and $\Omega_t^2 = 0$. We solve equations

$$\widehat{\mu_t} = \rho + (1-\beta) \cdot g + \varphi - \frac{\alpha \cdot A \cdot [x_t]^{\alpha-1} + (\eta \cdot \varphi - \delta)}{P_1},$$

$$\dot{x}_t = \frac{A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t}{P_1} - (\varphi + n + g) \cdot x_t,$$

$$\frac{\mu_t}{P_1} = u'(c_t).$$

We set $\omega_t^2 = \mu_t - \eta \cdot u'(c_t) > 0$ all t.

Step 2. Suppose that the stable arm from a cuts the graph of (A11) at point b, as shown in Figure 1. Terminal conditions from the stable arm determine $\omega_b^1 = 0$; $\omega_b^2 = \mu_b - \eta \cdot u'(c_b)$; $\Omega_b^1 = u'(c_t)$; $\Omega_t^2 = 0$; $\mu_b = P_1 \cdot u'(c_b)$. We follow arc $\overline{bc}$ along the graph of (A11).

On $\overline{bc}$ we set

$$j_t = 0 = j_t^S = A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t.$$

Using the terminal conditions above as starting values, solve

$$\dot{x}_t = -(\varphi + n + g) \cdot x_t,$$

$$c_t = A \cdot [x_t]^\alpha + (\eta \cdot \varphi - \delta) \cdot x_t,$$

$$\mu_t = (\rho + (1-\beta) \cdot g + \varphi) \cdot \mu_t - u'(c_t) \cdot (\alpha \cdot A \cdot [x_t]^{\alpha-1} + \eta \cdot \varphi - \delta).$$

Solving these equations backward in time, we stop when we reach x_0 - provided it lies on $\overline{bc}$.

We solve for c using Figure 2. The graph of μ_t/η starts above $u'(c_t) = \mu_t/P_1$ because $P_1 > \eta$. At c, the graphs of $u'(c_t)$ and μ_t/η cross. If they never cross, $x_c = \infty$, and we are done. Figure 2 presents these conditions. Conditions for the Lagrange multipliers are

$$\Omega_t^1 + \Omega_t^2 = u'(c_t),$$

$$\frac{\mu_t}{P_1} - \Omega_t^1 + \frac{\omega_t^1}{\theta \cdot P_1} = 0,$$

$$\frac{\mu_t}{\eta} - \Omega_t^1 - \Omega_t^2 - \frac{\omega_t^2}{\eta} = 0.$$

Substituting from the first into the third, we set ω_t^2 from

$$\frac{\omega_t^2}{\eta} = \frac{\mu_t}{\eta} - u'(c_t).$$

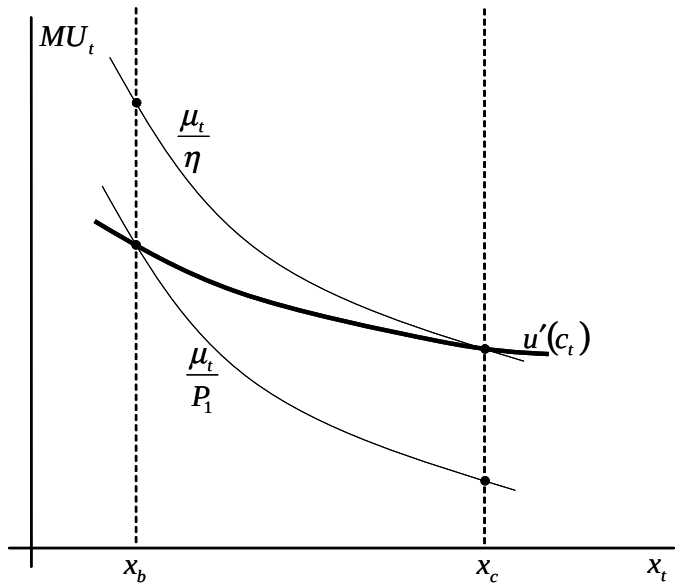

Figure 2: Solving for c.

This starts at the terminal level from Step 1 and ends with $\omega_t^2 = 0$. We set $\omega_t^1 = 0$ and

$$\Omega_t^1 = \frac{\mu_t}{P_1}\,.$$

Finally, we set

$$\Omega_t^2 = u'(c_t) - \Omega_t^1\,.$$

Both Ω_t^1 and Ω_t^2 start at the correct level.

If Ω_t^2 becomes negative on $\overline{bc}$, we must adjust Ω_t^1 downward to prevent this.

Step 3. We follow $\overline{cd}$. We have $j_t = 0$; $\eta \cdot j_t^S = A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t < 0$; and, $\omega_t^2 = 0$.

The equations are

$$\dot{x}_t = \frac{A \cdot [x_t]^\alpha - c_t + (\eta \cdot \varphi - \delta) \cdot x_t}{\eta} - (\varphi + n + g) \cdot x_t\,,$$

$$\mu_t = (\rho + (1-\beta) \cdot g + \varphi) \cdot \mu_t - u'(c_t) \cdot (\alpha \cdot A \cdot [x_t]^{\alpha-1} + \eta \cdot \varphi - \delta)\,,$$

$$\frac{\mu_t}{\eta} - u'(c_t) = 0\,.$$

For the Lagrange multipliers,

$$\frac{\mu_t}{P_1} = \Omega_t^1\,,$$

$$\omega_t^1 = 0\,,$$

$$\Omega_t^2 = u'(c_t) - \Omega_t^1\,.$$

Because we have $j_t^S < 0$, c_t rises above the graph of (A11), as shown in Figure 1. ■

Proof of Proposition 3. Consider the isoclines in regime 1 of Figure 1. Let

$$\phi(x) \equiv A \cdot [x_t]^{\alpha} + (\eta \cdot \varphi - \delta) \cdot x_t \,.$$

Then

$$\begin{aligned}
\dot{x}_t = \theta j_t - (\varphi + n + g)\, x_t = 0 &\Longleftrightarrow \\
\theta j_t = (\varphi + n + g)\, x_t &\Longleftrightarrow \\
\frac{\phi(x_t) - c_t}{P_*} = (\varphi + n + g)\, x_t &\Longleftrightarrow \\
\phi(x_t) - P_* (\varphi + n + g)\, x_t = c_t & \qquad \text{(A12)}
\end{aligned}$$

(A12) shows this isocline shifts downward when P^* rises.

The other isocline is vertical. We have

$$\frac{\dot{\mu}_t}{\mu_t} = 0 \Longleftrightarrow$$

$$P_* (\zeta + \varphi + n + g) = \phi'(x_t) = \alpha A x_t^{\alpha - 1} + (\eta\varphi - \delta) \qquad \text{(A13)}$$

So, a higher P^* causes this isocline to shift left.

Figure 1 then establishes Proposition 3. ■

Proof of Proposition 4. We have

$$i_* = \theta P_* j_* - (\eta\varphi - \delta)\, x_* = (\varphi + n + g)\, x_* P_* - (\eta\varphi - \delta)\, x_*$$

Thus, the long-run elasticity for i^*/x^* will, in fact, always be positive. ■