

# The Role of Annuitized Wealth in Post-Retirement Behavior\*

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## Abstract

This paper develops a tractable model of post-retirement behavior with health status uncertainty and state verification difficulties. The model distinguishes between annuitized and non-annuitized wealth and features Medicaid assistance with nursing-home care. The analysis shows how to solve the potentially complex dynamic problem analytically, making it possible to characterize optimal behavior with phase diagrams. Results reveal that annuitization promotes self-insurance and that Medicaid has a strong influence on incentives to hold annuities. We show the model can explain both rising cohort-average wealth after retirement and retirees' reluctance to fully annuitize their liquid wealth in practice.

## 1 Introduction

Interest in the life-cycle behavior of retired households has increased with population aging and the associated strain on public programs for the elderly.<sup>1</sup> Yet post-retirement behavior has proved challenging to understand. Standard theories, for example, are hard to reconcile with evidence that shows a lack of wealth depletion after retirement — the “retirement-saving puzzle” — and a low demand for annuities at retirement — the “annuity puzzle.” Analytic difficulties emerge as well. Some come from the fact that social insurance programs for older people tend to have elaborate rules, and the incentives that these rules generate often cannot be studied with the standard toolkit. Other difficulties arise from interactions of health uncertainty with incomplete financial and insurance markets. The purpose of this paper is to develop a parsimonious model that incorporates important features of the economic environment, yet retains sufficient tractability to

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<sup>1</sup>E.g., Hubbard *et al.* [1994, 1995], Palumbo [1999], Sinclair and Smetters [2004], Reichling and Smetters [2013], Dynan *et al.* [2004], Scholz *et al.* [2006], Scholz and Seshadri [2009], Ameriks *et al.* [2011, 2015a, 2015b], DeNardi *et al.* [2010, 2013], Lockwood [2014], Love *et al.* [2009], Laibson [2011], Finkelstein *et al.* [2011], and Poterba *et al.* [2011, 2012], Pashchenko [2013].

be useful for qualitative, as well as quantitative, analysis. The model emphasizes the distinction between annuitized and non-annuitized wealth. With it, we are able to study the mechanisms through which portfolio composition interacts with public programs and affects retiree behavior.

The model captures uncertain health and the correlation of major health changes with changes in mortality risk. Importantly, it assumes informational asymmetries that lead to incomplete private markets for long-term care insurance. It also incorporates a means-tested public alternative, Medicaid nursing-home care, which households can use as a fall-back during poor health. The model takes into account the inflexible nature of annuities as a form of wealth, as well as their treatment under Medicaid.

Despite its richness, the model is analytically tractable. One key to the tractability is the model’s continuous-time formulation, which enables it to sidestep technical challenges related to non-convexities (challenges arising from the Medicaid means test — and leading most of the literature to numerical analysis). A second key is the case-by-case analytic approach that our formulation allows: although the model’s elements and assumptions generate a variety of optimal behavioral patterns, we can partition the domain of observable initial conditions in such a way that outcomes are relatively straightforward on each (partition) element.

We use the model to study 3 specific topics. The “retirement-saving puzzle,” to take the first example, has bedeviled analysts of the basic life-cycle model of household behavior for decades (see the literature review below). The puzzle is that, in practice, a cohort’s average wealth often remains constant, or even rises, long into retirement. This seemingly contradicts a core idea of the life-cycle model, namely, that households save during working years in order to dissave thereafter.

The model offers new insights into the puzzle. Post-retirement behavior depends on both annuity level and portfolio composition. If its annuity income is below a threshold, a healthy household chooses to dissave and accept Medicaid quickly once it reaches poor health. For annuity incomes above the threshold — corresponding to the middle class in practice — households save if and only if their initial portfolio *share* of annuitized wealth is also high enough. Middle-class households want to postpone the time when they will run out of liquid wealth and need to rely either on annuity income or public assistance. In an environment with incomplete financial markets, liquid wealth provides the flexibility to do so.

The analysis then offers an intuitive explanation of why the retirement-saving puzzle may emerge. Combining optimizing households with stochastic processes for health and mortality, the model characterizes cohort-average wealth-age trajectories. It shows how and why rising trajectories are possible (though not inevitable).

We provide numerical illustrations. They suggest that our focus on portfolio composition can improve the quantitative performance of existing models that seek to rationalize retiree wealth holdings. For example, several existing studies use a fairly high (3.5-4.0) risk aversion coefficient and/or assume a bequest motive in addition to health status risk. Our model, by contrast, only needs a modest (1.0-2.0) risk aversion coefficient to generate rising (with age) cohort-average wealth, and it does not have to rely on a bequest motive to generate empirically relevant post-retirement wealth trajectories.

Second, we analyze the effect of household annuitization on the cost of the Medicaid program. There are important policy implications at stake: we want, for instance, to be able to assess how trends toward diminished annuitization at retirement — as with reductions in the popularity of DB pensions — could affect demand for Medicaid in the future.

The model shows that demand for Medicaid is shaped in important ways by household responses

to the Medicaid means test, which requires recipients to forfeit their annuity income. The means test makes it less attractive for heavily annuitized households to accept Medicaid, giving them more incentive to self-insure. We find, however, that the effect is largest for the middle class — for the poor, the advantages of Medicaid are so great that they virtually have a “corner solution;” the richest households, on the other hand, never use Medicaid at all.

For low-resource households, numerical illustrations show only a tiny effect of lower initial annuitization on Medicaid demand. For the middle class, in contrast, lower (higher) initial annuitization tends to generate higher (lower) Medicaid spending. Depending on the cross-sectional distribution of household types, across-the-board reductions in annuity shares at retirement could thus increase total demand for Medicaid assistance.

Third, economists have long been interested in explanations for households’ apparent reluctance to annuitize all, or most, of their wealth at retirement. Households, for instance, often claim Social Security benefits at or below the age for full retirement benefits, thereby forgoing additional actuarially fair annuitization (Brown [2007]). To study this issue, we depart from our benchmark specification, in which annuities are given by initial endowments, and instead allow households the chance to adjust their portfolio composition optimally at retirement.

The model shows that Medicaid crowds out the demand for annuities for all but the wealthiest households. In general, Medicaid reduces household incentives to accumulate private wealth. Importantly, however, disincentives are especially strong in the case of annuitized wealth. Our analysis shows that availability of public assistance can rationalize low annuity demand at retirement.

Numerical examples provide evidence that the interactions of portfolio composition and the Medicaid means test can be quantitatively large. Households that would desire almost 100% annuitization without Medicaid want a much larger share of liquid assets when Medicaid is available. Surprisingly, calibrations based on actual distributions of wealth suggest that Social Security and DB pensions alone can leave many households more annuitized at retirement than their optimum.

## 1.1 Relation to the literature

This subsection describes the two puzzles above in slightly more detail and compares our approach to other recent work.

In the standard life-cycle model, households smooth their lifetime consumption by accumulating wealth prior to retirement and decumulating it thereafter. At least since Mirer [1979], evidence has seemed at variance with the model’s post-retirement prediction. Kotlikoff and Summers noted,

“Decumulation of wealth after retirement is an essential aspect of the life cycle theory. Yet simple tabulations of wealth holdings by age ... or savings rates by age ... do not support the central prediction that the aged dissave. [1988, p.54]

Recent work with panel data confirms that mean and median cohort wealth, for either singles or couples, can be stationary or rising for many years after retirement (Poterba *et al.* [2010]).<sup>2</sup>

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<sup>2</sup>See also, for instance, Ameriks *et al.* [2015], who observe, “The elementary life-cycle model predicts a strong pattern of dissaving in retirement. Yet this strong dissaving is not observed empirically. Establishing what is wrong with the simple model is vital ....” See also DeNardi *et al.* [2013, fig.7] as well as Smith *et al.* [2009], Love *et al.* [2009], and many others.

Recent analyses of post-retirement saving such as Ameriks *et al.* [2011, 2015a, 2015b] and DeNardi *et al.* [2010, 2013] include a number of the same elements as our framework, namely, health changes and mortality risk, out-of-pocket expenses in poor health, government guaranteed consumption floors (in our case, Medicaid nursing-home care), and fixed annuity income. Since consumption floors can induce non-convexities, Ameriks *et al.* and DeNardi *et al.* rely upon numerical solutions. In explaining household wealth trajectories, both recognize the potential importance of post-retirement precautionary saving.

As stated, our formulation sidesteps non-convexities. The advantage is that the solution can be characterized with first-order conditions that can provide intuitions and comparative-static results. Non-convexities also raise the possibility that households will seek actuarially fair gambles to maximize their lifetime utility (e.g., Laitner [1988]), and our approach avoids that complication.

What is more, our model offers several important refinements for the study of precautionary saving. On the one hand, we show that a (healthy) household’s desire to save after retirement depends upon its portfolio composition: given two healthy households with identical total net worth, our model shows that the one with the higher fraction of annuities in its portfolio is the more likely to continue saving. On the other hand, the analysis explains why the behavior of households in good health is pivotal in driving cohort average wealth upward long into retirement.

DeNardi *et al.* [2013] present evidence that wealthier households tend to access Medicaid assistance later in life. Our results are consistent with this finding, and we can characterize Medicaid take-up timing analytically and provide further interpretations of the data.

Ameriks *et al.*, DeNardi *et al.*, and Lockwood [2014] consider the possible role of intentional bequests in sustaining private wealth holdings late in life — i.e., in helping to explain the “retirement-saving puzzle.” Our analysis, in contrast, does not require intentional bequests to fit the same evidence. The bequests that emerge in our model are by-products of incomplete annuitization.

One interpretation is that our work shows that intentional bequests are not needed for analyzing this paper’s issues and so, in the spirit of Occam’s razor, we omit them. Another is as follows. Survey evidence on intentional bequests is mixed: respondents to direct questions about leaving a bequest split approximately equally between answering that bequests are important and not important (Lockwood [2014], Laitner and Juster [1996]). Our analysis allows one to rationalize the post-retirement behavior of the latter group (as well as those for whom an “important” bequest could be a modest family heirloom).

Since the seminal work of Yaari [1965], many economists have sought explanations for why households do not fully annuitize their private wealth at retirement. Benartzi *et al.* write,

“The theoretical prediction that many people will want to annuitize a substantial portion of their wealth stands in sharp contrast to what we observe. [2011, p.149]

There is a rich literature on this “annuity puzzle” (e.g., Finkelstein and Poterba [2004], Davidoff *et al.* [2005], Mitchell *et al.* [1999], Friedman and Warshawski [1990], Benartzi *et al.* [2011], and many others).

Both this paper and Reichling and Smetters [2015] offer new interpretations of the “annuity puzzle.” While the studies have a number of assumptions in common, the institutional settings differ. Reichling and Smetters allow a household whose current health and/or mortality hazards have changed to purchase new annuities reflecting the revised status. In our model, state-verification problems preclude health-contingent annuities. Nonetheless, a household suffering a decline in

health status can access Medicaid nursing-home care, and that option alone, we show, can substantially reduce the demand for annuities at retirement.

Some explanations of the “annuity puzzle” (e.g., Friedman and Warshawsky [1990]) give intentional bequests a prominent role. As in the case of the retirement-saving puzzle, our analysis does not rely upon intentional bequests.

Ameriks *et al.* [2015a] present simulations of a formulation that has health changes and state-dependent utility. Given a 10% load factor on annuities and households with \$50-100,000 of existing income and bond wealth up to \$400,000, they find essentially no demand for extra annuities at retirement (Ameriks *et al.* [2015a, fig.10]). We show that this outcome is consistent with the qualitative implications of our model, and we show how and why household initial conditions, health-status realizations, and interest rates affect outcomes.

The organization of this paper is as follows. Section 2 presents our assumptions and compares our formulation with others in the literature. Sections 3-4 analyze our model. Section 5 explains how the interest rate, the annuitization and the interaction of annuitization with Medicaid influences saving. Section 6 presents a calibration and numerical examples further illustrating the usefulness of the model and its quantitative performance. Section 7 concludes.

## 2 Model

As indicated in the introduction, we follow the recent literature in subdividing a household’s post-retirement years into intervals with good and poor health.

We study single-person, retired households. At any age  $s$ , a household’s health state,  $h$ , is either “high,”  $H$ , or “low,”  $L$ . The household starts retirement with  $h = H$ . There is a Poisson process with hazard rate  $\lambda > 0$  such that at the first Poisson event the health state drops to low. Once in state  $h = L$ , a second Poisson process begins, with parameter  $\Lambda > 0$ . At the Poisson event for the second process, household’s life ends.

We focus on the general “health state” of an individual, rather than his/her medical status. Think of “health state” as referring to chronic conditions. Consider, for example, troubles with activities of daily living (ADLs), such as eating, bathing, dressing, or transferring in and out of bed. Individuals with such difficulties may need to hire assistance or move to a nursing home. The expense can be substantial. It may, in practice, be the largest part of average out-of-pocket (OOP) medical expenses (see, for instance, Marshall *et al.* [2010], Hurd and Rohwedder [2009]).

STATE-DEPENDENT UTILITY We assume that health state affects behavior through state-dependent utility. In our framework, there are no direct budgetary consequences from changes in  $h$  – all retirees have access to Medicare insurance that covers the medical part of long-term care needs. By contrast, we treat all non-medical long-term care (LTC) expenses (i.e., health-related expenses not covered by Medicare – such as long nursing-home stays) as part of consumption. A household with  $h = H$  and consumption  $c$  has utility flow

$$u(c) = \frac{[c]^\gamma}{\gamma}.$$

Following most empirical evidence, let

$$\gamma < 0.$$

We assume there is a household production technology for transforming expenditure,  $x$ , to a consumption service flow,  $c$ :

$$c = \begin{cases} x, & \text{if } h = H \\ \omega x, & \text{if } h = L \end{cases} . \quad (1)$$

We also assume that the low health state is an impediment to generating consumption services from  $x$ ; thus,

$$\omega \in (0, 1).$$

The loss of consumption services that occurs upon reaching the low health state may be substantial: the agent in need of LTC might lose capacity for home production related to ADLs, and her quality of leisure (implicitly included in  $c$ ) may decline precipitously. Utility from consumption expenditure  $x$  while in health state  $h = L$  is

$$U(x) \equiv u(\omega x) \equiv \omega^\gamma u(x). \quad (2)$$

Since  $\omega^\gamma > 1$ , an agent in the low health state has lower utility but higher marginal utility of expenditure.

**AVAILABLE INSURANCE INSTRUMENTS** We assume that state verification problems for  $h$  are much greater than for medical status. An agent knows when he/she enters state  $h = L$ , but the transition from  $h = H$  is not legally verifiable. That prevents agents from obtaining health-state insurance.<sup>3</sup> Marshall *et al.* write,

“Indeed, the ultimate luxury good appears to be the ability to retain independence and remain in one’s home ... through the use of (paid) helpers .... These types of expenses are generally not amenable to insurance coverage .... [p.26]

In contrast, all of our model’s households have (Medicare) medical insurance.

Annuities dependent upon the health state are similarly unavailable. In fact, in our baseline case, the analysis treats annuities as exogenously fixed at retirement. However, when discussing the “annuity puzzle,” we allow households choose their initial portfolio composition. Throughout, we assume that households cannot borrow against their annuities.

**MEANS-TESTED PUBLIC ASSISTANCE** In our framework, a household with health status  $h = L$  can qualify for Medicaid-provided nursing home care. The means test for this program requires the household to forfeit all of its bequeathable wealth and annuities to qualify for assistance.<sup>4</sup> Let Medicaid nursing home care correspond to expenditure flow  $X_M > 0$ . In practice, elderly households often view Medicaid nursing-home care as a relatively unattractive option.<sup>5</sup> Accordingly, the model incorporates disamenities of Medicaid by assuming that the utility flow from Medicaid nursing home care is  $U(\bar{X})$ , where  $\bar{X} \leq X_M$  is the expenditure flow adjusted for disamenities.

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<sup>3</sup>On the use of long-term care insurance, which is analogous to health-state insurance in our model, see Miller *et al.* [2010], Brown and Finkelstein [2007, 2008], Brown *et al.* [2012], CBO [2004], and Pauly [1990]. Private insurance covers less than 5% of long-term care expenditures in the US (Brown and Finkelstein [2007]). For a discussion of information problems and the long-term care insurance market, see, for example, Norton [2000].

<sup>4</sup>In practice, a household may be able to maintain limited private assets after accepting Medicaid – for example, under some circumstances a recipient can transfer her residence to a sibling or child (see Budish [1995, p. 43]). This paper disregards these program details.

<sup>5</sup>Ameriks *et al.* [2011] refer to disamenities of Medicaid-provided nursing home care as *public care aversion*. Indeed, the level of service is very basic, access is rigorously means tested, and many households strongly prefer to live in familiar surroundings and to maintain a degree of control over their lives (Schafer [1999]).

HOUSEHOLD FINANCIAL ASSETS Households retire with endowments of two assets, annuities, with income  $a$ , and bequeathable net worth  $b$ . Major components of annuitized wealth include Social Security, defined benefit pension, and Medicare benefits. Bequeathable wealth  $b$  pays real interest rate  $r > 0$ . Let  $\beta \geq 0$  be the subjective discount rate. We assume  $r \geq \beta$ . If we think of the analysis as beginning at age 65, the average interval of  $h = H$  might be about 12 years, and the average duration of  $h = L$  about 3 years.<sup>6</sup> With a Poisson process, average duration is the reciprocal of the hazard. We assume  $\Lambda > \lambda > r - \beta$ .

LTC EXPENDITURE Our specification of household preferences assumes the simplest form of state-dependence: utility is  $u(x)$  in the high health state and  $\omega^\gamma u(x)$  in the low health state, where  $x$  is a single consumption category that includes the non-medical part of LTC expenditure.<sup>7</sup> The single-good assumption is not as restrictive as one might think. In fact, a richer model where non-medical LTC expenditure is a separate, endogenous variable would produce an indirect utility function of form (2). To see this, assume that a household has two remaining periods of life and that  $h = H$  in the first period and  $h = L$  in the last period.<sup>8</sup> Set  $r = 0$  and  $\beta = 1$ ; disregard annuities, Medicaid, and uncertain mortality. Then a newly retired household solves

$$\max_x \{u(x) + U(b - x)\}. \quad (3)$$

To endogenize the choice of LTC expenditure,  $l$ , replace  $U(b - x)$  in (3) with

$$U(b - x) \equiv \kappa \cdot \max_l \{\varphi \cdot u(b - x - l) + (1 - \varphi) \cdot u(l)\}, \quad (4)$$

where  $\kappa > 0$  and  $\varphi \in (0, 1)$  are preference parameters. Maximization with respect to  $l$  in (4) yields exactly the reduced form utility function (2):

$$U(b - x) = \omega^\gamma \cdot u(b - x), \\ \omega^\gamma \equiv \kappa \cdot \left( [\varphi]^{1-\gamma} + [1 - \varphi]^{1-\gamma} \right).$$

NON-CONVEXITY Continuing with the two-period example, let Medicaid nursing-home care provide a consumption expenditure flow  $\bar{X}$ . Accordingly, objective function (3) becomes<sup>9</sup>

$$\max_x \left( u(x) + \max\{U(b - x), U(\bar{X})\} \right). \quad (5)$$

Figure 1 depicts the corresponding second-period utility. We can see the non-convexity that Medicaid introduces. Depending on  $b$ , the optimal solution to (5) is either  $x^* = b$  or  $x^* < b - \bar{X}$ . In other words, the household either consumes all of its wealth while healthy and accepts Medicaid in the second period, or it saves enough so that second period consumption exceeds the Medicaid floor  $\bar{X}$ . It is never optimal to set  $x^* \in (b - \bar{X}, b)$ .

<sup>6</sup>E.g., Sinclair and Smetters [2004].

<sup>7</sup>Hubbard *et al.* [1995] and DeNardi *et al.* [2010] use a similar specification of preferences but assume that non-medical LTC expenditure is an exogenously fixed parameter not subject to choice, and not directly affecting utility.

<sup>8</sup>The two-period example is also convenient for direct comparisons with other two-period models, such as Finkelstein *et al.* [2013] and Hubbard *et al.* [1995].

<sup>9</sup>This example is similar to the two-period model used in Hubbard *et al.* [1995].

This introduces complications in a multi-period discrete time framework even if the problem is solved numerically. Furthermore, the non-concave utility function of Figure 1 makes a lottery over wealth levels in the appropriate range an attractive way to maximize utility.

Fortunately, by switching to continuous time, our formal model can circumvent both complications above. Doing so allows us to make the age at which liquid wealth is optimally depleted a continuous choice variable (called  $T$ ) separate from expenditure level  $x$ . Then we can characterize the solution analytically – using standard optimal control methods.

**SUMMARY** Recapping our baseline assumptions:

**a1:** “Health state” is not verifiable; hence, there is no health-state insurance. Annuities are exogenously set at retirement.

**a2:** If  $b_s$  is bequeathable net worth when  $h = H$  and  $B_s$  is the same for  $h = L$ , we have  $b_s \geq 0$  and  $B_s \geq 0$  all  $s \geq 0$ .

**a3:**  $\gamma < 0$ , and  $\omega \in (0, 1)$ .

**a4:** A household transitions from  $h = H$  to  $h = L$  with Poisson hazard  $\lambda$ , and from health state  $h = L$  to death with Poisson hazard  $\Lambda$ . We assume  $\Lambda > \lambda$ .

**a5:** The real interest rate is  $r$ , with  $0 \leq \beta \leq r < \lambda + \beta$ .

**a6:** A household in the low health state and having no liquid wealth can turn to Medicaid nursing-home care. The consumption value of the latter is a flow  $\bar{X}$ .

### 3 Low Health Phase

We solve our model backward, beginning with the last phase of life when the household is in the low health state  $h = L$ . In its last phase, a household faces mortality hazard  $\Lambda$ . Without loss of generality, scale the age at which the  $h = L$  state begins to  $t = 0$ . At  $t = 0$ , let bequeathable net worth be  $B \geq 0$ . Annuity income is  $a \geq 0$ ,  $X_t$  is consumption expenditure at age  $t$ , and  $U(X_t)$  the corresponding utility flow. The expected utility of the household is

$$\int_0^\infty \Lambda e^{-\Lambda \cdot S} \int_0^S e^{-\beta t} U(X_t) dt dS = \int_0^\infty e^{-(\Lambda+\beta)t} U(X_t) dt$$

Below, we show that the household will optimally plan to exhaust its liquid wealth in finite time, which we denote by  $T$ . If the household is alive at age  $T$ , it is liquidity constrained and has two options: it can either relinquish its annuity income  $a$  and accept Medicaid-provided consumption flow  $\bar{X}$ , or consume its annuity income for the remainder of its life. Households with  $a \geq \bar{X}$  will prefer to live off their annuity income (case (i) below), while households with  $a < \bar{X}$  will accept Medicaid assistance (case (ii)). To simplify the exposition, it is convenient to analyze the two cases separately.

**Case (i):**  $a \geq \bar{X}$  Starting from initial wealth level  $B$ , the household chooses a consumption expenditure path  $X_t$  all  $t \geq 0$  to solve

$$V(B) \equiv \max_{X_t} \int_0^\infty e^{-(\Lambda+\beta)t} U(X_t) dt \tag{6}$$

$$\text{subject to } \dot{B}_t = r \cdot B_t + a - X_t,$$

$$B_t \geq 0 \quad \text{all } t \geq 0,$$

$$B_0 = B \quad \text{and } a \text{ given.}$$

The present-value Hamiltonian for (6) is

$$\mathcal{H} \equiv e^{-(\Lambda+\beta)t}U(X_t) + M_t(rB_t + a - X_t) + N_tB_t, \quad (7)$$

with costate  $M_t$ , and Lagrange multiplier  $N_t$  for the state-variable constraint  $B_t \geq 0$ . Provided  $M_t \geq 0$ , first-order conditions will be sufficient for optimality provided the transversality condition holds:

$$\lim_{t \rightarrow \infty} M_t \cdot B_t = 0 \quad (8)$$

The strict concavity of problem (6) ensures that if a solution exists, it is unique.

We start by formally showing that a household with  $B = 0$  will optimally set  $X_t = a$  for the remainder of its life.

**Lemma 1:** *If  $a \geq \bar{X}$ ,  $(B_t^*, X_t^*) = (0, a)$  is a stationary solution to (6).*

**Proof:** See Appendix.

The idea of the proof is that households in (6) behave as if their subjective discount rate is  $\Lambda + \beta > r$ ; so, a household without a binding liquidity constraint desires a falling time path of consumption expenditure. When  $B_t = 0$ , only  $X_t \leq a$ , however, is feasible. At that point, a permanently falling time path cannot be optimal because the household's liquid wealth would expand until death. Lemma 1 shows that the solution is instead to maintain the constrained outcome forever.

Given Lemma 1, we can construct the general solution to (6) as follows. Suppose the state-variable constraint does not bind until after  $t = T$ . Then for  $t \leq T$ , omit the term  $N_tB_t$  from the Hamiltonian. The first-order condition for optimal expenditure is

$$\frac{\partial \mathcal{H}}{\partial X_s} = 0 \iff e^{-(\Lambda+\beta)s} \cdot U'(X_s) = M_s, \quad (9)$$

and the costate equation is

$$\dot{M}_s = -\frac{\partial \mathcal{H}}{\partial B_s} \iff \dot{M}_s = -r \cdot M_s. \quad (10)$$

Substituting (9) into (10) shows that the optimal expenditure falls at a constant rate:

$$\begin{aligned} & -(\Lambda + \beta)e^{-(\Lambda+\beta)s}U'(X_s) + e^{-(\Lambda+\beta)s}U''(X_s)\dot{X}_s = \\ & = \dot{M}_s = -r \cdot M_s = -re^{-(\Lambda+\beta)s} \cdot U'(X_s) \iff \\ & (\gamma - 1) \frac{\dot{X}_s}{X_s} = -(r - (\Lambda + \beta)) \iff \\ & \frac{\dot{X}_s}{X_s} = \sigma, \text{ where } \sigma \equiv \frac{r - (\Lambda + \beta)}{1 - \gamma} < 0. \end{aligned} \quad (11)$$

Taking into account the household budget constraint, the candidate solutions are depicted on phase diagram Figure 2. Each dotted curve is a trajectory satisfying the budget constraint and

(11). Equation (11) shows that along each trajectory,  $X_t > 0$  all  $t$ . Nevertheless, we can rule out the optimality of most of the trajectories *a priori*. A given trajectory intersects the vertical line at  $B_0 = B > 0$  at two points. The higher is preferred. But following the trajectory is then inferior to stopping at the intersection with the line  $X_s = rB_s + a$ . Yet the latter cannot be optimal since bequeathable wealth is never exhausted. The exception is the trajectory that intersects the vertical axis at  $(0, a)$ . Lemma 1 suggests that latter stopping point can be part of an optimal path.

In fact, we can show the transversality condition is then satisfied.

**Proposition 1:** *The trajectory in Figure 2 that reaches  $(B_t, X_t) = (0, a)$  from above and then remains at  $(0, a)$  forever solves problem (6). The solution  $(B_t^*, X_t^*)$ ,  $t \geq 0$ , is continuous in  $t$ . There exists  $T^* = T^*(B, a) \in [0, \infty)$  such that both  $B_t^*$  and  $X_t^*$  are strictly decreasing in  $t$  for  $t \leq T^*$ , but  $(B_t^*, X_t^*) = (0, a)$  for  $t > T^*$ .*

**Proof:** See Appendix.

The next proposition provides additional characterization and establishes solution properties needed for the subsequent phase diagram analysis.

**Proposition 2:** *Let  $T^*$ ,  $B_t^*$ , and  $X_t^*$  be as in Proposition 1. Then  $T^*(B, a)$  is strictly increasing and continuous in  $B$ ,*

$$T^*(0, a) = 0 \text{ and } \lim_{B \rightarrow \infty} T^*(B, a) = \infty.$$

We have

$$X_t^* = ae^{\sigma(t-T^*)} \quad \text{for } t \in [0, T^*].$$

As a function of  $B$ ,  $X_0^* = X_0^*(B, a)$  is continuous, strictly increasing, and strictly concave;

$$X_0^*(0, a) = a; \text{ and, } \lim_{B \rightarrow \infty} \frac{\partial X_0^*(B, a)}{\partial B} = r - \sigma > 0.$$

The optimal value function  $V(B)$  in (6) is strictly increasing and strictly concave.

**Proof:** See Appendix.

**Case (ii):**  $a < \bar{X}$  Case (ii) obtains when the value of Medicaid nursing-home care exceeds a household's annuity income.

In Lemma 1, a household with  $B = 0$  chooses  $X_t = a$  forever. In case (ii), the same household could do better by turning to Medicaid. Once Medicaid care is accepted, there is no incentive to ever leave it. In particular, if a household ever exited Medicaid assistance, it would have to start with zero liquid wealth. Subsequent optimal, privately-financed behavior would entail  $X_t = a$  forever. Yet, Medicaid offers, in case (ii), a better alternative, namely,  $X_t = \bar{X} > a$ .

Let  $T$  denote the age when the household exhausts its liquid wealth and turns to Medicaid. Then the case (ii) household behavior can be described with a standard free endpoint problem (Kamien and Schwartz [1981, sect.7]):

$$V(B) = \max_{X_t, T} \left( \int_0^T e^{-(\Lambda+\beta)t} U(X_t) dt + e^{-(\Lambda+\beta)T} \frac{U(\bar{X})}{\Lambda + \beta} \right) \quad (12)$$

$$\text{subject to } \dot{B}_t = r \cdot B_t + a - X_t,$$

$$B_t \geq 0 \quad \text{all } t \geq 0,$$

$$B_0 = B \quad \text{and } a \text{ given.}$$

Setting  $T = \infty$  in (12) recovers case (i), where accepting Medicaid is never optimal. Note also that formulating problem (12) in continuous time separates the choices of  $T$  and  $X_t$  and eliminates the non-convexity that would appear if the model were instead cast in discrete time. The following propositions characterize the optimal solution and establish properties necessary for phase diagram analysis.

**Proposition 3:** *Problem (12) has a unique solution,  $(B_t^*, X_t^*)$ ,  $t \geq 0$ . There exists  $T^* = T^*(B, a) \in [0, \infty)$  such that both  $B_t^*$  and  $X_t^*$  are strictly decreasing in  $t$  for  $t \leq T^*$ , but  $(B_t^*, X_t^*) = (0, \bar{X})$  for  $t > T^*$ .  $(B_t^*, X_t^*)$  is continuous in  $t$  except at  $t = T^*$ .*

Let

$$\check{X} = \lim_{t \rightarrow T^* - 0} X_t^*.$$

There is a unique  $\check{X} = \check{X}(a) \in (\bar{X}, \infty)$ , independent of  $B$ , such that

$$X_t^* = \begin{cases} \check{X} \cdot e^{\sigma(t-T^*)}, & \text{for } t \in [0, T^*] \\ \bar{X}, & \text{for } t > T^* \end{cases}.$$

**Proof:** See Appendix.

The analog of Proposition 2 to be used in further analysis is

**Proposition 4:** *Let  $T^*$ ,  $B_t^*$ , and  $X_t^*$  be as in Proposition 3. Then  $T^*(B, a)$  is strictly increasing and continuous in  $B$ ,*

$$T^*(0, a) = 0, \text{ and } \lim_{B \rightarrow \infty} T^*(B, a) = \infty.$$

As a function of  $B$ ,  $X_0^* = X_0^*(B, a)$  is continuous (except at  $B = 0$ ) and strictly increasing; we have

$$X_0^*(B, a) = \begin{cases} \text{convex in } B, & \left(1 - \frac{a}{\check{X}}\right) \left(1 - \frac{r}{\sigma}\right) > 1 \\ \text{concave in } B, & \left(1 - \frac{a}{\check{X}}\right) \left(1 - \frac{r}{\sigma}\right) < (0, 1) \end{cases}, \text{ all } B > 0,$$

and,

$$\lim_{B \rightarrow \infty} \frac{\partial X_0^*(B, a)}{\partial B} = r - \sigma > 0.$$

The optimal value function  $V(B)$  in (6) is strictly increasing and strictly concave.

**Proof:** See Appendix.

DISCUSSION A primary difference between cases (i) and (ii) is in the behavior of optimal consumption at age  $T^*$  when the liquid wealth is exhausted. Figure 3 illustrates. In case (i),  $X_t^*$  continuously approaches its long-run limit  $a$ . In case (ii), by contrast, optimal consumption jumps down at  $t = T^*$ .

The discontinuity arises in case (ii) because at age  $T^*$ , the household exchanges its annuity income flow  $a$  for a Medicaid-provided consumption flow  $\bar{X} > a$ . Consider the household's trade-offs. Its current optimal consumption at age  $T^*$  is  $\check{X}$ . Postponing Medicaid for a short time  $dt$  forfeits utility  $U(\bar{X})dt$ . The short-term gain is  $U(\check{X})dt$  less the cost of resources expended.

Annuities represent a sunk cost. The variable private cost is  $[\check{X} - a]dt$ . In utility terms, the cost is  $U'(\check{X}) \cdot [\check{X} - a] dt$ . The corresponding first-order condition is

$$U(\check{X})dt - U'(\check{X}) \cdot [\check{X} - a]dt = U(\bar{X})dt,$$

which gives an equation for  $\check{X}$  as a function of  $\bar{X}$  and  $a$ :

$$U(\check{X}) - U(\bar{X}) = U'(\check{X}) \cdot [\check{X} - a]. \quad (13)$$

Since the optimal consumption expenditure never drops below  $\bar{X} > a$ , in (13) we have  $\check{X} > a$ . Thus, expression (13) implicitly defines an increasing function  $\check{X}(a)$  to be used in construction of our phase diagrams (see Proposition 5).

The solution methodology illustrates the advantage of our formulation. The lumpiness and means test of Medicaid introduce a non-convexity in a discrete-time formulation – as in Figure 1 – making multi-period analysis complicated. Our model, by contrast, circumvents the complications by allowing the household to select the timing of its Medicaid take-up in such a way that it exactly exhausts its liquid wealth first. The discontinuity of  $X_t^*$  in Figure 3 might be considered a symptom of the non-convexity of Figure 1. Nevertheless, our solution procedure is able to rely upon first-order conditions.

## 4 High Health State Phase

Turn next to households in the healthy phase of their retirement, where  $h = H$ . Without loss of generality, rescale household ages to  $s = 0$  at the start of this phase. A household's annuity income is  $a > 0$ , and its initial bequeathable net worth is  $b \geq 0$ . With Poisson rate  $\lambda$ , the household's health state changes to  $h = L$ , and it receives (recall Section 3) the continuation value  $V(b_s)$ , where  $b_s$  is its liquid wealth at the time of the transition. Accordingly, a household in state  $h = H$  solves<sup>10</sup>

$$v(b) = \max_{x_s} \left( \int_0^\infty e^{-(\lambda+\beta)s} [u(x_s) ds + \lambda V(b_s)] ds \right) \quad (14)$$

$$\text{s.t. } \dot{b}_s = r \cdot b_s + a - x_s,$$

$$b_s \geq 0 \quad \text{all } s \geq 0,$$

$$a \geq 0 \quad \text{and} \quad b_0 = b \quad \text{given.}$$

Concavity of  $V(\cdot)$  shown in the previous section implies that the integrand in (14) is strictly concave in  $(x_s, b_s)$ . Analogous to Section 3, the effective rate of subjective discounting is  $\lambda + \beta > r$ .

Disregarding the state-variable constraint  $b_s \geq 0$  for the moment, the present-value Hamiltonian is

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<sup>10</sup>Write the expected utility as

$$v(b) = \max_{x_s} \left( \int_0^\infty \lambda e^{-\lambda S} \left[ \int_0^S e^{-\beta s} U(x_s) ds + e^{-\beta S} V(b_S) \right] dS \right)$$

and change the order of integration to obtain (14).

$$\mathcal{H} \equiv e^{-(\lambda+\beta)t} \cdot [u(x_s) + \lambda \cdot V(b_s)] + m_s \cdot [r \cdot b_s + a - x_s], \quad (15)$$

with  $m_s$  the costate variable. The first-order condition for  $x_s$  is

$$\frac{\partial \mathcal{H}}{\partial x_s} = 0 \iff e^{-(\lambda+\beta)s} \cdot u'(x_s) = m_s. \quad (16)$$

The costate equation is

$$m_s = -\frac{\partial \mathcal{H}}{\partial b_s} = -e^{-(\lambda+\beta)s} \cdot \lambda \frac{\partial V(b_s)}{\partial b_s} - r m_s. \quad (17)$$

The law of motion for liquid wealth is

$$\dot{b}_s = r \cdot b_s + a - x_s. \quad (18)$$

We construct a phase diagram for  $(b_s, x_s)$ . Let  $X_0^*(B, a)$  be the initial consumption for the household as it enters the low health state at age  $s$  with liquid wealth  $B = b_s$ . The envelope theorem shows that

$$V'(B) = U'(X_0^*(B, a)). \quad (19)$$

Eqs (15)-(19) imply

$$u''(x_s) \cdot \dot{x}_s = - (r - (\lambda + \beta)) \cdot u'(x_s) - \lambda \cdot \omega^\gamma \cdot u'(X_0^*(b_s, a)). \quad (20)$$

Eqs (18) and (20) determine the phase diagram. The isoclines of the phase diagram are

$$\dot{b} = 0 : \quad x = \Gamma_b(b) \equiv r \cdot b + a, \quad (21)$$

$$\dot{x} = 0 : \quad x = \Gamma_x(b) \equiv \theta \cdot X_0^*(b, a), \quad (22)$$

where

$$\theta \equiv \left[ \frac{\lambda + \beta - r}{\lambda \cdot \omega^\gamma} \right]^{\frac{1}{1-\gamma}} \in (0, 1). \quad (23)$$

Several distinct phase portraits can arise depending on the shape of  $\Gamma_x(b)$  and the values of exogenous parameters. We begin our analysis of phase diagrams with a lemma that allows us to limit the eventual number of cases.

**Lemma 2:**  $\Gamma_x(b)$  and  $\Gamma_b(b)$  cross at most once.

**Proof:** See Appendix.

Given Lemma 2, the phase portrait of the high health state period depends on the relative magnitudes of  $\Gamma_b(0)$  and  $\Gamma_x(0)$ , and on their asymptotic slopes  $\Gamma_b'(\infty)$  and  $\Gamma_x'(\infty)$ . Recall that Propositions 1 and 3 imply

$$\Gamma_b(0) = a, \Gamma_x(0) = \begin{cases} \theta a, & a \geq \bar{X} \\ \theta \bar{X}(a), & a < \bar{X} \end{cases}.$$

Below we show that there exists  $\bar{a} \in (0, \bar{X})$  such that

$$\Gamma_b(0) < \Gamma_x(0) \iff a < \bar{a}. \quad (24)$$

Turning to the asymptotic slopes of the isoclines, Propositions 2 and 4 and (21) show that

$$\Gamma'_b(\infty) < \Gamma'_x(\infty) \Leftrightarrow r < \theta(r - \sigma). \quad (25)$$

It can be shown that inequality (25) will hold when the interest rate is below a threshold (note that  $\theta$  in (23) is also a function of  $r$ ). Accordingly, four phase portraits are possible depending on the signs of inequalities (24) and (25). We will distinguish between the high annuity case (labelled **A**) and low annuity case (labelled **a**) based on the sign of (24). Similarly, the standard interest rate case (labelled **r**) will obtain when (25) holds, and the high interest rate case (labelled **R**) will obtain when (25) does not hold. Summarizing, we have

**Proposition 5:** *The optimal solution  $(x_s^*, b_s^*)$  to (14) is a dotted trajectory on one of the four phase diagrams on Figure 4. The phase portrait depends on the parameter values as follows:*

		High annuity	Low annuity
		$a > \bar{a}$	$a < \bar{a}$
Standard interest rate	$r < \theta(r - \sigma)$	<b>(Ar)</b>	<b>(ar)</b>
High interest rate	$r > \theta(r - \sigma)$	<b>(AR)</b>	<b>(aR)</b>

where

$$\bar{a} = \theta \cdot \bar{X} \cdot (1 - \gamma(1 - \theta))^{-\frac{1}{\gamma}}.$$

**Proof:** See Appendix.

Proposition 5 shows that household total wealth (i.e. its liquid wealth plus capitalized annuity income) is not sufficient to predict whether a household in good health will save or dissave after retirement. It is the *level* of annuity income, in fact, that plays the pivotal role:  $a$  determines which phase diagram applies regardless of the initial  $b$ . Section 5 discusses the intuitions for Figure 4.

Most of our analysis centers on the standard interest rate case (see the calibration results in Sections 6.1-6.2). The following proposition shows that for a fixed annuity income level, wealth accumulation then obtains if and only if the initial ratio of liquid wealth to annuities is below a threshold.

**Proposition 6** *Assume  $r < \theta(r - \sigma)$  and let*

$$\rho(a) = \frac{b_\infty^*(a)}{a} = \frac{1}{a} \lim_{t \rightarrow \infty} b_t^*(b, a)$$

*be the long-run optimal ratio of liquid wealth to annuities. Then*

$$\rho(a) = \begin{cases} 0, & a \leq \bar{a}, \\ \varsigma(a) & a \in (\bar{a}, \bar{X}), \\ \bar{\rho} & a \geq \bar{X}. \end{cases}$$

*where  $\varsigma'(a) > 0$ ,  $\varsigma(\bar{a}) = 0$ ,  $\varsigma(\bar{X}) = \bar{\rho}$ , and*

$$\dot{b}_t^* > 0 \Leftrightarrow \frac{b}{a} < \rho(a).$$

**Proof:** See Appendix.

Figure 5 illustrates the results of Proposition 6. It shows that  $b_\infty^*(a)$  partitions the set of initial conditions  $(b, a)$  into two regions corresponding to saving or dissaving after retirement. Directional arrows in each region show the sign of  $\dot{b}$ . To illustrate qualitative results, it is informative to show how optimal behavior depends on household total wealth. Let

$$r_A = \frac{(\lambda + r)(\Lambda + r)}{\lambda + \Lambda + r} \quad (26)$$

denote the actuarially fair rate of return used to capitalize annuity income (see the Appendix for the derivation of (26)). The dotted line on the figure depicts the locus of  $(b, a)$  that corresponds to a fixed total wealth endowment  $w$  at retirement:

$$\mathcal{S}(w) \equiv \{(a^w, b^w) : a^w \geq 0, b^w \geq 0, a^w/r_A + b^w = w\}. \quad (27)$$

We see that for any  $w > \bar{a}/r_A$ , saving behavior conditional on  $w$  is dichotomous: households with the total wealth level  $w$  save after retirement if and only if the share of annuity wealth in their initial portfolio is large enough – i.e.,  $(b, a)$  falls on the segment of the dotted line that is below  $b_\infty^*(a)$ . The next section discusses qualitative properties of the optimal solution and explains the economic intuitions behind the shape of  $b_\infty^*(a)$  on Figure 5.

## 5 Qualitative results

This section examines qualitative implications of the model and explains the key trade-offs that shape household optimal behavior. Our focus is the relative attractiveness of liquid wealth and annuities, and how incentives for post-retirement saving can emerge. Important determinants of our results are the inflexible nature of annuities, the Medicaid means test, and the interaction of the two.

Three key factors determine whether a household in good health will accumulate wealth after retirement: its annuity income level; the initial ratio of liquid wealth to annuities,  $b/a$ ; and the interest rate on bonds.

**ANNUITY INCOME AND MEDICAID MEANS TEST** One novelty in our results is that a household’s annuity income rather than its total net worth determines which phase diagram applies. A reason is the treatment of annuity income by the Medicaid means test. Roughly speaking, a household’s net benefit from Medicaid take-up is  $\bar{X} - a$ . When  $\bar{X} - a$  is high, the incentive to shift private resources toward the low health state is weak. This explains the decumulation behavior of households with  $a < \bar{a}$ . In the standard interest rate case, the corresponding phase diagram (**ar**) obtains regardless of the initial  $b$ . If a household’s good health lasts for a long time, it enjoys a period of relatively high consumption, financed from spending both the principal of, and the interest income on, its bond wealth, together with its annuity income. Ultimately, it runs out of liquid wealth; its trajectory then approaches a “corner solution” with consumption expenditure  $a$  until the onset of poor health. At the arrival of low health state, the household quickly accepts Medicaid, and its expenditure jumps to  $\bar{X}$ . If, on the other hand, the good health phase is short, the household may enter the low health state with substantial liquid wealth and delay Medicaid take-up.

**INFLEXIBLE NATURE OF ANNUITIES AND SELF-INSURANCE MOTIVE** Annuity income influences the wealth accumulation decision for reasons other than the means test. To understand the intuitions,

it is convenient to isolate the group of households with  $a \geq \bar{X}$ . They never find it optimal to use Medicaid. The saving motives of this high-annuity group derive from seeking the optimal balance of liquid wealth and annuities.

Consider a household with  $B > 0$  at the onset of the low health state. High mortality makes it optimal to choose a steeply falling consumption profile, but this is not feasible when the  $B \geq 0$  constraint binds. The household chooses to exhaust its liquid wealth in finite time  $T^*(B, a)$ , and subsequently it enters the liquidity constrained state with flat consumption at  $a$ . Recall from Proposition 2 that optimal consumption during the low health phase is proportionate to  $a$ , so  $T^*$  depends only on the ratio of  $B/a$ .

The household realizes that it can postpone the liquidity constrained state by controlling the ratio of its liquid wealth to annuities while in good health. The optimal allocation of lifetime resources across health states then requires that the household (while in good health) target a specific ratio  $b/a$ . For the high annuity group  $a \geq \bar{X}$ , this ratio is  $\bar{\rho}$ , independent of household total wealth. When a household's annuity income is fixed, the adjustment from the initial condition  $(b, a)$  to the long-run optimum  $(b_\infty^*(a), a)$  requires accumulation or decumulation of bonds.

Annuities and bonds then function as complements: the former offer longevity protection, and the latter offer flexibility to adjust expenditure timing. In line with this, Proposition 6 shows that  $b_\infty^*(a)$  is strictly increasing in  $a$  (for  $a > \bar{a}$ ).

**THE INTERACTION OF SELF-INSURANCE MOTIVE AND THE MEANS TEST** While the low-annuity group  $a < \bar{a}$  prefers public LTC insurance through Medicaid and the high-annuity group  $a \geq \bar{X}$  chooses self-insurance, the middle group, with  $a \in (\bar{a}, \bar{X})$ , chooses a mixture of the two. A middle-annuity household finds the Medicaid-provided living standard relatively unattractive, and wants to accumulate liquid wealth to postpone Medicaid take-up. But, it also wants to use Medicaid as a fall-back option for the event that its longevity turns out to be great.

For the middle group, the anticipated public benefit discourages liquid wealth accumulation relative to the top group (i.e.  $b_\infty^*(a) < \bar{\rho}a$ ). At the same time, the self-insurance motive for the middle group is more sensitive to the annuity income level:  $b_\infty^*(a)$  rises more than proportionately with  $a$  for  $a \in (\bar{a}, \bar{X})$ . The steep rise results from the interaction of the means test and the self-insurance motive: as  $a$  rises, the means test makes the Medicaid option less valuable, strengthening incentive for self-insurance.

Our results for the standard interest rate case indicate that – holding total wealth constant – *annuitization promotes self-insurance* for two reasons. First, annuities and liquid wealth play complementary roles in meeting a household's health status and longevity insurance needs. Second, annuity income reduces the net benefit from public assistance.

**THE ROLE OF THE INTEREST RATE** While it may be beneficial for a healthy household to hold liquid wealth in anticipation of the low health state, the cost of maintaining this wealth depends on the interest rate  $r$ . If  $r$  is high, liquid wealth is an attractive investment. At first, the household may desire more liquid wealth in preparation for poor health. As it adds to  $b_s$ , its interest income rises. Even if the household raises its current consumption, it can save a portion of the new income to increase its low-health-state consumption in step. In fact, on phase diagram (**AR**) saving continues as long as high health status lasts.

For the high interest rate case, the only group that dissaves after retirement consists of households whose annuity income and liquid wealth are both low:  $a < \bar{a}$  and  $b < b_\infty^*(a)$  – see phase diagram (**aR**). This yields a saving dichotomy reminiscent of Hubbard *et al.* [1995]: low-resource households decumulate wealth preemptively to take full advantage of public support, whereas high-

resource households try to delay reliance upon Medicaid.

The fact that the high interest rate stimulates saving is, by itself, not surprising. What is perhaps surprising is that in our calibrated examples (see Section 6), the threshold interest rate,  $\bar{r} = \theta(\bar{r}) \cdot (\bar{r} - \sigma)$ , separating the standard and high interest rate cases in Proposition 5, often falls in the range 0.02-0.04. According to the model, therefore, shifts in the long-run interest rate from changes in population growth, technological progress, or international capital flows might well, in practice, be sufficient to cause qualitative changes in post-retirement household behavior.<sup>11</sup>

**THE ROLE OF INCOMPLETE MARKETS** The saving behavior described above obtains when markets are incomplete; the incompleteness arises, in our model, from asymmetric information about the health state and from restrictions on borrowing against future annuity income.

In a first-best environment with symmetric information and complete insurance markets, a household would optimally rely on state-contingent annuities and insurance contracts as follows: (i) At retirement, the household would buy an annuity paying a fixed benefit stream for the duration of the high health state. (ii) The household would also buy an insurance policy paying a lump-sum benefit when the high-health state ends. (We refer to this as “long-term care insurance.”) (iii) The household would use the insurance payout to purchase a low-health-state annuity (we presume the return on the latter would reflect the household’s low-health status mortality rate  $\Lambda$ ). All financial transactions could be completed at the moment of retirement, and retirees would not demand any liquid wealth.

In our model, asymmetric information precludes contracts contingent on health status — including those of items (i)-(iii) above. If we allow households to freely purchase additional annuities — which condition upon health status at the moment they are issued — we would have the model of Reichling and Smetters [2015]. Our analysis, however, highlights the roles of Medicaid and self-insurance given an environment with a less complete set of markets.

**THE TIMING OF MEDICAID TAKE-UP** DeNardi *et al.* [2013] provides evidence that even households with relatively high annuity income sometimes use Medicaid nursing-home assistance very late in life. Households with lower  $a$ , on the other hand, tend to access Medicaid more frequently and at younger ages. Our model offers an intuitive explanation for these outcomes and can provide other insights as well.

Proposition 3 shows that any household with  $a < \bar{X}$  will access Medicaid if it survives long enough. The model determines the Medicaid take-up time as a function of a retiree’s initial conditions  $(b, a)$  and age at the onset of poor health. If  $S$  is the time spent in good health, then the optimal age of Medicaid take-up is  $S + T^*(b_S^*(b, a), a)$ . The model thus provides a mapping between wealth components at retirement, household health history,  $S$ , and Medicaid take-up age, making a comprehensive treatment possible.

**BEQUEST BEHAVIOR** Households in our model leave accidental bequests if they die before spending down their liquid wealth. Survey questions suggest that such accidental bequests may be important in practice, while evidence on intentional bequests has been more mixed.<sup>12</sup>

Our model suggests that the incidence of bequests will not be arbitrary, and that annuity income will play a role even after controlling for total wealth at retirement and health history. For example, if we take two households with identical total wealth at retirement and identical health histories, Proposition 6 shows that the household with a higher annuity income will be more likely to leave a

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<sup>11</sup>See Kopecky and Koreshkova [2014] for an example of a general equilibrium analysis of public policy on LTC.

<sup>12</sup>E.g., Altonji, Hayashi, and Kotlikoff [1992, 1997], Laitner and Ohlsson [2001], and others.

bequest.

## 6 Quantitative results

This section uses data on the cost of long-term care to calibrate the model’s exogenous parameters (see Section 6.1). The calibration allows numerical exercises that demonstrate the empirical relevance of the model and the quantitative significance of the qualitative results that our analysis derives. Sections 6.2-6.3 then turn to the retirement saving and the annuity puzzles outlined in the Introduction, and Section 6.4 examines what might happen to the public cost of the Medicaid nursing-home program if annuitization at retirement were, in the future, to be lower. For expositional convenience, we assume the standard interest rate case and  $a < \bar{X}$ , unless otherwise stated.

### 6.1 Calibration

Our model has a limited number of parameters. We set  $\lambda = 0.0833$  and  $\Lambda = 0.3333$ , corresponding to time intervals of 12 and 3 years, respectively, as in Sinclair and Smetters [2004]. The literature has a variety of estimates of  $\gamma \leq 0$  (see, for example, Laitner and Silverman [2012]) and generally uses  $\beta \in [0, 0.04]$ . We consider values  $\gamma \in [-0.5, -3.0]$ , corresponding to a coefficient of relative risk aversion  $1 - \gamma \in [1.5, 4]$ , and values  $r, \beta \in [0.02, 0.03]$ .

The model includes two parameters that are less familiar:  $\omega$ , which measures the productivity of expenditure dollars for a household in low health status, and  $\bar{X}$ , which measures the value to a recipient household of Medicaid nursing-home care. We calibrate both using information other than post-retirement wealth holdings, leaving the latter for comparisons with the model (see Section 6.2). The proposed calibration exploits the fact that Medicaid is a social-insurance program. Theoretically,  $\bar{X}$  might be thought of as a choice variable for a social planner who seeks to insure the target recipient of public long-term care. Accordingly, a comparison of with the normal expenditure of a healthy target recipient identifies the difference in marginal utility across states that would rationalize  $\bar{X}$ .

Think of the target recipient as a household that would quickly turn to Medicaid upon reaching the low health state, and let  $\bar{x}$  denote the recipient’s expenditure level while still healthy. Efficiency requires equalizing marginal utilities of expenditure across health states:

$$U'(\bar{X}) = u'(\bar{x}). \tag{28}$$

In the model, households that are quick to accept Medicaid enter the low health state, say, at age  $s$ , with nearly zero liquid wealth,  $b_s = B \simeq 0$  (see phase diagram **(ar)**). Since  $b_s \simeq 0$ , the typical recipient’s consumption just prior to  $s$  must be  $\bar{x} \simeq a$ , so that  $U'(\bar{X}) = u'(a)$  in (28). Optimality condition (28) then relates  $\bar{X}$  and  $\omega$  as follows: for any  $X$ ,

$$U'(X) = \frac{\partial u(\omega X)}{\partial X} = \omega u'(\omega X) = u'\left(\frac{X}{\Omega}\right), \text{ where } \Omega = [\omega]^{1-\gamma} > 1. \tag{29}$$

Evaluating (29) at  $X = \bar{X}$  and using (28) and  $\bar{x} = a$ ,

$$[\omega]^{1-\gamma} = \Omega = \frac{\bar{X}}{a}. \tag{30}$$

Condition (30) enables us to use data on Medicaid nursing-home reimbursement amounts and target-recipient annuity incomes to evaluate  $\Omega$  and, hence, for a specified  $\gamma$ , to calibrate  $\omega$ .

Table 1 reports components of annuitized and non-annuitized wealth for single-person households taken from Poterba *et al.* [2011, Table 2] and calculates the corresponding annuity income flow  $a$ .<sup>13</sup> We assume, as above, that a target Medicaid recipient has low initial liquid wealth and an annuity income substantially below the population median. Based on data in Table 1, we set the latter income to  $\underline{a} = 10,000$ , which is about one-half of population median, and about 2/3 of the annuity income at the population's 30th income percentile.

To estimate the effective long-term care consumption flow  $\bar{X}$ , we start with a direct measure of nursing home care cost,  $X_M$ . In Met Life [2009], annual average expenditures for nursing-home care in 2008 are \$69,715 for a semi-private room, and \$77,380 for a private room.<sup>14</sup> Accordingly, we set  $X_M = 70,000$ . Prior studies (e.g., Ameriks *et al.* [2011], Schafer [1999]) suggest that  $\bar{X}$  might be much lower than  $X_M$ . Reasons might include the disutility of living in an institution and/or accepting government welfare. Accordingly, for a fixed  $X_M = 70,000$ , let

$$\bar{X} = \xi \cdot X_M, \quad \xi \leq 1.$$

Then, setting  $a = \underline{a}$  and using (30),

$$\omega = \Omega^{\frac{1}{\gamma}-1} = \left[ \xi \frac{X_M}{\underline{a}} \right]^{\frac{1}{\gamma}-1}.$$

We report results for  $\xi \in \{0.5, 0.75, 1\}$ , which imply  $\bar{X} \in \{35000, 52500, 70000\}$ .

**CALIBRATION CONSISTENCY CHECK** Because the choice of  $\bar{X}$  can markedly affect our numerical results, we digress to compare our  $\bar{X}$  with recent estimates in the literature.

DeNardi *et al.* [2010] model separate out-of-pocket medical expenditure, say,  $x^P$ , from total expenditure  $x$ , and base household utility on  $x - x^P$ . They model a government-provided consumption floor that applies in good health as well as bad. Because of differences in the treatments of state-dependent utility, comparisons of utility in good health are easiest. The floor in our model is directly tied to Medicaid nursing-home care; hence, it only applies when  $h = L$ . Nonetheless, condition (30) enables us to value it in terms of the consumption expenditure of a healthy household. De Nardi *et al.* estimate a consumption floor for  $x - x^P$  of about 3400 (2008 dollars), and an average out-of-pocket medical expenditure of about 4600 (2008 dollars). Thus, the consumption floor in the De Nardi *et al.* analysis that should roughly correspond to our  $\underline{a} = 10000$  is 8000.

Ameriks *et al.* [2011] have 3 states: good health, moderately good, and nursing home eligible. The first two share a consumption floor of 8400 (2008 dollars). As above, this can be compared to our  $\underline{a} = 10000$ . Moreover, in Ameriks *et al.* [2011], the nursing-home eligible state has its own floor, in which out-of-pocket medical and non-medical consumption total about 56300 (2008 dollars). We might compare that directly to our  $\bar{X}$ .

In the end, despite differences in the models, the consumption floors seem similar, especially if we select  $\bar{X} = 52,500$ .

<sup>13</sup>Poterba *et al.* [2011] use the actuarially fair rate of return on annuities to capitalize annuity flows. Consistent with this, we use the actuarially fair rate of return  $r_A$  from (26) to convert between annuity wealth and income flow.

<sup>14</sup>MetLife Mature Market Institute, "Market Survey of Long-Term Care Costs" 2009.

NUMERICAL RESULTS Table 2 provides calculations that illustrate Proposition 5 and the qualitative results of Section 5. Each panel of the table corresponds to a distinct vector of exogenous parameters  $(r, \beta, \Omega, \bar{X})$  consistent with (30) and reports the values of  $\bar{a}$  and  $b_\infty^*$  for a set  $\gamma \in \{-0.5, -1, -2, -3\}$ .

We can see that all four phase diagrams of Figure 4 obtain for empirically relevant parameter values. In the standard interest rate case, we can also see that phase diagram **(ar)**, corresponding to wealth decumulation in good health, always emerges for households in the bottom 30% of the distribution of net worth from Table 1. In contrast, households in the upper 50% always have diagram **(Ar)**.

In the case of phase diagram **(Ar)**, comparing initial wealth  $b$  in Table 1 with  $b_\infty^*$  in Table 2 we can see that either outcome  $b < (>) b_\infty^*$  can occur. Table 2 also shows that  $b_\infty^*(a)$  tends to rise rapidly (recall Proposition 6), and that  $b_\infty^*(a)$  can be very high relative to initial wealth  $b$  from Table 1.

Finally, the threshold  $\bar{r}$  is evidently highly dependent upon both  $1 - \gamma$  and  $\bar{X}$ , with high values of either making a low  $\bar{r}$  possible.

## 6.2 Retirement saving Puzzle

Although the standard life-cycle model implies that households should systematically dissave after retirement, survey data often reveals stationary or rising cohort average bequeathable wealth. The clearest evidence on post-retirement wealth accumulation behavior comes from panel data (which is not affected by mortality selection) — see, for example, Poterba *et al.* [2012, Fig. 1.1-1.2].<sup>15,16</sup> This section examines our model’s implications for cohort age-wealth trajectories.

The trajectory for the cohort average wealth depends on the saving behavior of households with  $h = L$  and  $h = H$ , as well as on the time-varying cohort composition by health state. Section 3 shows that all households in low health state should dissave; Section 4 shows that initial conditions  $(b, a)$  determine whether a household will save or dissave while in good health. A cohort wealth trajectory aggregates the behavior of individual households, using the stochastic processes for health and mortality.

COHORT COMPOSITION Consider a cohort of retirees. By assumption, all households begin retirement with  $h = H$ . Each household subsequently transits to  $h = L$ , then to death. If the  $h = L$  group became an ever larger fraction of the surviving total, average wealth in the cohort would inevitably begin a permanent decline. The high mortality within the low health group prevents this outcome, however: as the healthy group continuously loses members to low health state, the low health group simultaneously loses members to mortality.

Let  $f_H(t)$  be the fraction of households remaining alive and in good health  $t$  years after retirement. Then

$$f_H(t) = e^{-\lambda t}.$$

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<sup>15</sup>Poterba *et al.* [2012] study single households and couples separately. They link average bequeathable wealth holdings in adjoining survey waves, including only households with data in both waves. Their procedure should be immune to birth-cohort effects and to correlations of survival probabilities with portfolio size. For either means or medians, their figures show stationary or rising-with-age trajectories of bequeathable wealth for both singles and couples. See also Love *et al.* [2009] and Smith *et al.* [2009].

<sup>16</sup>The figures in Poterba *et al.* [2011] also seem to imply that when a spouse dies, a household’s wealth drops noticeably. That seems generally consistent with our model’s assumption of substantially higher consumption expenditure needs during poor health.

Similarly, the fraction alive at  $t$  but in low health state is

$$f_L(t) \equiv \int_0^t \lambda \cdot e^{-\lambda \cdot s} \cdot e^{-\Lambda(t-s)} ds.$$

Combining expressions, the fraction of survivors in high health state is

$$f(t) \equiv \frac{f_H(t)}{f_H(t) + f_L(t)} = \frac{1}{1 + \frac{\lambda}{\Lambda - \lambda} \cdot (1 - e^{-(\Lambda - \lambda) \cdot t})}.$$

We can see that  $f(0) = 1.0$  and that  $f(t)$  monotonically falls to  $f(\infty) = (\Lambda - \lambda)/\Lambda$ . With  $\lambda$  and  $\Lambda$  as in Section 6.1,  $f(\infty) = 0.75$ . The convergence of  $f(t)$  to  $f(\infty)$  is rapid:  $f(t)$  reaches 0.76 within 11 years of retirement. As explained above, the convergence of  $f(t)$  is important: as cohort composition stabilizes, the impact of healthy households on the average does not disappear in the limit.

**COHORT AVERAGE WEALTH** Consider a cohort of single-person households beginning retirement in the high health state. For simplicity, assume that all households start with the same initial condition  $(b, a)$ , which may be set from Table 1 data. We focus on liquid wealth — in the baseline model the capitalized value of a household’s primary annuities is unchanging.

Normalize the cohort size to 1. Using the notation of Sections 3-4, a household remaining in high health status  $t$  periods after retirement has liquid wealth  $b_t^* = b_t^*(b, a)$ . The total wealth of agents who remain healthy  $t$  periods into retirement is

$$b_H(t; b, a) = e^{-\lambda t} \cdot b_t^*(b, a).$$

The wealth of survivors in the low health state depends on the age when their health changed. If a household enters the low health state  $s \leq t$  periods after retirement, its initial wealth upon entering that state is  $B = b_s^*(b, a)$ . The household subsequently follows the low-health-state optimal wealth trajectory. At time  $t$ , it has passed  $t - s$  years in the low health state, and its wealth is  $B_{t-s}^*(B, a)$ . The fraction of households entering the low health state at age  $s$  and surviving until age  $t$  is  $\lambda e^{-\lambda s} e^{-\Lambda(t-s)}$ . Accordingly, the total wealth of agents who are in low health  $t$  periods into retirement is

$$b_L(t; b, a) = \int_0^t \lambda e^{-\lambda s} e^{-\Lambda(t-s)} B_{t-s}^*(b_s^*(b, a), a) ds. \quad (31)$$

The cohort average wealth level is the total wealth divided by the total number of survivors:

$$\bar{b}(t; b, a) = \frac{b_H(t; b, a) + b_L(t; b, a)}{f_H(t) + f_L(t)}. \quad (32)$$

It turns out that the asymptotic behavior of cohort average wealth,  $\bar{b}(t; b, a)$ , can be characterized analytically. For the case we consider in simulations, we have

**Proposition 7** *Assume phase diagram (Ar) and  $b \leq b_\infty^*(a)$ . Then*

$$\lim_{t \rightarrow \infty} \bar{b}(t; b, a) \in b_\infty^*(a) \left[ \frac{\Lambda - \lambda}{\Lambda}, 1 \right].$$

**Proof:** See Appendix.

The proof depends upon the properties of  $f(t)$  above and the shape of phase diagram **(Ar)**.

High annuity income is essential for post-retirement saving. If the endowment were instead  $(b, a)$  with  $a < \bar{a}$ , we would have dissaving during both  $h = H$  and  $L$ . Thus,  $\bar{b}(t; b, a)$  would steadily decline to 0 as  $t$  increases.

Middle-class households having  $b < b_\infty^*(a)$  in phase diagram **(Ar)**, in contrast, choose to accumulate bonds during good health. If  $b_\infty^*(a) \cdot (\Lambda - \lambda) / \Lambda > b$ , Proposition 7 then shows that  $\bar{b}(t; b, a) - b$  is asymptotically positive. Time-varying cohort composition can complicate the trajectory of cohort-average wealth early in retirement, but cohort composition soon stabilizes, after which the behavior of households in good health begins to dominate, and the retirement-saving-puzzle outcome emerges.

The next section shows that the middle-class group behaving according to Proposition 7 can be large relative to the low annuity group in practice.

**SIMULATED COHORT WEALTH TRAJECTORIES** For simulations, we set  $(b, a) = (100, 21)$ , which corresponds to the population median in Table 1. We assume that all retirees in the cohort start from the same initial condition.

Figure 6 plots average wealth relative to its initial level,  $\bar{b}(t)/\bar{b}(0)$ . We set  $\gamma = \{-0.75, -1.0, -2.0\}$ , and we take other parameters from Table 2, Panel 5. Table 2 shows that  $b < b_\infty^*(a)$  in each case. When  $\gamma = \{-0.75, -1\}$ , rapid initial changes in cohort composition make average wealth decline with age at first. As cohort composition stabilizes, the two wealth trajectories in Figure 6 start rising.

The plot for  $\gamma = -2$  illustrates the effect of the phase diagram switch on the cohort wealth trajectory: when  $\gamma$  changes from  $-1$  to  $-2$ , the phase diagram type switches from **(Ar)** to **(AR)** (see Panel 5 of Table 2). This pushes up the saving rate of healthy households; now the wealth accumulation of the healthy becomes a dominant driver of the cohort average, and the mean wealth never falls below  $\bar{b}(0) = b$ . The simulation with  $\gamma = -1.0$  provides the closest match to the empirical wealth trajectories in Poterba *et al.*

The assortment of wealth trajectories on Figure 6 seems to encompass observed cohort wealth accumulation rates; thus, our model offers a resolution of the retirement-saving puzzle.

### 6.3 Annuity puzzle

Standard life-cycle theories have been hard to reconcile with households' apparent lack of demand for complete, or nearly complete, annuitization at retirement — i.e., the “annuity puzzle.” In order to study this issue, this section examines what happens if we allow households to reset the composition of their portfolio at retirement. We find that Medicaid can dramatically lower the demand for annuities.

Let  $w$  denote a household's total wealth level, and let

$$\alpha = \frac{a/r_A}{w} = \frac{a}{a + r_A b}$$

denote its share of annuitized wealth at retirement. This section allows households to choose their most preferred initial annuity share  $\hat{\alpha}$ :

$$\hat{\alpha} = \arg \max_{\alpha \in [0,1]} v((1 - \alpha)w, \alpha w). \tag{33}$$

If  $\hat{\alpha}$  exceeds the endowment  $\alpha$ , a household will exhibit demand for annuities at retirement.

We use Figure 7 to illustrate our analysis. Figure 7 compares the initial portfolios at retirement from Table 1 to optimal portfolios from (33). Each panel on the Figure corresponds to a separate wealth endowment – i.e., the 30th, 50th and 70th percentiles of the Table 1 retirement wealth distribution. The total wealth for group  $p = \{30, 50, 70\}$  is  $w^p = a^p/r_A + b^p = \{177, 328, 641\}$  (000s of 2008 dollars). The lower (dashed) curves correspond to expected utility at retirement with no Medicaid. Thus, the dashed curve on panel  $p$  shows  $v((1 - \alpha)w^p, \alpha w^p)$  as a function of  $\alpha$ , setting  $\bar{X} = 0$ . Similarly, the solid curves show  $v(\cdot)$  as function of  $\alpha$ , for  $\bar{X} = 52.5$ . There are two markers on each curve: the circle depicts the initial portfolio for group  $p$ , and the diamond shows the optimal portfolio.

**DEMAND FOR ANNUITIES WITHOUT MEDICAID** For the sake of comparison, consider the case without Medicaid first, i.e., the special case with  $\bar{X} = 0$ . Proposition 5 shows that  $\bar{X} = 0$  implies  $a > \bar{a} = 0$ ; hence, phase diagram **(Ar)** applies. Without Medicaid, the model is homothetic in  $(b, a)$ , and the optimal share of annuitized wealth at retirement,  $\hat{\alpha}$ , is independent of household total wealth. Accordingly, the dashed solid dots on each panel have the same abscissa,  $\hat{\alpha} = 0.94$ . Then households prefer almost full annuitization.<sup>17</sup> Evidently, absent Medicaid, the model exhibits the annuity puzzle: on all of the three dashed curves, the diamonds are to the right of the circles (endowment).

**DEMAND FOR ANNUITIES WITH MEDICAID** Qualitatively, the presence of a Medicaid option lowers annuity demand for all households for two reasons. First, Medicaid provides longevity protection and, as such, it substitutes for primary annuities. Second, the means test essentially confiscates annuity income, making annuities a less attractive form of wealth.

For the  $p = 30$  group, Medicaid also reduces the incentive to hold bonds. At the circle on the dashed curve, we have noted that phase diagram **(Ar)** applies — retirees use bonds to help offset the high marginal utility of low health status. At the corresponding circle on the solid curve, on the other hand, Table 2 verifies that  $a < \bar{a}$ ; so,  $p = 30$  households in poor health are happy to substitute Medicaid for private spending. With a reduced incentive to hold either annuities or bonds, the net change in  $\hat{\alpha}$  as we move to the solid curve can be small. In Figure 7, the desired share of annuities falls only from 0.94 to 0.86.

For the  $p = 50$  and  $p = 70$  groups, phase diagram **(Ar)** applies at circle markers on both curves (see Table 2). Hence, we cannot tell whether the demand for bonds at retirement falls or rises as we add Medicaid to the model. When  $p = 30$ , Medicaid diminishes incentives to hold both annuities and bonds, so the incentive changes counterbalanced one another. For  $p = 50$  or 70, only the reduced incentive to hold annuities is unambiguously powerful. In the figure, the effect on annuity demand is dramatic: in shifting from the dashed to the solid curve,  $\hat{\alpha}$  falls from 0.94 to 0.46 (0.51) for the  $p = 50$  ( $p = 70$ ) group.

**SUMMARY** Outcomes on Figure 7 illustrate that for households with  $a < \bar{X}$ , the annuity puzzle can be resolved by introducing Medicaid nursing-home care. Specifically, in each panel of Figure 7, the diamond lies to the left of the circle on each solid curve. For  $p = 30$  households, high initial annuitization, in practice, plays a major role in the outcome. At the median or upper quartile, on the other hand, the model shows that Medicaid can have a potent effect on household incentives.

<sup>17</sup>Our framework differs from Yaari [1965] in that mortality hazard is correlated with (state-dependent) marginal utility, and this explains why households desire less than 100 % annuitization. However, the deviation from full annuitization is slight — an outcome that is reminiscent of other recent analyses, e.g. Davidoff *et al.* [2005].

## 6.4 Public spending on Medicaid

Returning to our baseline specification with exogenous initial portfolio composition, this section analyzes the influence of annuity endowments on the demand for Medicaid. If present trends continue, annuitization at retirement may be lower in the future due to a decrease in prevalence of DB pensions, and our model suggests that the cost of the Medicaid program may then rise.

To illustrate, consider a household with net worth  $w$  at retirement and compare its behavior under two different feasible allocations of initial wealth,  $(b, a) \in S(w)$  and  $(b', a') \in S(w)$  with  $a' < a$  and  $b' > b$ .

The model identifies two potentially important channels through which the cost of the Medicaid program might be affected when a household starts with the less annuitized portfolio. The direct effect is that the government's (net) annual cost of Medicaid care per enrollee would rise from  $X_M - a$  to  $X_M - a'$ . There is an additional, indirect, channel, as well: a household's net flow benefit from accepting a year of Medicaid assistance would rise from  $\bar{X} - a$  to  $\bar{X} - a'$ , increasing incentives for Medicaid participation.

Suppose first that  $a' < a < \bar{a}$ . Section 5 shows that a low annuity household is virtually at the "corner solution" where prompt acceptance of Medicaid in the low health state is a non-marginal choice. Accordingly, the indirect channel will be inoperative. By contrast, consider a household with  $a > \bar{a}$ . Section 5 shows that reduced annuitization will weaken the incentive to self-insure. Consequently, the indirect channel will operate, and it will add to increased governmental cost of the Medicaid program.

To measure the magnitude of the demand change, it is convenient to build on the example of the previous section. Figure 8 considers the same three groups of households,  $p = \{30, 70, 50\}$ , and for each group, depicts the expected present value (at retirement) of future government outlays on Medicaid as a function of the initial annuitized wealth share  $\alpha$ . On each line, the circle is the initial portfolio from Table 1, and the diamond is the optimal portfolio from (33) that we take to represent  $a' < a$ .

For the  $p = 30$  household, moving from the initial portfolio to the optimal portfolio makes the Medicaid cost negligibly decline. For households with  $p = 50$  and  $p = 70$ , the corresponding cost rises by roughly \$7000 and \$3000, respectively. As noted, the direct channel operates in all cases, but the additional, indirect, channel only operates for  $p = 50$  and  $p = 70$ . Evidently, the indirect channel is more potent.

In sum, the model suggests that for the middle-class, lower primary annuitization can lead to substantially higher demand for Medicaid. If primary annuitization were to decrease generally, then demand for Medicaid might well rise.

## 7 Conclusion

This paper presents a life-cycle model of post-retirement household behavior emphasizing the roles of changing health status (correlated with changes in mortality), annuitized wealth, and Medicaid assistance with long-term care. Despite the presence of health-status uncertainty and the non-convexity introduced by the Medicaid means test, our analysis yields a standard optimal control problem where the solution can be characterized with a set of phase diagrams.

Qualitatively (and quantitatively in calibrated examples), we show the model is consistent with

the gently rising cohort wealth trajectories after retirement that tend to appear in data. Similarly, we show that a sizeable fraction of households may not wish to buy additional annuities at retirement — with both Medicaid LTC and existing primary annuitization from Social Security and DB pensions playing important roles in the outcome. The model can, in other words, offer explanations for two long-standing empirical puzzles, the “retirement-saving puzzle” and the “annuity puzzle.”

The model shows that after retirement but while in good health, middle-class households may want to maintain, or continue to build, their non-annuitized net worth. Households value primary annuities for the income that they provide, bonds for flexibility of access to funds, and Medicaid LTC for backstop protection against extreme longevity (in the low health state). Primary annuities and bonds can assume complementary roles: middle-class households with high primary-annuity income may, during good health, save part of it to (temporarily) support a higher living standard in poor health than Medicaid nursing-home care provides. In the model, this behavior can be understood to be a consequence of state-dependent utility and incomplete financial and insurance markets.

The model establishes a two-way link between primary annuitization and public expenditures on Medicaid. The two are related both directly, as annuity income affects the net public cost of Medicaid, and indirectly, through the dependence of incentives for self-insurance on annuitization. In the future, the latter incentives may weaken if, for instance, DB pensions become less prevalent. The analysis indicates, moreover, that increases in demand for Medicaid may be even stronger if interest rates remain low.

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## Appendix

**Proof of Lemma 1.** Suppose  $(B_t, X_t) = (0, a)$  all  $t$ . Consider Hamiltonian (7). The first-order condition for  $X_t$  yields

$$e^{-(\Lambda+\beta)t} \cdot U'(a) = M_t. \quad (34)$$

The costate equation yields, after substituting from (34),

$$\begin{aligned} \dot{M}_t &= -rM_t - N_t \Leftrightarrow \\ -(\Lambda + \beta)e^{-(\Lambda+\beta)t} \cdot U'(a) &= -r \cdot e^{-(\Lambda+\beta)t} \cdot U'(a) - N_t \Leftrightarrow \\ N_t &= -[r - (\Lambda + \beta)]e^{-(\Lambda+\beta)t} \cdot U'(a) \end{aligned}$$

By assumption,  $-[r - (\Lambda + \beta)] > 0$ . So,  $N_t \geq 0$ . And, the time-path of the Lagrange multiplier is continuous.  $B_t = 0$  all  $t$  in this lemma. Hence,  $N_t \cdot B_t = 0$ . Similarly, we can see that transversality condition (8) holds. ■

**Proof of Proposition 1** Refer to Hamiltonian (7). Let  $(B_t^*, X_t^*)$  be the trajectory that converges to  $(0, a)$  from above. Equation (11) shows the vertical motion in Figure 2 is strictly negative. Let  $T^* < \infty$  be the time  $(B_t^*, X_t^*)$  reaches  $(0, a)$ . For  $t \leq T^*$ , the budget constraint of (6) together with (11) determine the shape of  $(B_t^*, X_t^*)$ ; (9) determines  $M_t$ . Set  $N_t = 0$ .

For  $t > T^*$ , set  $N_t, M_t, X_t^*$ , and  $B_t^*$  as in the proof of Lemma 1. Then the first-order condition for  $X_t$ , the costate equation, the budget equation, and the state-variable constraint all hold for  $t \geq 0$ ; we have  $N_t \geq 0$  all  $t$ ; the path of  $N_t$  is piecewise continuous;  $N_t \cdot B_t = 0$  all  $t$  by construction; the costate variable is non-negative all  $t$  and continuous by construction; and, transversality condition (8) holds. Hence,  $(B_t^*, X_t^*)$  is optimal. ■

**Proof of Proposition 2** Expression (11) shows

$$X_{T^*}^* = X_0^* \cdot e^{\sigma T^*}.$$

By construction,  $X_{T^*}^* = a$ . So,

$$X_0^* = a \cdot e^{-\sigma \cdot T^*}. \quad (35)$$

Budget accounting then implies

$$B = \int_0^{T^*} e^{-rt} (a \cdot e^{-\sigma(T^*-t)} - a) dt, \quad (36)$$

which determines  $T^* = T^*(B, a)$ . From (36), we can see that  $T^*(B, a)$  is a strictly increasing and continuous function of  $B$ , with

$$\lim_{B \rightarrow \infty} T^*(B, a) = \infty, \quad (37)$$

and

$$T^*(0, a) = 0. \quad (38)$$

Turning to the properties of  $X_0^*(B, a)$ , we can then see from (35) that  $X_0^*(B, a)$  is continuous and strictly increasing in  $B$ ; (38) implies  $X_0^*(0, a) = a$ .

Differentiating (36) with respect to  $B$  gives  $\frac{\partial T^*}{\partial B}$ :

$$1 = \int_0^{T^*} a \cdot e^{-rt} \cdot (-\sigma) \cdot e^{-\sigma(T^*-t)} \cdot \frac{\partial T^*}{\partial B} \cdot dt \Leftrightarrow$$

$$\frac{\partial T^*}{\partial B} = \frac{1}{-a \cdot \sigma \cdot e^{-\sigma T^*} \cdot \int_0^{T^*} e^{(\sigma-r)t} dt}.$$

Differentiating (35),

$$\frac{\partial X_0^*}{\partial B} = -\sigma \cdot a \cdot e^{-\sigma \cdot T^*} \cdot \frac{\partial T^*}{\partial B}.$$

Combining the last two expressions,

$$\frac{\partial X_0^*}{\partial B} = \frac{1}{\int_0^{T^*} e^{(\sigma-r)t} dt}. \quad (39)$$

Since  $T^*(B, a)$  is increasing in  $B$ , (39) implies  $X_0^*$  is concave in  $B$ . Given (37), (39) also establishes

$$\lim_{B \rightarrow \infty} \frac{\partial X_0^*(B, a)}{\partial B} = r - \sigma.$$

Finally, the envelope theorem shows

$$V'(B) = U'(X_0^*(B, a)).$$

Hence,  $V(B)$  is continuously differentiable; and, because  $X_0^*$  is strictly increasing in  $B$ ,  $V(B)$  is strictly concave. ■

### Proof of Proposition 3

Step 1. Fix  $a$  and  $\bar{X}$ . In case (ii), we have  $a < \bar{X}$ . Define a function

$$\pi(X) \equiv U(X) - U(\bar{X}) + U'(X) \cdot (a - X), \quad \text{all } X > a. \quad (40)$$

This function is continuous and strictly increasing in  $X$ , and it has opposite signs at the ends of the interval  $[\bar{X}, \infty)$ :

$$\pi'(X) = U''(X) \cdot (a - X) > 0,$$

$$\pi(\bar{X}) = U'(\bar{X}) \cdot (a - \bar{X}) < 0,$$

$$\lim_{X \rightarrow \infty} \pi(X) = -U(\bar{X}) + \lim_{X \rightarrow \infty} \left( U(X) + \gamma \cdot U(X) \cdot \frac{a - X}{X} \right) = -U(\bar{X}) > 0.$$

It follows that on  $(\bar{X}, \infty)$ ,  $\pi(X)$  has a unique root. Denote this root  $\check{X}$ .

Step 2. Optimality requires that once  $B_t = 0$ , the household permanently accepts Medicaid. Prior to that, the Hamiltonian is (7) with  $N_t = 0$ . The first-order condition, costate equation, and budget equation are as in case (i). Hence, the phase diagram is as in Figure 2. For any initial  $B \geq 0$ , choose the trajectory at the top of the diagram that converges to  $(0, \check{X})$ . As in case (i), convergence takes a finite time (which we denote  $T^*$ ). Assume Medicaid take-up for  $t > T^*$ , with  $X_t^* = \bar{X}$ .

The concavity of the problem and the discussion in the text show that we know optimal behavior conditional on  $T^*$ . Let

$$W(T) = e^{-(\Lambda+\beta)T} \frac{U(\bar{X})}{\Lambda + \beta}$$

denote the continuation value of accepting Medicaid at date  $T$ . Kamien and Schwartz [1981, p. 143] show that the first-order conditions for the optimal acceptance date  $T^*$  are

$$B_{T^*} \geq 0, \quad M_{T^*} \geq \frac{\partial W(T^*)}{\partial B_{T^*}} \geq 0, \quad B_{T^*} \cdot \left[ M_{T^*} - \frac{\partial W(T^*)}{\partial B_{T^*}} \right] = 0, \quad (41)$$

$$\mathcal{H}_{t=T^*} + \frac{\partial W(T^*)}{\partial T} = 0, \quad (42)$$

where we use the Hamiltonian from (7) without the state-variable constraint. Our proposed solution has

$$B_{T^*} = 0. \quad (43)$$

From (9),  $M_{T^*} > 0$ .  $W(T)$  is not a function of  $B_T$ , making its partial derivative 0. Hence, our proposed solution is consistent with (41). Evaluating (42) at  $T = T^*$  yields

$$\pi(X_{T^*}) \cdot e^{-(\Lambda+\beta)T^*} = 0.$$

Hence, Step 1 establishes (42).

By construction, we have  $\check{X} = \lim_{t \rightarrow T^*-0} X_t^*$  and

$$X_t^* = \begin{cases} \check{X} \cdot e^{\sigma(t-T^*)}, & \text{for } t \in [0, T^*] \\ \bar{X}, & \text{for } t > T^* \end{cases}.$$

It remains to show that the first-order condition for  $T^*$  is sufficient. We have argued that the root of  $\pi(\cdot)$  is unique. Suppose we choose a larger (smaller)  $T^*$ . The trajectories of Figure 2 remain as before. Thus budgetary accounting implies we must lower (raise)  $\check{X}$  for our stationary point accordingly, leading to  $\pi(\check{X}) < (>)0$ . Hence, the right-hand side of first-order condition (42) yields a maximum at our original  $T^*$ . ■

**Proof of Proposition 4.** The proof follows that of Proposition 2. We concentrate here on the convexity/concavity of  $X_0^*(B, a)$  and its asymptotic behavior.

As in the proof of Proposition 2,  $X_0^* = X_{T^*}^* \cdot e^{-\sigma T^*}$ . In this case,  $X_{T^*}^* = \check{X}$ . So,

$$X_0^* = \check{X} \cdot e^{-\sigma T^*} \quad (44)$$

and

$$\frac{\partial X_0^*}{\partial B} = -\sigma \cdot \check{X} \cdot e^{-\sigma T^*} \cdot \frac{\partial T^*}{\partial B}. \quad (45)$$

Budgetary accounting implies

$$B = \int_0^{T^*(B,a)} e^{-rt} \cdot (X_0^*(B, a) e^{\sigma t} - a) dt.$$

Differentiating the above with respect to  $B$  yields

$$1 = e^{-rT^*} (X_0^* \cdot e^{\sigma T^*} - a) \frac{\partial T^*}{\partial B} + \frac{\partial X_0^*}{\partial B} J(T^*), \text{ where } J(T) \equiv \int_0^T e^{-(r-\sigma)t} dt \quad (46)$$

Substituting from (44)-(45) into (46), we have

$$\frac{\partial X_0^*}{\partial B} = \frac{1}{D(T^*)} \quad \text{where} \quad D(T^*) \equiv -\frac{1}{\sigma} \cdot \frac{\check{X} - a}{\check{X}} \cdot e^{-(r-\sigma)T^*} + J(T^*).$$

The asymptotic behavior of  $\partial X_0^*/\partial B$  follows:

$$\lim_{B \rightarrow \infty} D(T^*(B, a)) = \lim_{T^* \rightarrow \infty} D(T^*) = \lim_{T^* \rightarrow \infty} J(T^*) = \frac{1}{r - \sigma}.$$

The convexity or concavity of  $X_0^*(B, a)$  follows as well:

$$D'(T^*) = \frac{r - \sigma}{\sigma} \cdot \frac{\check{X} - a}{\check{X}} \cdot e^{-(r-\sigma)T^*} + e^{-(r-\sigma)T^*} = \left[ 1 - \left( 1 - \frac{a}{\check{X}} \right) \left( 1 - \frac{r}{\sigma} \right) \right] \cdot e^{-(r-\sigma)T^*}.$$

■

**Proof of Lemma 2.** Suppose  $(b^*, x^*)$  is a solution to (21)-(22) for a fixed  $a$  and  $\bar{X}$ .

Step 1. Suppose  $a \geq \bar{X}$ .

Proposition 4 and (22) imply

$$x^* = \theta \cdot X_0^*(b, a) = \theta \cdot \check{X} \cdot e^{-\sigma T}$$

where  $T = T^*(b^*, a)$ . Let  $Z = \check{X}/a$ . Then the equation for  $b^*$  reads

$$\theta a Z e^{-\sigma T} = r b^* + a \iff \frac{b^*}{a} = \frac{1}{r} [\theta Z e^{-\sigma T} - 1]. \quad (47)$$

As in the proof of Proposition 4, budgetary accounting yields

$$\begin{aligned} b^* &= \int_0^T e^{-rt} (a Z e^{\sigma t} e^{-\sigma T} - a) dt \iff \\ \frac{b^*}{a} &= Z \frac{e^{-\sigma T} - e^{-rT}}{r - \sigma} - \frac{1 - e^{-rT}}{r} \end{aligned} \quad (48)$$

Equating  $\frac{b^*}{a}$  in (47)-(48), we have

$$\begin{aligned} \frac{1}{r} (\theta Z e^{-\sigma T} - 1) &= Z \frac{e^{-\sigma T} - e^{-rT}}{r - \sigma} - \frac{1 - e^{-rT}}{r} \iff \\ e^{-rT} \left( \frac{Z}{r - \sigma} - \frac{1}{r} \right) &= Z e^{-\sigma T} \left( \frac{1}{r - \sigma} - \frac{\theta}{r} \right) \iff \\ e^{(r-\sigma)T} Z \left( \frac{1}{r - \sigma} - \frac{\theta}{r} \right) &= \left( \frac{Z}{r - \sigma} - \frac{1}{r} \right) \end{aligned} \quad (49)$$

The last expression depends on  $b^*$  only through  $T$ . (49) either has a unique solution  $T > 0$  or no solution. If  $T > 0$  exists, Proposition 4 shows that  $T = T^*(b, a)$  is strictly increasing in  $b$ ; hence  $b^*$  must be unique if  $T$  is unique.

Step 2. If  $a < \bar{X}$ , repeat Step 1 argument setting  $Z = 1$  – recall Proposition 2. ■

**Proof of Proposition 5**

Step 1 Suppose  $a \geq \bar{X}$ . Then

$$a \geq \bar{X} > \theta \cdot \bar{X} \cdot (1 - \gamma(1 - \theta))^{-\frac{1}{\gamma}} = \bar{a},$$

and

$$\Gamma_b(0) = a > \theta X^*(0, a) = \theta a = \Gamma_x(0),$$

so we have left-hand side diagrams on Figure 4. The asymptotic slope from Proposition 2 establishes the cases that obtain for  $r < (>) \theta \cdot (r - \sigma)$ .

Step 2 Suppose  $a < \bar{X}$ . We show that there exists a unique  $\bar{a} \in (0, \bar{X})$  such that

$$\Gamma_b(0) = a < \theta \check{X}(a) = \Gamma_x(0) \Leftrightarrow a < \bar{a}.$$

Consider  $\pi(\cdot)$  from (40) and make a change of variables

$$\check{X}(a) = aZ(a). \tag{50}$$

Since  $\pi(\check{X}(a)) = 0$  implies that  $\check{X}(a) > a$ , we have  $Z(a) > 1$ . Using (2) and (50), equation  $\pi = 0$  can be written as

$$(1 - \gamma) Z^\gamma + \gamma Z^{\gamma-1} = \left(\frac{a}{\bar{X}}\right)^{-\gamma} \tag{51}$$

The left-hand side of (51) is strictly decreasing in  $Z$  for all  $Z \geq 1$ , with

$$Z(\bar{X}) = 1, \lim_{a \rightarrow 0} Z(a) = \infty \text{ and } Z'(a) < 0.$$

Hence, there is a unique  $\bar{a} \in (0, \bar{X})$  with

$$\bar{a} = \theta \check{X}(\bar{a}) \Leftrightarrow Z(\bar{a}) = \frac{1}{\theta} \tag{52}$$

Evaluating (51) at  $Z = 1/\theta$  and  $a = \bar{a}$  gives

$$\bar{a} = \bar{X} \cdot \theta (1 - \gamma(1 - \theta))^{-\frac{1}{\gamma}}.$$

Since  $Z(a)$  is strictly decreasing,

$$a < \theta \check{X}(a) \iff \frac{1}{\theta} < Z(a) \iff a < \bar{a}.$$

Step 3 Step 2 shows that  $a > (<) \bar{a}$  separates the left and right-hand side diagrams in Figure 4. Lemma 2 and the asymptotic slopes in Proposition 4 complete the proof. ■

**Proof of Proposition 6** Suppose  $\theta \cdot (r - \sigma) > r$ . Let  $\check{X}(a) = aZ(a)$  as in (50). Define  $Z(a)$  for all  $a \geq \bar{a}$  as follows:

$$Z(a) = \begin{cases} \frac{1}{a} \check{X}(a), & \text{if } a \in [\bar{a}, \bar{X}) \\ 1, & \text{if } a \geq \bar{X} \end{cases}.$$

Then Proposition 5 shows  $Z(a)$  is continuous for all  $a \geq \bar{a}$  and strictly decreasing for  $a \in [\bar{a}, \bar{X})$ . From (52) we have

$$1 \leq Z(a) \leq \frac{1}{\theta}. \quad (53)$$

In the proof of Lemma 2, (47) shows

$$\frac{b^*}{a} = \frac{\theta Z(a)e^{-\sigma T^*} - 1}{r}. \quad (54)$$

And, (49) relates  $T^*$  and  $Z$ :

$$e^{(r-\sigma)T^*} = \frac{\frac{1}{r} - \frac{Z}{r-\sigma}}{\frac{\theta Z}{r} - \frac{Z}{r-\sigma}}.$$

In the low interest rate case,  $\theta \cdot (r - \sigma) > r$  and (53) imply that both the numerator and the denominator of the above expression are positive. Define

$$\psi(Z) \equiv Ze^{-\sigma T^*} = Z \left[ \frac{\frac{1}{Zr} - \frac{1}{r-\sigma}}{\frac{\theta}{r} - \frac{1}{r-\sigma}} \right]^{-\frac{\sigma}{r-\sigma}}.$$

Then, from (54)

$$\frac{d}{da} \left( \frac{b^*}{a} \right) = \frac{\theta}{r} \psi'(Z) \cdot Z'(a).$$

Showing that  $\psi'(Z) < 0$  for all  $Z \in (1, 1/\theta]$  and  $\psi'(1) = 0$  would complete the proof. Indeed,

$$\frac{d}{dZ} \ln \psi(Z) = \frac{1}{Z} + \frac{\sigma}{r - \sigma} \frac{\frac{1}{rZ^2}}{\frac{1}{rZ} - \frac{1}{r-\sigma}} = \frac{1}{Z} \frac{\frac{1}{Z} - 1}{\frac{r-\sigma}{rZ} - 1} < 0.$$

The numerator of the above expression is negative for all  $Z > 1$  and zero for  $Z = 1$ . The denominator is positive when  $r < \theta(r - \sigma)$  and  $Z < 1/\theta$ . ■

**Proof of Proposition 7:** The idea of the proof is to construct the bounds on  $b_L(t; b, a)$ . Starting from (31) and multiplying and dividing by  $e^{-\lambda t} b_t^*(b, a)$ ,

$$b_L(t; b, a) = e^{-\lambda t} b_t^*(b, a) \int_0^t \lambda e^{-\lambda(s-t)} e^{-\Lambda(t-s)} \frac{B_{t-s}^*(b_s^*(b, a), a)}{b_t^*(b, a)} ds.$$

Since the household enters the low health state with wealth  $b_s^*(b, a)$  and dissaves afterwards,  $B_{t-s}^*(b_s^*(b, a), a) \leq b_s^*(b, a)$ . From the initial condition is  $b \leq b_\infty^*(a)$ , Proposition 5 implies saving in good health, so that  $b_s^*(b, a) \leq b_t^*(b, a)$ . Accordingly,  $B_{t-s}^*(b_s^*(b, a), a) \leq b_t^*(b, a)$ , and  $b_L(t; b, a)$  can be bounded as follows:

$$\begin{aligned} 0 \leq b_L(t; b, a) &\leq e^{-\lambda t} b_t^*(b, a) \int_0^t \lambda e^{-(\Lambda-\lambda)(t-s)} ds \leq e^{-\lambda t} b_t^*(b, a) \frac{\lambda}{\Lambda - \lambda}, \\ &0 \leq \frac{b_L(t; b, a)}{e^{-\lambda t} b_t^*(b, a)} \leq \frac{\lambda}{\Lambda - \lambda}. \end{aligned} \quad (55)$$

Now use (55) to bound  $\lim_{t \rightarrow \infty} \bar{b}(t; b, a)$ :

$$\begin{aligned} \lim_{t \rightarrow \infty} \bar{b}(t; b, a) &= \lim_{t \rightarrow \infty} \frac{b_H(t; b, a) + b_L(t; b, a)}{f_H(t) + f_L(t)} = \lim_{t \rightarrow \infty} \frac{e^{-\lambda t} b_t^*(b, a)}{f_H(t) + f_L(t)} \left( 1 + \frac{b_L(t; b, a)}{e^{-\lambda t} b_t^*(b, a)} \right) = \\ &= \frac{\Lambda - \lambda}{\Lambda} b_\infty^*(a) \left( 1 + \lim_{t \rightarrow \infty} \frac{b_L(t; b, a)}{e^{-\lambda t} b_t^*(b, a)} \right). \end{aligned}$$

Applying the bounds in (55) to the above expression gives

$$\frac{\Lambda - \lambda}{\Lambda} b_\infty^*(a) \leq \lim_{t \rightarrow \infty} \bar{b}(t; b, a) \leq b_\infty^*(a).$$

■

**Derivation of the actuarially fair rate of return on annuities** Let  $A$  be the market value of an annuity with income  $a$ . Then

$$a = Ar_A \tag{56}$$

If  $\mathbb{E}_T[\cdot]$  is the expectation over the stochastic life-span  $\tilde{T}$ , we have

$$\begin{aligned} A &= \mathbb{E}_T \left[ \int_0^{\tilde{T}} e^{-rt} dt \right] = a \int_0^\infty \lambda e^{-\lambda T} \int_0^T e^{-rt} dt dT + \\ &+ a \int_0^\infty \lambda e^{-\lambda T} \int_T^\infty \Lambda e^{-\Lambda(S-T)} \int_T^S e^{-rs} ds dS dT. \end{aligned} \tag{57}$$

The first right-hand side term registers annuity income during the healthy phase of retirement, the second term gives income during the last phase of life. Performing the integration and combining (56)-(57), we have

$$r_A = \frac{(\lambda + r)(\Lambda + r)}{\lambda + \Lambda + r}.$$

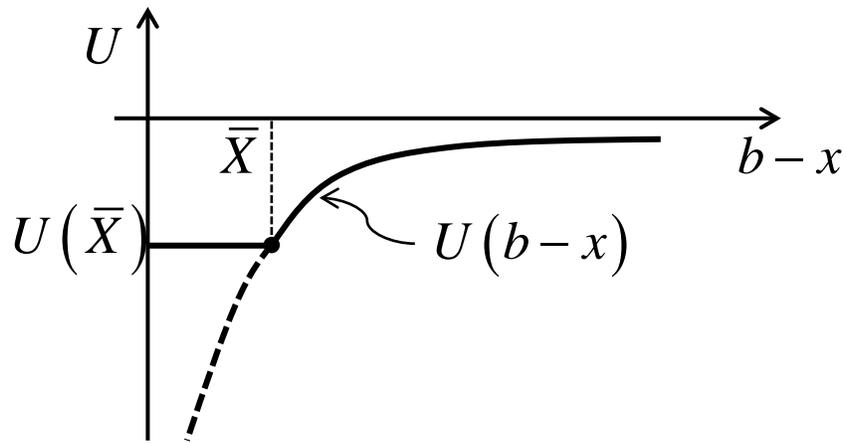


Figure 1. Second period utility for optimization problem (5)

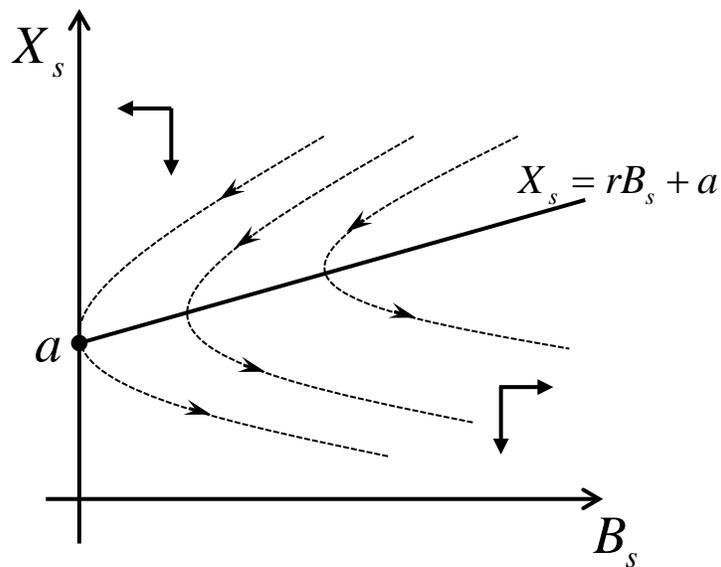


Figure 2. Trajectories obeying first-order conditions and the budget constraint, but not the transversality condition.

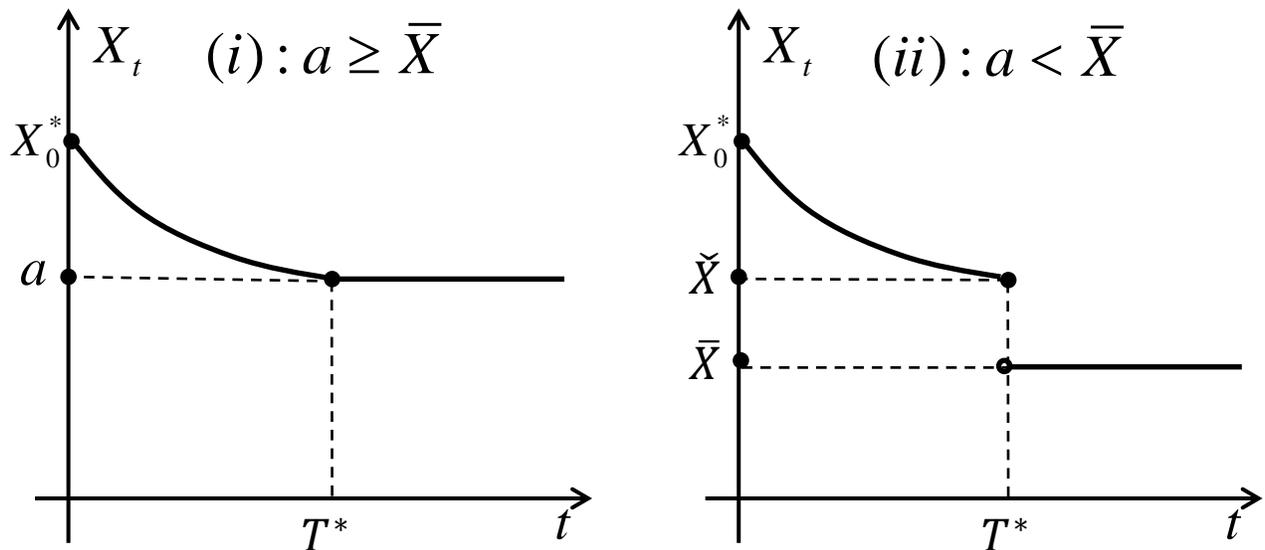


Figure 3. Optimal consumption path in low health state. Case (i):  $a \geq \bar{X}$ , Case (ii):  $a < \bar{X}$ .

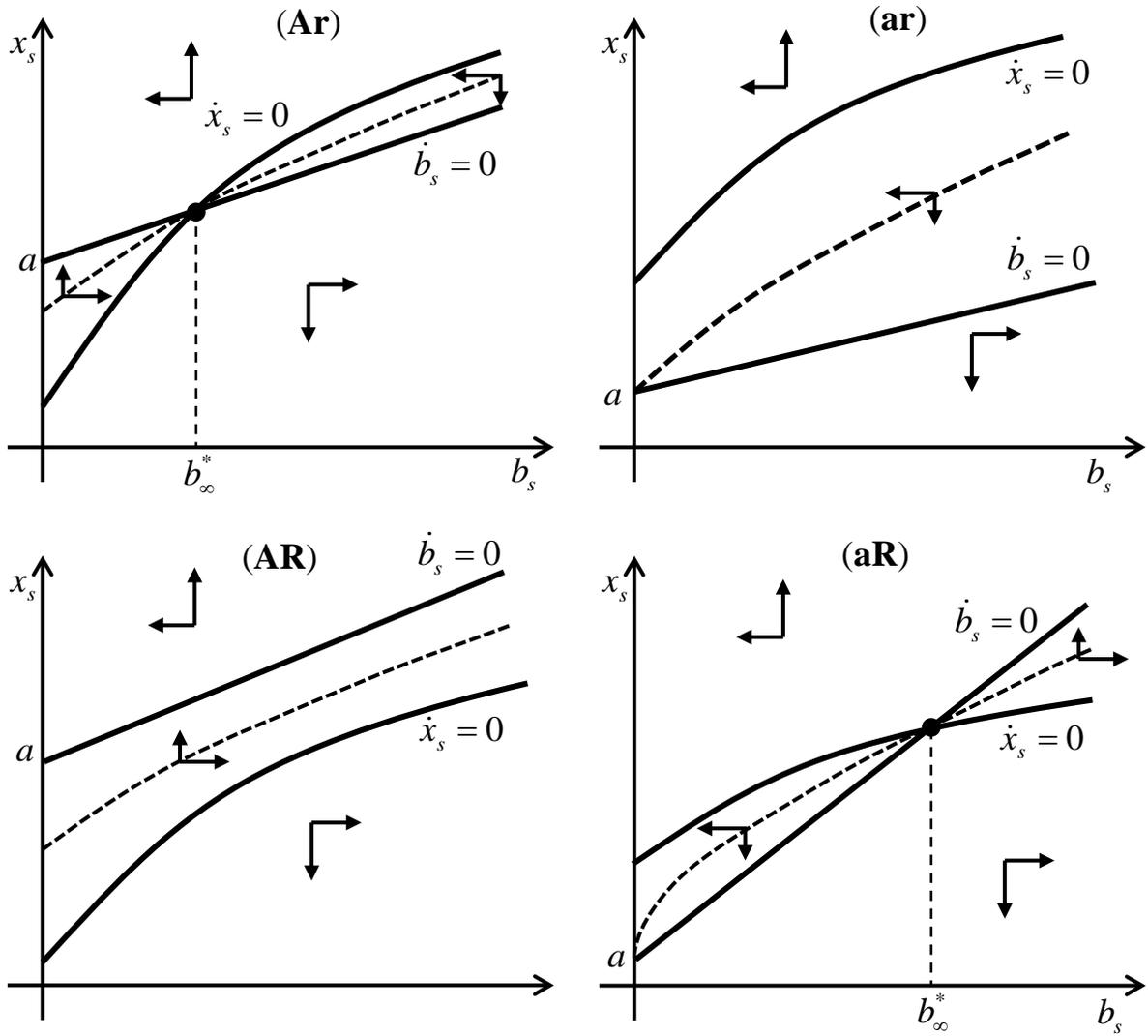


Figure 4. Possible phase diagrams for optimal behavior in high health state. See Proposition 5.

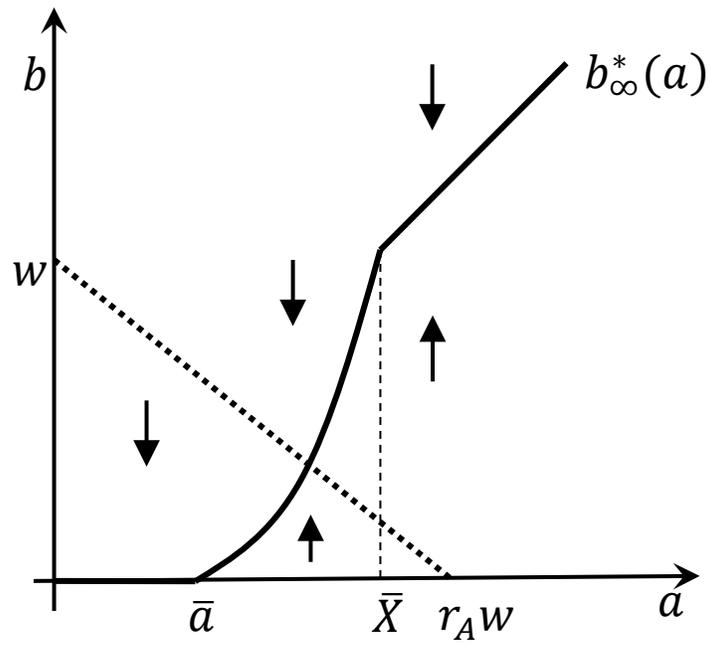


Figure 5. Wealth accumulation behavior for various initial conditions  $(b, a)$ .  
Standard interest rate case.

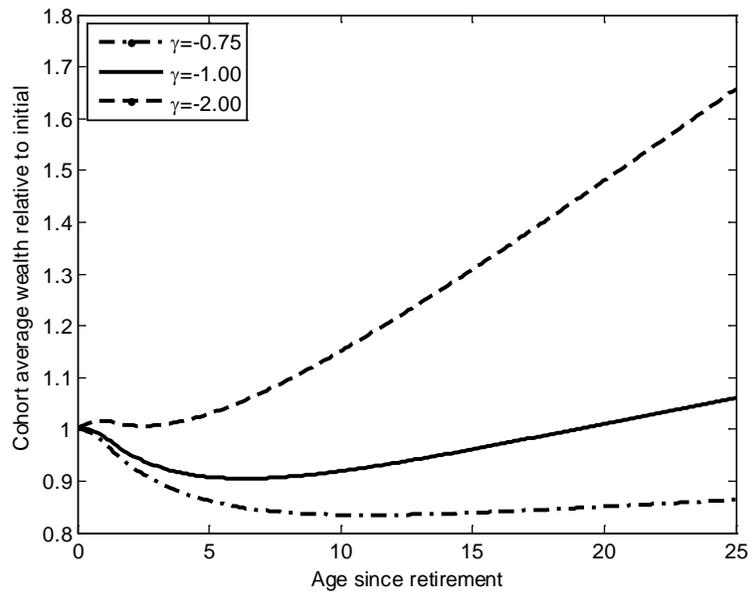


Figure 6. Cohort average wealth trajectories for different values of  $\gamma$ .  
Other parameters:  $\lambda = \frac{1}{12}, \Lambda = \frac{1}{3}, r = \beta = 0.03, \Omega = 52.5, \bar{X} = 52.5$ .

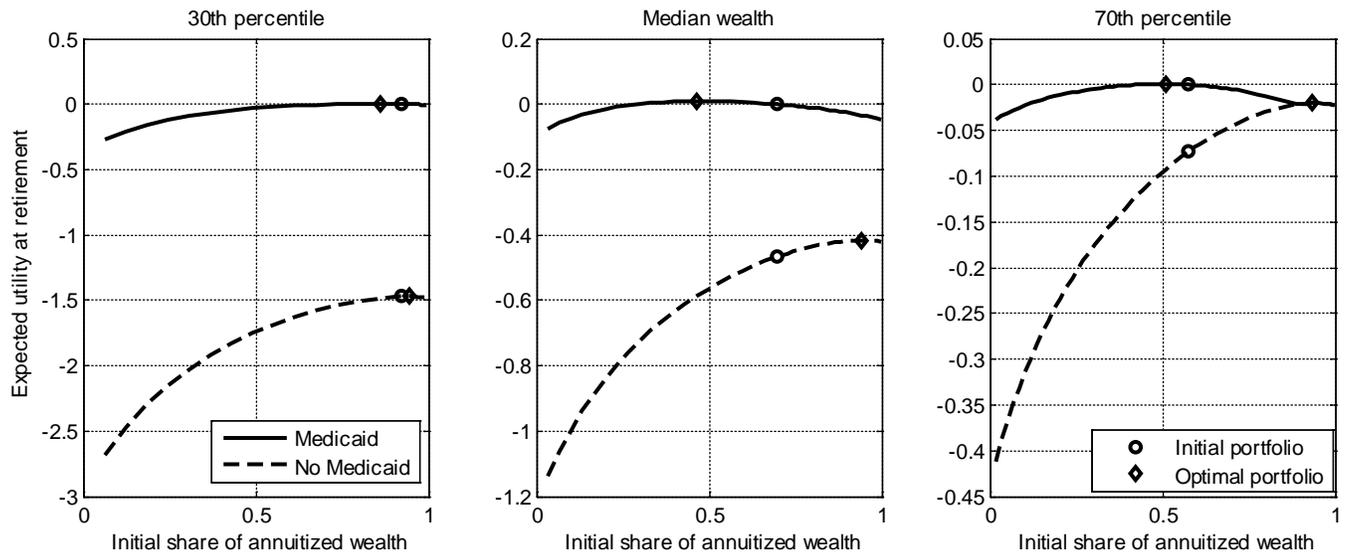


Figure 7. Expected utility at retirement as function of initial portfolio. Exogenous parameters:

$$\gamma = 1, \lambda = \frac{1}{12}, \Lambda = \frac{1}{3}, r = \beta = 0.03, \Omega = 52.5, \bar{X} = \{0, 52.5\}.$$

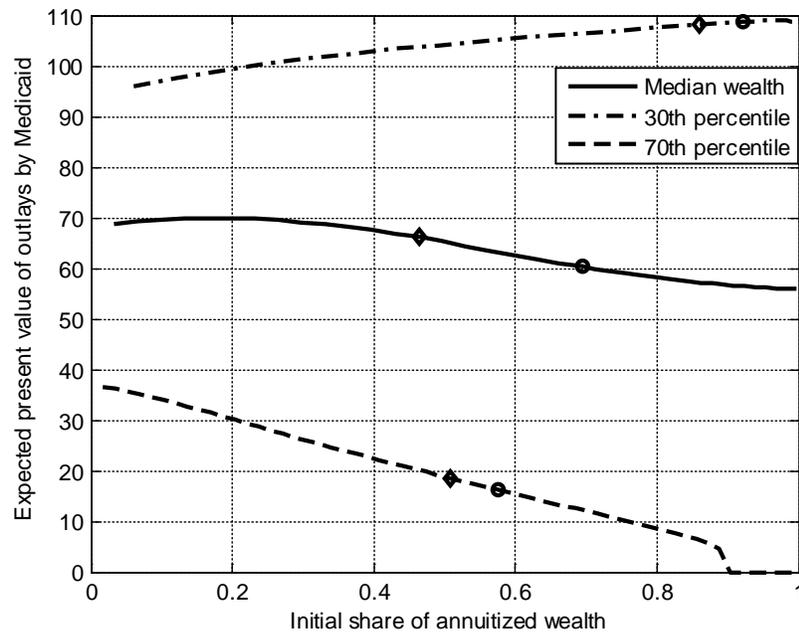


Figure 8. Expected present value of Medicaid LTC outlays per enrollee as function of retiree initial portfolio composition.

Net Worth	Annuitized Wealth		Annuity Income, $a$	Bequeathable Wealth, $b$
	Soc. Security, $b_{SS}$	DB Pension, $b_{DB}$		
30 <sup>th</sup> Percentile	166	0	15	14
Median	230	0	21	100
70 <sup>th</sup> percentile	299	73	34	272
Mean	226	89	29	334

Table 1. Primary annuities and bequeathable wealth (000s of 2008 dollars) for single-person households Aged 65-69.  
Source: Poterba *et al.* [2011], Table 2 and p. 99.

$\gamma$	$\bar{a}$	$\bar{r}$	$a = 15$		$a = 21$		$a = 34$		$a = 29$	
			PD type	$b_{\infty}^*$						
Panel 1	$r = 0.02, \beta = 0.02, \Omega = 7.0, \bar{X} = 70$									
-0.5	20.4	0.032	ar	0	Ar	38.4	Ar	809.6	Ar	526.0
-1	18.6	0.026	ar	0	Ar	326.5	Ar	1861.3	Ar	1300.2
-2	16.5	0.019	aR	980.2	AR	$\infty$	AR	$\infty$	AR	$\infty$
-3	15.3	0.014	aR	48.5	AR	$\infty$	AR	$\infty$	AR	$\infty$
Panel 2	$r = 0.02, \beta = 0.02, \Omega = 5.25, \bar{X} = 52.5$									
-0.5	19.7	0.040	ar	0	Ar	44.8	Ar	456.5	Ar	306.9
-1	18.1	0.034	ar	0	Ar	162.8	Ar	795.0	Ar	566.2
-2	16.2	0.025	ar	0	Ar	757.0	Ar	2411.1	Ar	1816.4
-3	15.1	0.020	aR	206.7	AR	$\infty$	AR	$\infty$	AR	$\infty$
Panel 3	$r = 0.02, \beta = 0.02, \Omega = 3.5, \bar{X} = 35$									
-0.5	18.4	0.052	ar	0	Ar	45.7	Ar	232.6	Ar	168.9
-1	17.1	0.047	ar	0	Ar	93.1	Ar	339.4	Ar	255.7
-2	15.6	0.038	ar	0	Ar	224.3	Ar	625.2	Ar	489.3
-3	14.6	0.031	Ar	28.7	Ar	442.1	Ar	1081.4	Ar	1779.6
Panel 4	$r = 0.03, \beta = 0.03, \Omega = 7.0, \bar{X} = 70$									
-0.5	20.4	0.035	ar	0	Ar	92.2	Ar	1830.0	Ar	1204.1
-1	18.6	0.028	aR	1545.0	AR	$\infty$	AR	$\infty$	AR	$\infty$
-2	16.5	0.020	aR	134.0	AR	$\infty$	AR	$\infty$	AR	$\infty$
-3	15.3	0.015	aR	18.4	AR	$\infty$	AR	$\infty$	AR	$\infty$
Panel 5	$r = 0.03, \beta = 0.03, \Omega = 5.25, \bar{X} = 52.5$									
-0.5	19.7	0.043	ar	0	Ar	64.5	Ar	639.1	Ar	432.7
-1	18.1	0.037	ar	0	Ar	328.5	Ar	1526.6	Ar	1099.6
-2	16.2	0.027	aR	314.2	AR	$\infty$	AR	$\infty$	AR	$\infty$
-3	15.1	0.021	aR	8.0	AR	$\infty$	AR	$\infty$	AR	$\infty$
Panel 6	$r = 0.03, \beta = 0.03, \Omega = 3.5, \bar{X} = 35$									
-0.5	18.4	0.057	ar	0	Ar	53.2	Ar	265.5	Ar	193.7
-1	17.1	0.051	ar	0	Ar	116.6	Ar	415.2	Ar	314.7
-2	15.6	0.040	ar	0	Ar	358.8	Ar	963.1	Ar	759.9
-3	14.6	0.032	Ar	110.9	Ar	1421.7	Ar	3123.6	Ar	2661.2

Table 2. Phase Diagram Types for Various Parameter Combinations (see Figure 4 and Proposition 5).  
Fixed parameters:  $\Lambda = 1/3, \lambda = 1/12$ .