Grothendieck Function-Sheaf Correspondence

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These notes are prepared for my talk in the seminar on geometric class field theory. It is worth noting that we will later apply our main theorem (4.2) to (various versions of) the Picard scheme of a curve.

In section 1-3, we develop some necessary background materials in the theory of étale fundamental groups and l-adic sheaves. As we assume readers have some experience with them, the contents here serve only as a reminder.

Section 4 is devoted to the statement and proof of the main theorem. In the last section, we provide an interesting corollary.

1 The étale fundamental group

The main references for this section are [Mi, I.5] and [FK, A.1]. Of course, [SGA1] would be the ultimate source.

Let X, Y be connected schemes and f be a finite étale morphism. The finite and étale condition implies that the image of f is open and closed, thus f is surjective. For that reason, such a map f is also called a finite étale covering. As the name suggests, this is an algebraic analogue of covering spaces in topology.

Let $\bar{x}$ be a geometric point of X, namely a morphism $\operatorname{Spec} k^s \to X$. We define

$\mathrm{Fib}_Y(\bar{x}) := \{\operatorname{Spec} k^s \to Y | f(\bar{y}) = \bar{x}\}$

It is known that for all $g \in \mathrm{Fib}_Y(\bar{x})$, the map $\varphi_g : \mathrm{Aut}_X(Y) \to \mathrm{Fib}_Y(\bar{x})$ defined by $\sigma \mapsto \sigma \circ g$ is injective. Here it is crucial to have the condition that Y is connected.

Definition 1.1. A finite étale covering $f : X \to Y$ is called **Galois** if the map φ_g is bijective for all $g \in \mathrm{Fib}_Y(\bar{x})$. If furthermore $f(\bar{y}) = \bar{x}$, then we say the pair $(Y, \bar{y})$ is a **pointed Galois covering** of $(X, \bar{x})$.

Definition 1.2. An **étale fundamental group** of X with a base point $\bar{x}$ is defined as $\pi_1(X, \bar{x}) := \varprojlim_{(Y,\bar{y})} \mathrm{Aut}_X(Y)$ where the limit is taken over all finite pointed Galois coverings $(Y, \bar{y})$.

Here we provide some facts about étale fundamental groups.

1. There is a natural isomorphism (unique up to inner automorphisms) $\pi_1(X, \bar{x}) \simeq \pi_1(X, \bar{x}')$ for any two geometric points $\bar{x}, \bar{x}' \in X$. For this reason, we sometimes write $\pi_1(X)$ for $\pi_1(X, \bar{x})$ when inner automorphisms don't cause any problem.

2. The functor π_1 is covariant in $(X, \bar{x})$, i.e. if $f : X \to X'$, and $f(\bar{x}) = \bar{x}'$, then there exists a natural induced map $f_* : \pi_1(X, \bar{x}) \to \pi_1(X, \bar{x}')$. Here we briefly explain how we may find the element $f_*(\alpha) \in \pi_1(X', \bar{x}')$ for a given $\alpha \in \pi_1(X, \bar{x})$. Let $U \to X'$ be a finite Galois covering. The choice of a geometric point $\bar{u}$ of $X \times_{X'} U$ over $\bar{x}$ induces a geometric point $\bar{u}'$ of U over $\bar{x}'$. Then the action of α on $\mathrm{Fib}_{X \times_{X'} U}(\bar{x})$ induces an action on $\mathrm{Fib}_U(\bar{x}')$, the latter being identified with $\mathrm{Aut}_X(U)$ by the choice of $\bar{u}'$. So α induces an element in $\mathrm{Aut}_X(U)$ for each finite étale cover U, so by taking limit we obtain $f_*(\alpha)$.

2 l-adic sheaves

Some good references for this section are [SGA4$\frac{1}{2}$, Rapport 2], [SGA5, Exp. VI] and [FK, §12]. It seems that [Mi, V.1] is not the best place to learn about l-adic sheaves.

Throughout this section, we pick a prime number l and let X be a connected scheme and X_{et} be the étale site on X. In the series of definitions in the current section, the base scheme is always assumed to be X if not specified.

As a preparation, recall that an étale sheaf $\mathscr{G}$ is called locally constant if it becomes a constant sheaf when restricted to an étale covering U over

X. Next, recall that an étale sheaf $\mathscr{G}$ is called constructible if the base X can be stratified into locally closed subschemes so that $\mathscr{G}$ defines a locally constant sheaf with finite stalks over each stratum.

Definition 2.1. A $\mathbb{Z}_l$**-sheaf** $\mathscr{F}$ (or an l**-adic sheaf**) on X_{et} is a projective system $(\mathscr{F}_n)_{n\in\mathbb{Z}_{>0}}$ of constructible sheaves $\mathscr{F}_n$ such that $\mathscr{F}_{n+1} \to \mathscr{F}_n$ induces an isomorphism $\mathscr{F}_{n+1} \otimes (\mathbb{Z}/l^n) \simeq \mathscr{F}_n$.

Remark 2.2. Note that we included *constructibility* in the definition of $\mathbb{Z}_l$-sheaves. In my talk, I followed [Mi] to define $\mathbb{Z}_l$-sheaves first without this condition and then defined constructible $\mathbb{Z}_l$-sheaves later. But in [SGA4$\frac{1}{2}$] and [SGA5], they define only constructible $\mathbb{Z}_l$-sheaves and go on. It seems that we gain a bad category if we have to include non-constructible sheaves as its objects.

As for terminology, [SGA5] uses the term "constructible $\mathbb{Z}_l$-sheaves" whereas [SGA4$\frac{1}{2}$] simply call them $\mathbb{Z}_l$-sheaves. We are following the latter convention.

The issue of constructibility was pointed out to me by Teruyoshi.

Remark 2.3. In the definition above, we don't have to confine ourselves to $\mathbb{Z}_l$-sheaves. Let K be a finite extension of $\mathbb{Q}_l$ and $\mathscr{O}_K, m_K$ be its ring of integers and its maximal ideal, respectively. Then we can use $(\mathscr{O}_K, \mathscr{O}_K/m_K^n)$ instead of $(\mathbb{Z}_l, \mathbb{Z}/l^n)$ to define sheaves of $\mathscr{O}_K$-modules.

Definition 2.4. A $\mathbb{Z}_l$-sheaf $\mathscr{F}$ is called **locally constant** (resp. **constant**) if each $\mathscr{F}_n$ is. The same definition works for $\mathscr{O}_K$-sheaves.

Example 2.5. The easiest example of $\mathbb{Z}_l$-sheaves is the constant sheaf $\mathbb{Z}_l := (\mathbb{Z}/l^n)_n$, which can be defined over any base. If l is invertible in X, we define its Tate twist $\mathbb{Z}_l(1) = \mu_{l^\infty} := (\mu_{l^n})_n$. The sheaf $\mathbb{Z}_l(1)$ is locally constant, but doesn't become a constant $\mathbb{Z}_l$-sheaf over a finite étale covering.

Let K, $\mathscr{O}_K$, m_K be as in 2.3. Now we define the category of K-sheaves as the quotient of the 'saturated' category of $\mathscr{O}_K$-sheaves by its torsion objects. But there is another way to think of this category, which is more explicit.

The category of K-sheaves has the same objects as the category of constructible $\mathscr{O}_K$-sheaves. For $\mathscr{O}_K$-sheaves $\mathscr{F}$ and $\mathscr{G}$, we formally denote their images in the category of K-sheaves by $\mathscr{F} \otimes K$ and $\mathscr{G} \otimes K$, respectively. We define the morphism as $\mathrm{Hom}_{K-sheaves}(\mathscr{F} \otimes K, \mathscr{G} \otimes K) := \mathrm{Hom}_{\mathscr{O}_K-sheaves}(\mathscr{F}, \mathscr{G}) \otimes_{\mathscr{O}_K} K$.

If we take the limit of the categories of K-sheaves as K runs through all finite extensions of $\mathbb{Q}_l$, we finally obtain the category of $\overline{\mathbb{Q}}_l$-sheaves. It is worth noting that everything started from $\mathscr{O}_K$-sheaves which are projective systems of *constructible* sheaves with natural compatibility condition.

Definition 2.6. Let $\mathscr{F}$ be a $\mathscr{O}_K$-sheaf. A K-sheaf $\mathscr{F} \otimes K$ is called **locally constant** if $\mathscr{F}$ is.

We say a $\overline{\mathbb{Q}}_l$-sheaf is **locally constant** if it can be written as a limit of locally constant K-sheaves. A locally constant $\overline{\mathbb{Q}}_l$-sheaf will also be called an **l-adic local system**.

Definition 2.7. Let $\mathscr{F}$ be an $\mathscr{O}_K$-sheaf on X. The stalk of $\mathscr{F}$ at a geometric point $\bar{x}$ of X is defined to be $\mathscr{F}_{\bar{x}} := \varprojlim_n (\mathscr{F}_n)_{\bar{x}}$, where $(\mathscr{F}_n)_{\bar{x}}$ are usual stalks in X_{et}. We define the stalk of $\mathscr{F} \otimes K$ at $\bar{x}$ to be $(\mathscr{F} \otimes K)_{\bar{x}} := (\mathscr{F}_{\bar{x}}) \otimes_{\mathscr{O}_K} K$. To define a stalk of a $\overline{\mathbb{Q}}_l$-sheaf, we simply take a direct limit of stalks of K-sheaves.

When $\mathscr{E}$ is a locally constant $\overline{\mathbb{Q}}_l$-sheaf on X, the number of copies of $\overline{\mathbb{Q}}_l$ in the $\overline{\mathbb{Q}}_l$-vector space $\mathscr{E}_{\bar{x}}$ is called the **rank** of $\mathscr{E}$, which makes sense locally over the base.

Remark 2.8. This kind of definition in terms of $\varprojlim$ may look familiar. Similarly, one defines the étale cohomology with coefficients in l-adic sheaves as a projective limit of the étale cohomologies on "finite levels".

When we work with l-adic local systems, the following categorical equivalence is very useful. See [FK, A.I.8.] for various equivalences of categories of the same flavor.

Proposition 2.9. *Let $\bar{x}$ be a geometric point on a connected scheme X. The "stalk functor" $\mathscr{F} \mapsto \mathscr{F}_{\bar{x}}$ defines the following categorical equivalence.*

$$\left\{ \begin{array}{l} l\text{-adic local systems} \\ \text{on } X \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{continuous representations of } \pi_1(X, \bar{x}) \\ \text{on finite-dimensional } \overline{\mathbb{Q}}_l\text{-vector spaces} \end{array} \right\}$$

Remark 2.10. Dennis suggested that we may be able to avoid the heavy machinery of l-adic sheaves or $\overline{\mathbb{Q}}_l$-sheaves for the purpose of geometric class field theory. Namely, we may regard l-adic sheaves as the representations of the fundamental group π_1 by the above proposition and work with the latter most of the time.

3 Frobenius action on stalks

Let X be a connected $\mathbb{F}_q$-scheme where q is some power of a prime number p.

Definition 3.1. By the **absolute Frobenius** morphism Fr_q on X, we mean the $\mathbb{F}_q$-morphism of X to itself which is identity on the underlying topological space of X and $x \mapsto x^q$ on the structure sheaf of rings $\mathscr{O}_X$.

The following general lemma will be very useful throughout this article.

Lemma 3.2. *Let $f : S \to T$ be a morphism of schemes and $\mathscr{F}$ be an étale sheaf on T. Let $\bar{s}$ be a geometric point of S. Then we have a natural isomorphism $(f^*\mathscr{F})_{\bar{s}} = \mathscr{F}_{\overline{f(s)}}$.*

Proof. [Mi, Thm II.3.2.] □

Consider a point $x \in X(\mathbb{F}_{q^n})$, namely a morphism $s_x : \operatorname{Spec} \mathbb{F}_{q^n} \to X$. We choose its geometric point $\bar{x}$, which is a morphism $\operatorname{Spec} \overline{\mathbb{F}}_q \to X$ factoring through s_x.

Now let $\mathscr{E}$ be an l-adic local system of rank r on X where $r \in \mathbb{Z}_{>0}$. Then there are three ways to define Frobenius action on $\mathscr{E}_{\bar{x}}$.

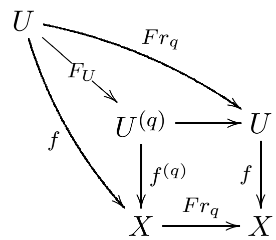
1. We have natural identifications $\overline{\mathbb{Q}}_l^r \simeq \mathscr{E}_{\bar{x}} \simeq (s_x^*\mathscr{E})(\bar{x}) = (s_x^*\mathscr{E})_{\bar{x}}$. The last space gets a natural action by $\pi_1(\bar{x}) \simeq \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Then the arithmetic Frobenius $F_x := x \mapsto x^q$ in the Galois group defines an automorphism of $\overline{\mathbb{Q}}_l^r$.

2. The map s_x induces the map $(s_x)_* : \pi_1(\bar{x}) \to \pi_1(X, \bar{x})$. The element $F_x \in \pi_1(\bar{x})$ defines an image in $\pi_1(X, \bar{x})$, which will also be called F_x by abuse of notation. As an element of $\pi_1(X, \bar{x})$, F_x acts on $\mathscr{E}_{\bar{x}}$ naturally. We may justify the abuse of notation by the exercise 3.3.

3. The absolute Frobenius Fr_q naturally induces a map of sheaves $Fr_q^*\mathscr{E} \to \mathscr{E}$. Accept this for now. Then by taking stalks at $\bar{x}$, we have a natural map $\mathscr{E}_{\overline{Fr_q(x)}} \simeq (Fr_q^*\mathscr{E})_{\bar{x}} \to \mathscr{E}_{\bar{x}}$. In particular, if $x \in X(\mathbb{F}_q)$, it is an endomorphism of $\mathscr{E}_{\bar{x}}$ which turns out to be an automorphism.

   We need to explain how to define the map $Fr_q^*\mathscr{E} \to \mathscr{E}$. Since Fr_q^* is the left adjoint of Fr_{q*}, or $\operatorname{Hom}(\mathscr{E}, Fr_{q*}\mathscr{E}) \simeq \operatorname{Hom}(Fr_q^*\mathscr{E}, \mathscr{E})$, it is enough to find a natural map $\mathscr{E} \to Fr_{q*}\mathscr{E}$. For any étale morphism

$U \xrightarrow{f} X$, let $U^{(q)} \xrightarrow{f^{(q)}} X$ be the base change of f by $X \xrightarrow{Fr_q} X$. Then our task is to define a map $\Gamma(U, \mathscr{E}) \to \Gamma(U, Fr_{q*}\mathscr{E}) = \Gamma(U^{(q)}, \mathscr{E})$. First we obtain a map $F_U : U \to U^{(q)}$ from $U \xrightarrow{Fr_q} U$ and $U \xrightarrow{f} X$ by the universal property of fiber products. But the map F_U turns out to be an isomorphism whenever $U \xrightarrow{f} X$ is étale (see [SGA5, Exp.XV prop.2.c.2.]), hence its inverse $F_U^{-1} : U^{(q)} \to U$ will induce the map $\Gamma(U, \mathscr{E}) \to \Gamma(U^{(q)}, \mathscr{E})$. It also follows that the so-induced maps $\mathscr{E} \to Fr_{q*}\mathscr{E}$ and $Fr_q^*\mathscr{E} \to \mathscr{E}$ are both isomorphisms of sheaves.



Exercise 3.3. Choose a point $x \in X(\mathbb{F}_{q^n})$ as before. Show that the first two definitions of Frobenius actions on $\mathscr{E}_{\bar{x}}$ are same up to inner automorphisms. If $x \in X(\mathbb{F}_q)$, show that the third definition also gives the same Frobenius action, again up to inner automorphisms. Thus we conclude that the trace of Frobenius is independent of the choice among three definitions.

4 The main theorem

The main theorem to be proved in this section is a special case of the Grothendieck function-sheaf correspondence. One may read brief but more general accounts of the function-sheaf correspondence in [Fr, §4.2] and [Bu, §12.4 by Gaitsgory].

Let H be a connected separated commutative group scheme of finite type defined over $\mathbb{F}_q$. The group scheme H is equipped with the multiplication $m : H \times H \to H$, the inverse operation $i : H \to H$, and the identity section $e : \operatorname{Spec} \mathbb{F}_q \to H$, all defined over $\mathbb{F}_q$.

Let us choose a prime l different from p. First we define the notion of character sheaves.

Definition 4.1. A rank-one l-adic local system $\mathscr{E}$ on H is called an (l-adic) **character sheaf** if it satisfies $m^*\mathscr{E} \simeq \mathscr{E} \boxtimes \mathscr{E}$. Note that $\mathscr{E} \boxtimes \mathscr{E}$ is by definition $p_1^*\mathscr{E} \otimes p_2^*\mathscr{E}$ where p_1 and p_2 are two projection maps from $H \times_{\mathbb{F}_q} H$ to H.

Now we are ready to state the following main theorem of this article.

Theorem 4.2. *We keep the preceding notations in this section. Then we have a natural bijection*

$$\{\ l\text{-adic character sheaves on } H\ \} \longleftrightarrow \mathrm{Hom}_{ab.\,gp}(H(\mathbb{F}_q), \overline{\mathbb{Q}}_l^{\times})$$

Remark 4.3. By a natural bijection, we mean that there is a natural way to match objects from each side. This will become clear in the proof as we explain how to match them.

Proof. ($\Rightarrow$) Suppose that we are given a character sheaf $\mathscr{E}$. Then we may define a function $f : H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l^{\times}$ by $f(x) := F_x(1)$ since the stalk $\mathscr{E}_{\bar{x}} \simeq \overline{\mathbb{Q}}_l$ gets the Frobenius action F_x as we explained in the earlier section.

We need to show that f is a group homomorphism, i.e. $f(x+y) = f(x)f(y)$ for any $x, y \in H(\mathbb{F}_q)$. To see this, take stalks at $(x, y) \in (H \times H)(\mathbb{F}_q)$ of the isomorphism $m^*\mathscr{E} \simeq \mathscr{E} \boxtimes \mathscr{E}$. Then we get $\mathscr{E}_{\overline{x+y}} \simeq \mathscr{E}_{\bar{x}} \otimes \mathscr{E}_{\bar{y}}$, from which it is easy to deduce $F_{x+y}(1) = F_x(1)F_y(1)$, or $f(x+y) = f(x)f(y)$.

($\Leftarrow$) First consider the Lang isogeny φ, which is $y \mapsto Fr_q(y) - y$ on the functor of points and may also be defined to be the composite map

$$H \xrightarrow{(Fr_q, i)} H \times H \xrightarrow{m} H$$

This map is finite, étale and Galois. For the proof that it is finite, we refer readers to the remark below (4.5). To see it is étale, observe that the differential of φ is isomorphism, being the "negative" identity because the differential of Fr_q is 0 in characteristic p. We also know φ is Galois, since it is not hard to see $\ker\varphi = H(\mathbb{F}_q)$ is the Galois group of φ via $x \in H(\mathbb{F}_q) \mapsto T_x$. Here T_x is the translation-by-x map $y \mapsto y + x$, viewed as an automorphism of the covering $H \xrightarrow{\varphi} H$

Now we explain how to attach a character sheaf to a function $f \in \mathrm{Hom}(H(\mathbb{F}_q), \overline{\mathbb{Q}}_l^{\times})$. By the categorical equivalence (2.9), giving a rank-one l-adic local system on H is same as giving a map $\pi_1(H) \to \overline{\mathbb{Q}}_l^{\times} \simeq \mathrm{Aut}(\overline{\mathbb{Q}}_l)$. Thus the natural surjection $\pi_1(H) \twoheadrightarrow H(\mathbb{F}_q)$ composed with f gives rise to a rank-one l-adic local system $\mathscr{E}$. Note that $\varphi^*\mathscr{E}$ is a constant $\overline{\mathbb{Q}}_l$-sheaf by construction.

It remains to check that $\mathscr{E}$ satisfies $m^*\mathscr{E} \simeq \mathscr{E} \boxtimes \mathscr{E}$. Since $\varphi^*\mathscr{E}$ is constant, $(\varphi,\varphi)^* m^*\mathscr{E} = (m^*\varphi^*)\mathscr{E}$ is also constant. Similarly, $\mathscr{E} \boxtimes \mathscr{E}$ is constant as well. Thus it is enough to check that the Galois group $H(\mathbb{F}_q) \times H(\mathbb{F}_q)$ of the map $(\varphi,\varphi) : H \times H \to H$ acts on $m^*\mathscr{E}$ and $\mathscr{E} \boxtimes \mathscr{E}$ in the same manner. These actions can be compared on the stalk at a geometric point over the particular point $(0,0)$. Here we write 0 for the identity section $e : \operatorname{Spec} \mathbb{F}_q \to H$.

Consider the natural identification $m^*\mathscr{E}_{\overline{(0,0)}} \simeq \mathscr{E}_{\bar{0}}$. It is not hard to check that $m : H \times H \to H$ transfers the action of $(x,y) \in H(\mathbb{F}_q) \times H(\mathbb{F}_q)$ to $x + y \in H(\mathbb{F}_q)$ via m_*. But by the above construction, the action of $x + y$ is nothing but multiplication by $f(x + y)$.

Likewise, replacing the map m by p_1 and p_2, we see the action of (x,y) on $p_1^*\mathscr{E} \otimes p_2^*\mathscr{E}$ is multiplication by $f(x)f(y)$ since the tensor product is compatible with taking stalks. Therefore (x,y) acts on both $m^*\mathscr{E}$ and $\mathscr{E} \boxtimes \mathscr{E}$ as multiplication by $f(x + y) = f(x)f(y)$.

($\Rightarrow$ followed by $\Leftarrow$ is identity)

We start from a character sheaf $\mathscr{E}$ and attach a function f on $H(\mathbb{F}_q)$ as described above. It suffices to prove that $\varphi^*\mathscr{E}$ is a constant sheaf because in that case the sheaf $\mathscr{E}$ is pinned down by the action of the $H(\mathbb{F}_q)$, viewed as the Galois group of the étale covering φ, on its stalks, and such an action is exactly what f represents.

First we recall that $Fr_q^*\mathscr{E} \simeq \mathscr{E}$, which is true not only for character sheaves but also for any l-adic local systems. For the sketch of the proof, we refer readers to section 3 (p.5) where we explain the third way to define the Frobenius action.

Then we have a series of isomorphisms

$\varphi^*\mathscr{E} \simeq (Fr_q, i)^* m^*\mathscr{E} \simeq (Fr_q, i)^*(\mathscr{E} \boxtimes \mathscr{E}) \simeq Fr_q^*\mathscr{E} \otimes i^*\mathscr{E} \simeq \mathscr{E} \otimes i^*\mathscr{E} \simeq (id, i)^*(\mathscr{E} \boxtimes \mathscr{E}) \simeq (id, i)^*(m^*\mathscr{E}) \simeq f^*(e^*\mathscr{E})$

But $e^*\mathscr{E}$ on $\operatorname{Spec} \mathbb{F}_q$ is a constant $\overline{\mathbb{Q}}_l$-sheaf since the Frobenius action on it is trivial by $f(0) = 1$. Therefore, $\varphi^*\mathscr{E}$ is a constant sheaf as desired.

($\Leftarrow$ followed by $\Rightarrow$ is identity)

Starting from a function $f \in \operatorname{Hom}_{ab.\,gp}(H(\mathbb{F}_q), \overline{\mathbb{Q}}_l^\times)$, we could attach a character sheaf $\mathscr{E}$ on H to it. Our task will be showing that F_x acts on $\mathscr{E}_{\bar{x}}$ as multiplication by $f(x)$.

Observe that the F_x-action on $\mathscr{E}_{\bar{x}}$ is induced by the T_x-action on $\Gamma(\varphi, \mathscr{E}) \simeq \mathscr{E}_{\bar{x}}$ (see the lemma (4.4)). But the T_x-action on $\Gamma(\varphi, \mathscr{E})$ is exactly how

$x \in H(\mathbb{F}_q)$ acts on $\Gamma(\varphi, \mathscr{E})$, which is multiplication by $f(x)$, where $H(\mathbb{F}_q)$ is viewed as the Galois group for the étale covering φ.

□

Lemma 4.4. *Let H be as before and $\varphi : H \to H$ be the Lang isogeny which is $y \mapsto Fr_q(y) - y$ on the functor of points. Let $\mathscr{E}$ be any locally constant $\overline{\mathbb{Q}}_l$-sheaf of finite rank on H such that $\varphi^*\mathscr{E}$ is a constant $\overline{\mathbb{Q}}_l$-sheaf.*

Then via the natural isomorphism of $\overline{\mathbb{Q}}_l$-vector spaces $\mathscr{E}_{\bar{x}} \simeq \Gamma(\varphi, \mathscr{E})$, the F_x-action on $\mathscr{E}_{\bar{x}}$ is induced by the natural action of the translation map T_x on right side.

Proof. As we saw in section 3 that two different definitions of the F_x-action coincide, it suffices to check that the T_x-action induces the F_x-action if we take the fiber over $x \in X(\mathbb{F}_q)$ of the map φ.

On the functor of points, if $y \mapsto x$ by φ, it means that $x = Fr_q(y) - y$. Thus the map T_x maps y to $y + x = Fr_q(y)$ for each $y \in \varphi^{-1}(x)$. Now it is easy to see that this action reduces to the q-th power map F_x over $\operatorname{Spec} \mathbb{F}_q$, where the latter is viewed as the $\mathbb{F}_q$-point x of X.

□

Remark 4.5. During the proof, we promised to explain why the map φ is a finite morphism. Here is the sketch of the argument that Dennis suggested. First off, one can reduce to the case of abelian varities or affine group schemes over $\mathbb{F}_q$ since any group scheme over a field is an extension of an abelian variety by an affine group scheme.

In the case of abelian varieties, the map φ is really a self-isogeny since $\ker \varphi$ is finite. Then it is standard that any isogeny of abelian varities is a finite morphism. In the case of affine schemes, we prove that φ induces an isomorphism $H/\ker\varphi \simeq H$ and that the quotient map $H \to H/\ker\varphi$ is always a finite morphism.

Note that H must be of finite type; otherwise it may happen that the cardinality of $H(\mathbb{F}_q)$ is infinite and $\ker \varphi$ is not a finite group scheme.

5 Bonus material

We keep the notations in the last section. There is one interesting thing in the statement of the theorem (4.2). So to speak, the character sheaf $\mathscr{E}$ is determined by its associated function f which describes Frobenius actions on the geometric points only over $\mathbb{F}_q$-points of H. Thus the theorem implies that the Frobenius action F_x for $x \in H(\mathbb{F}_{q^n})$ must be determined by the function f. Will there be some explicit relations?

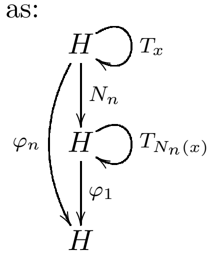
Let $N_n : H \to H$ be the map which is on the functor of points $y \mapsto y + Fr_q(y) + \cdots + Fr_{q^{n-1}}(y)$. Our question is answered by the following corollary of the main theorem.

Corollary 5.1. *If $x \in H(\mathbb{F}_{q^n})$, then the Frobenius action F_x on $\mathscr{E}_{\bar{x}}$ is the multiplication by $f(N_n(x))$.*

Remark 5.2. Note that $N_n(x) \in H(\mathbb{F}_q)$ since $N_n(x)$ is invariant under Fr_q.

Proof. The main theorem and its proof works for $x \in H(\mathbb{F}_{q^n})$ once we let q^n play the role of q. In particular, when we consider the étale covering given by the Lang isogeny $\varphi_n : H \to H$, $y \mapsto Fr_{q^n}(y) - y$, the Frobenius action F_x on $\mathscr{E}_{\bar{x}} = \Gamma(\varphi_n, \mathscr{E})$ coincides with the action of the translation map T_x (lemma (4.4)), which is viewed as a Galois action for the covering φ_n.

Let $\varphi_1 : H \to H$, $y \mapsto Fr_q(y) - y$ be the Lang isogeny for Fr_q. Then it is easy to see that φ_n factors as:



Furthermore, we know from the proof of the main theorem (back to $\mathbb{F}_q$-case) that $\varphi_1^*\mathscr{E}$ is already a constant sheaf, hence $\Gamma(\varphi_n, \mathscr{E}) = \Gamma(\varphi_1, \mathscr{E})$. On the other hand, since the Galois action of T_x for the covering φ_n induces the Galois action of $T_{N_n(x)}$ for the covering φ_1, the T_x-action on $\Gamma(\varphi_n, \mathscr{E})$ is same as the $T_{N_n(x)}$-action on $\Gamma(\varphi_1, \mathscr{E})$. As $N_n(x) \in H(\mathbb{F}_q)$, it follows from the lemma (4.4) that $T_{N_n(x)}$-action for the covering φ_1 induces the Frobenius action $F_{N_n(x)}$, which is precisely multiplication by $f(N_n(x))$.

□

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