Differential Equations I
(MATH 350)

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<u>LECTURE NOTES</u>

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Chapter 1

Definitions & Terminology

1.1 Introduction and Motivation

Mathematical models play a critial role in the sciences and engineering by helping us gain a better understanding of real life phenomena. The models are generally simplified versions of the actual physical phenomena under investigation. They are simplified because the parameters governing the natural phenomena may is often not completely understood or may be too complicated to be represented mathematically. The development of mathematical models usually result in an equation or a set of equations specifying how an unknown function (say $\phi(t)$) changes with respect to a variable (say t). Such an equation is referred to as a **differential equation**. For example, the variation in the population of mosquitoes in a certain village may be represented by the differential equation

$$\frac{dM}{dt} = kM, \tag{1.1}$$

where M is the number (population) of mosquitos at time t, and k is a known constant (obtained from observational or experimental data). Fortunately for us, equation (1.1) can be solved easily using integration techniques learned in your Calculus class. Re-writing the equation and integrating results in

$$\begin{aligned} \frac{dM}{M} &= kdt, \\ \int \frac{dM}{M} &= \int kdt, \\ \ln(M) &= kt + C_1, \\ M = e^{kt+C_1} &= e^{C_1}e^{kt} \\ M(t) &= Ce^{kt}. \end{aligned} \tag{1.2}$$

The integration constant C can be determined if we know the initial population of mosquitoes. Equation (1.2) then helps us to predict the population of mosquitoes at a future time. For instance, if $k = -5$, and if initially (at $t = 0$) the population of mosquitoes is 1000, then we have

$$1000 = Ce^{5\times 0} = C,$$
$$\implies M(t) = 1000e^{-5t}. \tag{1.3}$$

Thus, our simple model may now be used to predict the decline in the population of mosquitoes in the village. A differential equation (e.g. equation 1.1) with the initial value specified (e.g. $M = 1000$ at $t = 0$; as in the above example) is called an **Initial Value Problem** (IVP).

Notice that the solution to a differential equation is not a number but a function (see equation 1.2), and contains arbitrary "constants of integration". The presence of these arbitrary constants implies that there is generally no unique solution to a differential equation. Thus, equation (1.2) is referred to as the **general solution** since there are infinitely many solutions; there is a solution for every C. Three of such solutions are displayed in Figure 1.1. The solution for a particular value of C (as in equation 1.3) is called a **particular solution** to the differential equation.

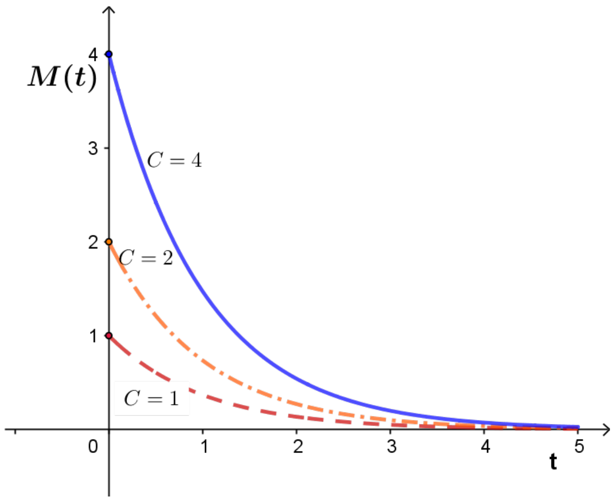

Figure 1.1: Three solutions of $M(t) = Ce^{kt}$ for $k = -1$.

1.2 Classifications and Terminology

The unknown function in a differential equation (e.g. M in equation 1.1) is called the **dependent variable** and the others (e.g. t in equation 1.1) is called the **independent variable**. It is usually clear from the equation which variable is dependent and which is independent.

Definition 1. [Differential Equation]
A differential equation is an equation involving the rate of change of a quantity. That is, it involves a function and its derivatives.

An **ordinary differential equation (ODE)** is a differential equation in which only the derivatives of the unknown function with respect to one independent variable appear in the equation. A **partial differential equation (PDE)** is a differential equation in which partial derivatives of the unknown function with respect to at least two independent variables appear in the equation. Examples of ODEs are

$$\begin{aligned} \frac{dy}{dt} + kt &= 10, \\ \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by &= x^2. \end{aligned} \tag{1.4}$$

The following are some examples of PDEs

$$\begin{aligned} \frac{\partial u^2}{\partial x^2} + \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial \phi}{dx} + b\frac{\partial \phi}{\partial y} + b &= 0. \end{aligned}$$

The **order** of a differential equation is the order of the highest derivative appearing in the equation. Examples of **first order equations** are

$$\frac{dx}{dt} + x = 0, \quad \text{and} \quad \frac{dy}{dx} + 3xy = 0,$$

and the following are **second order equations**:

$$\frac{d^2y}{dx^2} + cy = 0, \quad \text{and} \quad \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 = 0.$$

The constants appearing in a differential equation are called **coefficients or parameters**. For example the constant k appearing in equation (1.1), and the constants a and b in equation (1.4).

A first order ODE is called **explicit** if it can be written in the form

$$\frac{dy}{dx} = f(x, y), \tag{1.5}$$

for a real-valued function f of two variables. Otherwise it is said to be **implicit**. An example of a first order implicit differential equation is an equation of the form

$$F\left(x, y, \frac{dy}{dx}\right) = 0,$$

where F is a continuous function. If the above equation can be solved for dy/dx (or if dy/dx can be isolated), then we may obtain an explicit differential equation of the form in (1.5).

The general form for an nth-order differential equation can be expressed in the form

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0, \tag{1.6}$$

where it is assumed that the equation holds for all x in an open interval ($a < x < b$, where a or b could be infinite). If the highest-order term can be isolated, then we can write (1.6) in the explicit form

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}\right) = 0. \tag{1.7}$$

Equation (1.7) is often preferred to (1.6) for computational and theoretical purposes.

1.3 Solution of a differential equation

Definition 2. [Explicit Solution]
A function $\phi(x)$ that when substituted for y in equation (1.5) satisfies the equation for all x in an interval I is called an **explicit solution** to the equation on I.

Example 1. Show that $\phi(x) = x - 1 + ce^{-x}$ is an explicit solution to the linear equation

$$\frac{dy}{dx} + y = x,$$

where c is a constant.

The point is to show that $y = \phi(x)$ satisfies the given equation.

Solution. Let $y = \phi(x) = x - 1 + ce^{-x}$

$$\implies \frac{dy}{dx} = 1 - ce^{-x}$$

Note that the equations above are defined for all real x. Substituting into the differential equation gives

$$\begin{aligned} \frac{dy}{dx} + y &= (1 - ce^{-x}) + (x - 1 + ce^{-x}) \\ &= x. \end{aligned} \tag{1.8}$$

Hence, $\phi(x) = x - 1 + ce^{-x}$ is an explicit solution for all real x.

Example 2. Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0 \tag{1.9}$$

Solution. Let

$$y = \phi(x) = x^2 - x^{-1} \quad \implies \frac{dy}{dx} = 2x + x^{-2}$$

$$\implies \frac{d^2y}{dx^2} = 2 - 2x^{-3}.$$

Note that these expressions are defined for all $x \neq 0$. Substituting them into (1.9), we get

$$\begin{aligned} \frac{d^2y}{dx^2} - \frac{2}{x^2}y &= (2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) \\ &= (2 - 2x^{-3}) - (2 - 2x^{-3}) \\ &= 0. \end{aligned}$$

Since this is true for any $x \neq 0$, the function $\phi(x) = x^2 - x^{-1}$ is an explicit solution to (1.9) on $(-\infty, 0)$ and $(0, \infty)$.

Example 3. Check whether the function $y = \int_0^x \sqrt{1 + t^3}dt + C, \quad -1 < x < \infty$, is a solution to the linear equation

$$\frac{dy}{dx} = \sqrt{1 + x^3}$$

on the given interval, where C is a constant.

Solution. Applying the fundamental theorem of calculus, we can differentiate

$$y = \int_0^x \sqrt{1+t^3}dt + C,$$

to get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x \sqrt{1+t^3}dt\right) + 0 = \sqrt{1+x^3}.$$

So yes, the given function is a solution to the differential equation.

Definition 3 (Implicit Solution)**.** A relation $G(x, y) = 0$ is said to be an implicit solution to equation (1.6) on the interval I if it defines one or more explicit solutions on I.

Example 4. Verify that

$$x + y + e^{xy} = 0,$$

is an implicit solution to the equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0.$$

Solution. We assume that a function $y(x)$ exists and also differentiable (by the so-called **implicit function theorem**). Applying the technique of implicit differentiation, we differentiate both sides of $x + y + e^{xy} = 0$ to get:

$$\begin{aligned}\frac{d}{dx}(x + y + e^{xy}) &= 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right)\\ &= 1 + \frac{dy}{dx} + ye^{xy} + xe^{xy}\frac{dy}{dx}\\ &= (1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0,\end{aligned}$$

where the last expression is identical to the given differential equation. Thus, $x + y + e^{xy} = 0$ is an implicit solution on some interval.

1.4 Applications in different fields

In this section, we state some differential equations that arise in a variety of subject areas.

1. **Population model:** Let $p(t)$ be the population of a species at time t. The growth of the population is governed by the **logistic model** (or differential equation):

$$\boxed{\frac{dp}{dt} = k_1 p - k_2\frac{p(p-1)}{2}} \qquad (1.10)$$

or

$$\boxed{\frac{dp}{dt} = -Ap(p - p_1), \quad p(0) = p_0}, \tag{1.11}$$

where $A = k_2/2$ and $p_1 = 2k_1/k_2 + 1$ are constants to be determined. Note that if $k_2 = 0$, the logistic model reduces to the simple (or **Malthusian**) model we introduced earlier for the growth of mosquitoes (equation 1.1).

2. **Electricity:** The time variation of electric charge Q in an electric circuit with inductance L, resistance R, capacitance C, and with an externally applied voltage (or electromotive force) is governed by the differential equation

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).} \tag{1.12}$$

3. **Psychology:** One model in psychology of the learning of a task by an individual learner, p, is given by

$$\boxed{\frac{dy}{dt} = \frac{2p}{\sqrt{n}}\left[y^{3/2}\left(1 - y\right)^{3/2}\right],} \tag{1.13}$$

where y represents the learner's skill level as a function of time, t, and n is the nature of the task to be learned. Both p and n are constants.

4. **Black-Scholes equation:** The value $V(x, t)$ of an option to by or sell a stock with price x at time t is governed by the partial differential equation

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial t} - rV} \tag{1.14}$$

where the constant r is a prevailing risk-free interest rate, and σ^2 is a measure of the volatility of the investor's return on this particular stock.

5. **Predator-Prey model:** In ecology, the population dynamics of competing species in which one is a predator with population x_2, and the other its prey with population x_1 is governed by the **Lotka-Volterra**

system of equations

$$\frac{dx_1}{dt} = Ax_1 - Bx_1x_2,$$
$$\frac{dx_2}{dt} = -Cx_2 + Dx_1x_2,$$

where A, B, C, D are positive constants.

Chapter 2

First Order Differential Equations

2.1 Separable Equations

Definition 4 (**Separable Equations**). A first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

is said to be separable if $f(x, y)$ can be written as the product of a function $g(x)$ that depends on only x and a function $h(y)$ that depends on only y such that

$$\frac{dy}{dx} = h(y)g(x) \tag{2.1}$$

Method of Solution

To solve

$$\frac{dy}{dx} = h(y)g(x), \tag{2.2}$$

separate variables to get

$$\frac{dy}{h(y)} = g(x)dx, \quad h(y) \neq 0.$$

Now integrate both sides:

$$\int \frac{dy}{h(y)} = \int g(x)dx, \quad h(y) \neq 0.$$

The result of the integration above yields the solution to the differential equation.

Example 5. Solve the following differential equations

$$\frac{dy}{dx} = \frac{1}{xy^3}$$

Solution.

$$\frac{dy}{dx} = \frac{1}{xy^3}.$$

$$\Longrightarrow y^3 dy = \frac{dx}{x},$$

$$\Longrightarrow \int y^3 dy = \int \frac{dx}{x},$$

$$\Longrightarrow \frac{1}{4}y^4 = \ln(x) + C_1,$$

$$\Longrightarrow y^4 = [\ln(x) + 4C_1] = [\ln(x) + C],$$

$$y = [\ln(x) + C]^{1/4}$$

Example 6. Solve the following differential equations

$$\frac{dy}{dx} = y(2 + \sin x) \tag{2.3}$$

Solution.

$$\frac{dy}{dx} = y(2 + \sin x),$$

$$\Longrightarrow \int \frac{dy}{y} = \int (2 + \sin x)dx,$$

$$\Longrightarrow \ln(y) = 2x - \cos(x) + C_1,$$

$$\Longrightarrow y = e^{2x - \cos x + C_1} = e^{C_1} e^{2x - \cos(x)},$$

$$\therefore y = Ke^{2x - \cos(x)}.$$

Example 7. Solve

$$(1 + x)dy - ydx = 0$$

Solution.

$$
\begin{aligned}
(1+x)dy - ydx &= 0 \\
\Longrightarrow \frac{dy}{y} &= \frac{1}{1+x}dx \\
\Longrightarrow \int \frac{dy}{y} &= \int \frac{1}{1+x}dx \\
\Longrightarrow \ln|y| &= \ln|1+x| + C_1 \\
\Longrightarrow \ln|\frac{y}{1+x}| &= C_1 \\
\Longrightarrow \frac{y}{1+x} &= \pm e^{C_1} = C \\
\Longrightarrow y &= C(1+x),
\end{aligned}
$$

where C is an arbitrary constant.

Example 8. Solve

$$\frac{dy}{dx} = y^2 - 9$$

Solution.

$$\int \frac{dy}{y^2-9} = \int dx, \quad y \neq \pm 3$$

Now, we use partial fractions to write:

$$\frac{1}{y^2-9} = \frac{1}{(y-3)(y+3)} \equiv \frac{A}{y-3} + \frac{B}{y+3},$$

$$\Longrightarrow 1 = A(y+3) + B(y-3),$$

$$y = 3: \quad 6A = 1, \Longrightarrow A = \frac{1}{6}$$

$$y = -3: \quad 1 = -6B, \Longrightarrow B = -\frac{1}{6}$$

$$\frac{1}{y^2-9} = \frac{1/6}{y-3} + \frac{-1/6}{y+3}$$

So the integral becomes

$$\int \left[\frac{1/6}{y-3} + \frac{-1/6}{y+3}\right] dy = \int dx$$

$$\frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y+3| = x + C_1$$

$$\ln\left|\frac{y-3}{y+3}\right| = 6x + C_2, \quad C_2 = 6C_1$$
$$\frac{y-3}{y+3} = \pm e^{C_2}e^{6x} = Ce^{6x}$$

Thus,

$$y = 3\left(\frac{1+Ce^{6x}}{1-Ce^{6x}}\right).$$

2.2 Linear First-Order Equations

A linear first order differential equation is an equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

To solve the differential equation, we first write it in a standard form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2.4}$$

Now suppose that equation (2.4) could be simplified by multiplying by some function, say $\mu(x)$, such that

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \tag{2.5}$$

and that

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x). \tag{2.6}$$

Then it will be easy to integrate the equation above and obtain a solution to the differential equation. Notice that equations (2.5) and (2.6) imply that we can compute the function $\mu(x)$:

$$\mu'(x) = \mu(x)P(x)$$
$$\implies \int \frac{d\mu}{\mu} = \int P(x)dx$$
$$\implies \mu(x) = e^{\int P(x)dx}. \tag{2.7}$$

So from equation (2.6) we get the solution

$$\mu(x)y = \int \mu(x)Q(x)dx$$
$$\therefore y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx, \tag{2.8}$$

where $\mu(x)$ is given by equation (2.7). The function $\mu(x)$ is called the **integrating factor of the differential equation.**

$$\tag{2.9}$$

Below is a summary of the approach outlined above.

a) Write the linear equation in a standard form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2.10}$$

b) Calculate the integrating factor:

$$\mu(x) = e^{\int P(x)dx}$$

c) Multiply (2.10) by $\mu(x)$ such that

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x)$$

d) Integrate the equation above to get

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx$$

Example 9. Solve the following IVP

$$\frac{dy}{dx} + 4y - e^{-x} = 0, \quad y(0) = \frac{4}{3}.$$

Solution.

$$\frac{dy}{dx} + 4y - e^{-x} = 0$$

$$\Longrightarrow \frac{dy}{dx} + 4y = e^{-x}$$

$$\mu(x) = e^{\int 4dx} = e^{4x}$$

where we have ignored the constant of integration because it cancels out in the subsequent steps. Thus

$$e^{4x}\frac{dy}{dx} + 4xe^{4x} = e^{3x}$$

$$\Longrightarrow \frac{d}{dx}\left[e^{4x}y\right] = e^{3x}$$

$$\implies e^{4x}y = \int e^{3x}dx = \frac{1}{3}e^{3x} + C$$
$$\implies y = \frac{1}{3}e^{-x} + Ce^{-4x}$$

Applying the initial condition, we have

$$x = 0, y = \frac{4}{3}, \implies \frac{4}{3} = \frac{1}{3} + C, \implies C = 1$$
$$\therefore y = \frac{1}{3}e^{-x} + e^{-4x}.$$

Example 10. Solve the differential equation

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

Solution. Re-write in a standard form to get

$$\frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x, \quad x \neq 0$$
$$\mu = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = e^{\ln x^{-4}} = x^{-4}$$
$$\implies x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x$$
$$\implies \frac{d}{dx}\left[x^{-4}y\right] = xe^x$$
$$\implies x^{-4}y = \int xe^x dx$$

Using integration by parts, we let

$$I = \int xe^x dx$$
$$\implies I = xe^x - e^x + C = e^x(x-1) + C$$

Thus

$$x^{-4}y = e^x(x-1) + C$$
$$\therefore y = x^4 e^x(x-1) + Cx^4.$$

EXERCISE 1. Solve

1) $\dfrac{dy}{dx} - y = e^{3x}$

2) $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

3) $\dfrac{dy}{dx} = \dfrac{y}{x} + 2x + 1$

2.3 Exact Equations

Definition 5. Consider the total differential of a continuous function $F(x, y)$ such that

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M(x,y)dx + N(x,y)dy \tag{2.11}$$

in a rectangle R, where

$$M(x,y) = \frac{\partial F}{\partial x}, \qquad N(x,y) = \frac{\partial F}{\partial y}. \tag{2.12}$$

Equation (2.11) is an **exact equation** in R if

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0. \tag{2.13}$$

Remark. Equation (2.13) shows that the solution to the exact equation is the level curves

$$F(x,y) = C$$

where C is an arbitrary constant. The following theorem gives a simple test to determine if a given differential form, $M(x,y)dx + N(x,y)dy$, is exact.

Theorem 1 (Test for Exactness). Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous in a rectangle R. Then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y) \tag{2.14}$$

holds for all (x,y) in R.

Example 11. Determine whether the following equation is exact. If it is, then solve it.

$$(2xy + 3)dx + (x^2 - 1)dy = 0.$$

Solution. Let

$$M = (2xy + 3), \qquad N = (x^2 - 1)$$
$$\Longrightarrow \frac{\partial M}{\partial y} = 2x, \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$
$$\Longrightarrow \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Therefore, the equation is exact. Now

$$M = \frac{\partial F}{\partial x} = 2xy + 3$$

$$\Longrightarrow F = \int (2xy + 3)dx = x^2y + 3x + g(y)$$

where $g(y)$ is any arbitrary function of y. The "constant" of integration in this case is a function of y, $g(y)$, because $F(x, y)$ is a function of both x and y and we integrated with respect to x. We need to solve for $g(y)$ by using the fact that $\partial F/\partial y = N$:

$$\frac{\partial F}{\partial y} = x^2 + g'(y) = x^2 - 1,$$

where we first differentiated the previous equation and equated the result to N. Thus

$$g'(y) = -1, \quad \Longrightarrow g(y) = -y$$

Thus, we get

$$F = x^2y + 3x - y.$$

So the implicit solution, $F(x, y) = C$, to the equation is given by

$$\boxed{x^2y + 3x - y = C}$$

where C is an arbitrary constant. In this case, we may solve explicitly for y to get

$$\boxed{y = \frac{C - 3x}{x^2 - 1}}.$$

Remark. (1) In most cases it is not necessary to look for an explicit solution. The implicit solution $F(x, y) = C$ suffices.

(2) It is not necessary to include a constant of integration after solving, for example, $g'(y) = -1$. This is because any additional constant would finally be combined with C to get another constant.

(3) One good check of your solution procedure, in the present case, is to ensure that the resulting expression for $g(y)$ is a function of only y. If not, then there is something wrong with your algebra or solution approach.

(4) An alternative approach to the solution is to start from

$$\frac{\partial F}{\partial y} = x^2 - 1,$$

such that

$$F = \int (x^2 - 1)dy = x^2y - y + h(x),$$

where the "constant" of integration in this case, $h(x)$, must be a function of only x since $F(x, y)$ is a function of x and y and we integrated with respect to y. Now

$$\frac{\partial F}{\partial x} = M \Longrightarrow 2xy + h'(x) = 2xy + 3$$

$$\Longrightarrow h'(x) = 3, \quad \Longrightarrow h(x) = 3x.$$

Note how the equation for $h(x)$ contains only the variable x, telling us that we are on the right track. Hence, we get

$$F(x, y) = x^2y - y + 3x$$

which is the same as the expression for F obtained using the first approach. So the solution becomes $x^2y - y + 3x = C$ as expected.

Example 12. Solve the initial value problem

$$\begin{cases} \left(ye^{xy} - \frac{1}{y}\right) dx + \left(xe^{xy} + \frac{x}{y^2}\right) dy = 0 \\ \\ y(1) = 1. \end{cases}$$

Solution. We first check whether the differential form is exact. Let

$$M = \left(ye^{xy} - \frac{1}{y}\right), \quad N = \left(xe^{xy} + \frac{x}{y^2}\right).$$

So we have

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + \frac{1}{y^2}$$

and

$$\frac{\partial N}{\partial x} = e^{xy} + xye^{xy} + \frac{1}{y^2}.$$

Thus, the equation is exact since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Let

$$M = \frac{\partial F}{\partial x} = ye^{xy} - \frac{1}{y}$$

$$\Longrightarrow F(x, y) = \int \left(ye^{xy} - \frac{1}{y} \right) dx = e^{xy} - \frac{x}{y} + g(y).$$

Now

$$N = \frac{\partial F}{\partial y} \quad \Longrightarrow xe^{xy} + \frac{x}{y^2} + g'(y) = xe^{xy} + \frac{x}{y^2},$$

$$\Longrightarrow g'(y) = 0, \quad \Longrightarrow g(y) = C_1,$$

where C_1 is a constant. Thus, the solution $F(x, y) = C_2$ becomes

$$e^{xy} - \frac{x}{y} + C_1 = C_2$$

$$\therefore \boxed{e^{xy} - \frac{x}{y} = C}$$

where $C = C_2 - C_1$ is a constant. Using the initial condition $y(1) = 1$ (i.e. $x = 1, y = 1$), we have

$$e^{(1)(1)} - \frac{1}{1} = C \quad \Longrightarrow C = e - 1.$$

Hence, the solution becomes

$$\boxed{e^{xy} - \frac{x}{y} = e - 1.}$$

2.4 Substitutions and Transformations

So far we have studied separable, linear and exact equations. Some differential equations do not fall into any these categories. However, after applying some transformations and substitutions, it is possible to turn some them into a form that we know how to solve using the previous methods. We will look at some these transformations in this section. Specifically, we will learn about **homogeneous equations**, then **equations of the form** $dy/dx = G(ax + by)$, and finally **Bernoulli equations**.

2.4.1 Homogeneous equations

Definition 6. Consider the differential equation

$$\frac{dy}{dx} = f(x, y). \tag{2.15}$$

If $f(x, y)$ can be expressed as a ratio y/x alone, then equation (2.15) is said to be a **homogeneous equation**.

Remark (Test)**.** If

$$f(xt, yt) = f(x, y) \quad \text{for all} \quad t \neq 0$$

then the equation is homogeneous.

Solution approach

From (2.15) we have

$$\frac{dy}{dx} = G\left(\frac{y}{x}\right),$$

where $G(y/x)$ is the transformed version of $f(x, y)$. Let

$$v = \frac{y}{x} \quad \Longrightarrow \frac{dy}{dx} = G(v).$$

We can differentiate $y = vx$ using the product rule to get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = G(v),$$

which is now a separable equation. To see this, we re-write the equation as

$$x\frac{dv}{dx} = G(v) - v$$

$$\Longrightarrow \frac{dv}{G(v) - v} = \frac{dx}{x}.$$

We then integrate both sides of the equation:

$$\boxed{\int \frac{dv}{G(v) - v} = \int \frac{dx}{x},}$$

to get v. Once v is obtained, we can use the transformation $v = y/x$ to get an implicit or explicit solution in terms of x and y.

Example 13. Determine whether the differential equation is homogeneous. If it is, then solve it using the method for homogeneous first order differential equations.

$$\frac{dy}{dx} = \frac{y^2}{xy + y^2}.$$

Solution. Let

$$f(x, y) = \frac{y^2}{xy + y^2}$$

Then

$$f(xt, ty) = \frac{(ty)^2}{(tx)(ty) + (ty)^2} = \frac{t^2y^2}{t^2xy + t^2y^2} = \frac{y^2}{xy + y^2} = f(x, y).$$

So the equation is homogeneous. We can express the right hand side as ratio y/x alone, by dividing both the numerator and the denominator by x^2 to get

$$\frac{dy}{dx} = \frac{(y/x)^2}{(y/x) + (y/x)^2}$$

Let $v = y/x$, then

$$\frac{dy}{dx} = \frac{v^2}{v + v^2}.$$

Differentiating $y = vx$ using the product rule results in

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{v^2}{v + v^2}$$

$$\Longrightarrow v + x\frac{dv}{dx} = \frac{v}{1 + v}$$

$$\Longrightarrow x\frac{dv}{dx} = \frac{v}{1 + v} - v = \frac{-v^2}{1 + v}$$

$$\Longrightarrow \int \frac{1 + v}{v^2} dv = -\int \frac{1}{x} dx$$

$$\Longrightarrow \int \left(\frac{1}{v^2} + \frac{1}{v}\right) dv = -\ln|x| + C$$

$$\Longrightarrow -\frac{1}{v} + \ln|v| = -\ln|x| + C$$

Substitute $v = y/x$ into the equation above to get

$$-\frac{x}{y} + \ln\left|\frac{y}{x}\right| = -\ln|x| + C$$

$$\implies -\frac{x}{y} + \ln|y| - \ln|x| = -\ln|x| + C$$

The term $-\ln|x|$ cancels out, so we get

$$-\frac{x}{y} + \ln|y| = C$$

$$\implies -x + y\ln|y| = Cy$$

$$\therefore \boxed{y\ln|y| = Cy + x.}$$

Example 14. Solve the differential equation

$$(y^2 - xy)dx + x^2dy = 0.$$

Solution. It is straightforward to check that the equation is not exact. Now re-writing the equation, we get

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2,$$

so we see that the equation is homogeneous. Let $v = y/x$

$$\implies \frac{dy}{dx} = v - v^2.$$

Differentiating $y = vx$ using the product rule and equating the result to the equation above, we get

$$v + x\frac{dv}{dx} = v - v^2$$

$$\implies x\frac{dv}{dx} = -v^2$$

$$\implies -\int \frac{1}{v^2}dv = \int \frac{1}{x}dx$$

$$\implies \frac{1}{v} = \ln|x| + C \qquad \implies v = \frac{1}{\ln|x| + C}$$

But $y = vx$, so we get

$$\boxed{y = \frac{x}{\ln|x| + C}.}$$

2.4.2 Equations of the form $dy/dx = G(ax + by)$

Solution Approach

Consider a differential equation of the form

$$\frac{dy}{dx} = G(ax + by). \tag{2.16}$$

Make the substitution

$$z = ax + by \tag{2.17}$$

$$\implies \frac{dz}{dx} = a + b\frac{dy}{dx} \quad \implies \frac{dy}{dx} = \frac{1}{b}\frac{dz}{dx} - \frac{a}{b} \tag{2.18}$$

So from (2.16) we get

$$\frac{1}{b}\frac{dz}{dx} - \frac{a}{b} = G(z)$$

$$\implies \frac{dz}{dx} - a = bG(z) \quad \implies \frac{dz}{dx} = a + bG(z),$$

which is now a separable equation. So separating variables and integrating, we get

$$\boxed{\int \frac{dz}{a + bG(z)} = \int dx.}$$

After solving for z, we then plug it into (2.17) for the solution.

Example 15. Solve the differential equation

$$\frac{dy}{dx} = (x + y + 2)^2$$

Solution. Let

$$z = x + y + 2$$

$$\implies \frac{dz}{dx} = 1 + \frac{dy}{dx} \quad \implies \frac{dy}{dx} = \frac{dz}{dx} - 1.$$

Substituting into the differential equation yields

$$\frac{dz}{dx} - 1 = z^2 \quad \implies \frac{dz}{dx} = 1 + z^2$$

$$\int \frac{1}{1 + z^2} dz = \int dx$$

$$\implies \tan^{-1} z = x + C,$$

$$\implies z = \tan(x + C).$$

$$\implies x + y + 2 = \tan(x + C)$$

$$\therefore \boxed{y = \tan(x + C) - x - 2.}$$

2.4.3 Bernoulli equation

Definition 7. A first-order differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{2.19}$$

where $P(x)$ and $Q(x)$ are continuous on an interval (a, b) and n is a real number is called a **Bernoulli equation**.

Solution Approach

Note that when $n = 0$ or 1, the equation is either linear or separable and can easily be solved. For other values of n, first divide through by y^n to get the equation

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x),$$

and let

$$v = y^{1-n}.$$

$$\Longrightarrow \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\Longrightarrow \frac{1}{1-n}\frac{dv}{dx} = y^{-n}\frac{dy}{dx}, \quad n \neq 1$$

Substituting into the modified differential equation, we get

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\Longrightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Note that the above equation is now a linear equation in terms of v since $(1-n)$ is just a real number. To see this, you can let $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ and get the linear first order equation:

$$\frac{dv}{dx} + P_1(x)v = Q_1(x).$$

Example 16. Solve the following equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^2$$

Solution.

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^2$$

The y^2 on the right-hand side tells us that we are dealing with a Bernoulli equation. Dividing through by y^2, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = x^2$$

Let

$$v = y^{-1}$$

$$\implies \frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

$$\implies -\frac{dv}{dx} + \frac{1}{x}v = x^2 \quad \implies \frac{dv}{dx} - \frac{1}{x}v = -x^2.$$

Note that the final equation is a linear equation with integrating factor

$$\mu = e^{-\int 1/x} = e^{-\ln|x|} = e^{\ln|1/x|} = \frac{1}{x}.$$

Multiplying through by the integrating factor and re-arranging, we get

$$\implies \frac{d}{dx}\left[\frac{1}{x}v\right] = -x$$

$$\implies \frac{1}{x}v = -\int x dx = -\frac{1}{2}x^2 + C.$$

$$\implies v = -\frac{1}{2}x^3 + Cx$$

Since $v = y^{-1}$, we get

$$y = \frac{1}{\frac{1}{2}x^3 + Cx} = \frac{2}{x^3 + C_1 x}, \quad C_1 = 2C.$$

Chapter 3

Linear Second Order Differential Equations

Second order differential equations arise in many applications such as the vibrations of mass-spring oscillators. This chapter presents the analytical techniques for solving such equations.

3.1 Homogeneneous Linear Equations

Consider the equation

$$ay''(t) + by'(t) + cy(t) = f(t). \tag{3.1}$$

If $f(t) \equiv 0$, the resulting equation is called a homogeneous ordinary differential equation such that

$$ay''(t) + by'(t) + cy(t) = 0. \tag{3.2}$$

The function $f(t)$ is called the " nonhomogeneity" of the general equation. The following approach is used to obtain a general solution to (3.2):

a) Assume a solution of the form $y = e^{rt}$ and substitute into (3.2) to get the **characteristic** or **auxilliary** equation:

$$ar^2 + br + c = 0. \tag{3.3}$$

b) Find the roots of (3.3) to get

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(i) If $b^2 - 4ac > 0$, the roots r_1 and r_2 are **real and distinct**. Then the solutions are $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. Combining the solutions gives the general solution

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where C_1 and C_2 are unknown constants, often determined from given conditions on the equation.

(ii) If $b^2 - 4ac = 0$, then r_1 and r_2 are **real repeated roots**, and the solutions are given by $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$. The general solution is then given by

$$y = C_1 e^{r_1 t} + C_2 te^{r_1 t}$$

(iii) If $b^2 - 4ac < 0$, then r_1 and r_2 are **complex conjugate roots** such that $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. The solutions are $y_1 = e^{\alpha t}\cos(\beta t)$, and $y_2 = e^{\alpha t}\sin(\beta t)$, and the general solution is given by

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

Example 17. Find the general soltution to the following equations

a) $y'' - y' - 2y = 0$

b) $y'' + 8y' + 16y = 0$

c) $y'' - 6y' + 10y = 0$

Solution. (a)

$$y'' - y' - 2y = 0$$

Substitute $y = e^{rt}$ into the equation to get

$$\begin{aligned} r^2 - r - 2 &= 0 \\ \Longrightarrow (r-2)(r+1) &= 0 \\ \Longrightarrow r &= 2, -1 \end{aligned}$$

So we get

$$\boxed{y = C_1 e^{2t} + C_2 e^{-t}.}$$

(b)

$$\begin{aligned} y'' + 8y' + 16y &= 0 \\ \Longrightarrow r^2 + 8r + 16 = 0, \quad \Longrightarrow (r+4)^2 &= 0, \\ \Longrightarrow r &= -4, -4, \end{aligned}$$

are repeated roots.

$$\therefore \boxed{y = C_1 e^{-4t} + C_2 te^{-4t}.}$$

(c)

$$
\begin{aligned}
& y'' - 6y' + 10y = 0 \\
& \Longrightarrow r^2 - 6r + 10 = 0, \\
& \Longrightarrow r = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i \\
& \Longrightarrow \alpha = 3, \quad \beta = 1.
\end{aligned}
$$

Thus, the general solution is given by

$$\boxed{y = C_1 e^{3t}\cos(t) + C_2 e^{3t}\sin(t).}$$

Example 18. Solve the initial value problem:

$$
\begin{aligned}
y'' + 2y' - 8y = 0, \\
y(0) = 3, \quad y'(0) = -12.
\end{aligned}
$$

Solution.

$$
\begin{aligned}
& y'' + 2y' - 8y = 0, \\
& \Longrightarrow r^2 + 2r - 8 = 0, \quad \Longrightarrow (r+4)(r-2) = 0 \\
& \Longrightarrow r = -4, 2 \\
& \Longrightarrow y = C_1 e^{-4t} + C_2 e^{2t}
\end{aligned}
$$

Applying the initial conditions:

$$y(0) = 3 \Longrightarrow C_1 + C_2 = 3$$

$$
\begin{aligned}
y'(0) = -12 \Longrightarrow -4C_1 + 2C_2 = -12 \\
\Longrightarrow 2C_1 - C_2 = 6
\end{aligned}
$$

Solving the two equations $C_1 + C_2 = 3$ and $2C_1 - C_2 = 6$ simultaneously, we get

$$C_1 = 3, \quad \text{and} \quad C_2 = 0.$$

$$\therefore \boxed{y = 3e^{-4t}.}$$

3.2 Non-homogeneous Equations

If $f(t) \neq 0$ in equation (3.1), the equation is said to be non-homogeneous. There are two main analytical methods for solving non-homogeneous equations: The method of undetermined coefficients and the method of variation of parameters.

3.2.1 The Method of undetermined coefficients

By this method, we first determine the solution to the homogeneous form of the equation (say y_h) as discussed in the previous section. Depending on the form of the non-homogeneous term $f(t)$, there are various techniques of obtaining the **particular solution** (say y_p) to the nonhomogeneous part of the equation as explained below. Since the differential equation is linear, the general solution, y, is obtained by summing the homogeneous and particular solutions to get

$$y = y_h + y_p,$$

by employing the so called **superposition principle**.

Solution Approach

(a) To find a particular solution to the differential equation

$$ay'' + by' + cy = kt^m e^{rt}, \tag{3.4}$$

assume the following form for the particular solution:

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt} \tag{3.5}$$

with

(i) $s = 0$ if r is not a root of the **characteristic equation**

(ii) $s = 1$ if r is a root of the characteristic equation

(iii) $s = 2$ if r is a double root of the characteristic equation

(b) To find a particular solution to the differential equation of the form

$$ay'' + by' + cy = kt^m e^{\alpha t}\cos(\beta t)$$

or

$$ay'' + by' + cy = kt^m e^{\alpha t}\sin(\beta t),$$

assume the following form for the particular solution:

$$\begin{aligned} y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + \\ t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t) \end{aligned} \tag{3.6}$$

with

(i) $s = 0$ if $\alpha + i\beta$ is not a root of the characteristic equation.

(ii) $s = 1$ if $\alpha + i\beta$ is a root of the characteristic equation.

In the above, the $A's$ and $B's$ are constants.

Example 19. Find the particular solution of the following equations

(a) $y'' - y = -11t + 1$

(b) $2y'' + y = 9e^{2t}$

(c) $y'' - y = t\sin(t)$

Solution. (a)

$$y'' - y = -11t + 1$$

Comparing this equation to the form (3.4), we find that $r = 0$ and $m = 1$. Now the characteristic equation is given by

$$r^2 - 1 = 0,$$

and the roots of the characteristic equation are given by

$$r = 1, -1.$$

So the homogeneous solution is given by

$$y_h = C_1e^t + C_2e^{-t}.$$

In this case $s = 0$ in (3.5), since $r = 0$ is not a root of the characteristic equation (which we found to be 1 and -1). So assume the following form for the particular solution:

$$y_p = At + B.$$

This implies that we expect y_p to satisfy the differential equation since it's a solution. Now

$$y_p' = A \quad \text{and} \quad y_p'' = 0.$$

Substituting into the original equation results in

$$-y_p = -11t + 1$$

$$\Longrightarrow -At - B = -11t + 1$$

Comparing coefficients we get

$$A = 11, \quad B = -1.$$

So the particular solution is given by

$$\boxed{y_p = 11t - 1.}$$

Remark. (1) You may check to see if your answer is correct by substituting y_p into the differential equation.

(2) In this case, we are asked to find only the particular solution. But the general solution of the equation, if desired, is given by $y = y_h + y_p$:

$$y = C_1 e^t + C_2 e^{-t} + 11t - 1.$$

(b)

$$y'' - y = t\sin(t)$$

From the general form (3.6), we see that $\alpha = 0$ and $\beta = 1$ so that $\alpha \pm i\beta = \pm i$. Substituting $y = e^{rt}$ into the equation results in

$$\Longrightarrow r^2 - 1 = 0 \Longrightarrow r = \pm 1.$$

Thus, $s = 0$ in (3.6) and $m = 1$, so the form of the particular solution is $y_p = (A_1 t + A_0)\cos(t) + (B_1 t + B_0)\sin(t)$. For simplicity, we avoid using constants with subscripts and instead use the following form for the particular solution:

$$y_p = (At + B)\cos(t) + (Ct + D)\sin(t).$$

$$\Longrightarrow y_p' = (A + D + Ct)\cos(t) + (-At - B + C)\sin(t)$$

$$\Longrightarrow y_p'' = (-At - B + 2C)\cos(t) + (-2A - D - Ct)\sin(t)$$

Substitute into the original equation to get

$$(-2At - 2B + 2C)\cos(t) + (-2A - 2D - 2Ct)\sin(t) = t\sin(t)$$

Comparing coefficients yields:

$$-2C = 1 \Longrightarrow C = -\frac{1}{2}$$

$$-2A = 0 \Longrightarrow A = 0$$

$$-2A - 2D = 0 \Longrightarrow D = 0$$

$$-2B + 2C = 0 \Longrightarrow B = C = -\frac{1}{2}$$

Thus,

$$\boxed{y_p = -\frac{1}{2}\cos(t) - \frac{1}{2}t\sin(t).}$$

(c)

$$2y'' + y = 9e^{2t}$$

$$\implies 2r^2 + 1 = 0 \quad \implies r = \pm\frac{i}{\sqrt{2}}.$$

Let

$$y_p = Ae^{2t}$$

$$\implies y_p' = 2Ae^{2t}$$

$$\implies y_p'' = 4Ae^{2t}$$

Substitute into original equation to get

$$8Ae^{2t} + Ae^{2t} = 9e^{2t}$$

$$\implies 9A = 9 \quad \implies A = 1.$$

Thus

$$\boxed{y_p = e^{2t}.}$$

Example 20. Find the form of the particular solution to

$$y'' + 2y' - 3y = f(t)$$

where $f(t)$ is

(a) $7\cos(3t)$
(b) $2te^t\sin(t)$
(c) $t^2\cos(\pi t)$
(d) $5e^{-3t}$
(e) $3te^t$
(f) t^2e^t

Solution.

$$y'' + 2y' - 3y = 0$$

$$\implies r^2 + 2r - 3 = 0$$

$$(r+3)(r-1) = 0$$

$$\implies r = 1, -3.$$

(a) $f(t) = 7\cos(3t)$

$$y_p = A\cos(3t) + B\sin(3t)$$

(b) $f(t) = 2te^t\sin(t)$. The form of the particular solution is

$$y_p = (At+B)e^t\cos(t) + (Ct+D)e^t\sin(t).$$

Remark. Here $s = 0$ in (3.6) because $1 + i$ is not a root of the characteristic equation. The factors $(At + B)$ and $(Ct + D)$ are needed because $m = 1$ as a result of the linear term $2t$ in $f(t)$.

(c) $f(t) = t^2 \cos(\pi t)$

$$y_p = (At^2 + Bt + C)\cos(\pi t) + (Dt^2 + Et + F)\sin(\pi t).$$

Remark. Here $s = 0$ in (3.6) because $0 + i\pi$ (or $i\pi$) is not a root of the characteristic equation. The factors $(At^2 + Bt + C)$ and $(Dt^2 + Et + F)$ are needed because $m = 2$ as a result of the quadratic term t^2 in $f(t)$.

(d) $f(t) = 5e^{-3t}$
Note that $r = -3$ is a root of the auxilliary equation, so

$$y_p = Ate^{-3t}.$$

Remark. In this case, $s = 1$ in (3.5) because -3 is a root of the characteristic equation. The constant A is needed because $m = 0$ as a result of the constant 5 in $f(t)$.

(e) $f(t) = 3te^t$
Note that $r = 1$ is a root of the auxilliary equation so

$$y_p = t(At + B)e^t.$$

Remark. In this case, $s = 1$ in (3.5) because 1 is a root of the characteristic equation. The factor $(At + B)$ is needed because $m = 1$ as a result of the linear term $3t$ in $f(t)$.

(f) $f(t) = t^2e^t$

$$y_p = t(At^2 + Bt + C)e^t.$$

Remark. As in the previous case, $s = 1$ in (3.5) because 1 is a root of the characteristic equation. The factor $(At^2 + Bt + C)$ is needed because $m = 2$ as a result of the quadratic term t^2 in $f(t)$.

Example 21. Find the form of the particular solution to

$$y'' - 2y' + y = f(t)$$

where $f(t)$ is the same as those in example (20):

(a) $7\cos(3t)$ (c) $t^2\cos(\pi t)$ (e) $3te^t$

(b) $2te^t\sin(t)$ (d) $5e^{-3t}$ (f) t^2e^t

3.2.2 The Method of variation of parameters

In the previous section we developed the method of undetermined coefficients for finding particular solutions to the nonhomogeneous equations

$$ay''(t) + by'(t) + cy(t) = f(t), \tag{3.7}$$

where $f(t)$ must be of a particular functional form: polynomial, exponential, sine, cosine or a combination of these. For instance, the method could not be used to find a particular solution if $f(t) = \tan(t)$. Here, we present the method of variation of parameters which is more general than undetermined coefficients. To proceed, we revisit some of our previous elborations by first solving the corresponding homogeneous equation

$$ay''(t) + by'(t) + cy(t) = 0, \tag{3.8}$$

for the complementary solutions $y_1(t)$ and $y_2(t)$ to get the general solution

$$y_h = c_1y_1 + c_2y_2, \tag{3.9}$$

where c_1 and c_2 are constants. To obtain a particular solution y_p to (3.7) by the method of variation of parameters, we assume that c_1 and c_2 are now functions of the independent variable t such that $c_1 = w_1(t)$ and $c_2 = w_2(t)$ and (3.9) becomes

$$y_p = w_1(t)y_1 + w_2(t)y_2. \tag{3.10}$$

The idea by Lagrange, who invented this method, is to determine the funcitons $w_1(t)$ and $w_2(t)$ such that (3.10) is a particular solution to (3.7). Since there are two functions to determine, we expect to have two equations (or conditions) to solve. In other for (3.10) to satisfy (3.7), we need to find the derivatives of y_p. So from (3.10), we have

$$y_p' = w_1'y_1 + w_1y_1' + w_2'y_2 + w_2y_2'$$

$$\implies y_p' = (w_1'y_1 + w_2'y_2) + (w_1y_1' + w_2y_2').$$

To simplify the algebra after taking a second derivative of y_p, we assume that

$$w_1'y_1 + w_2'y_2 = 0, \tag{3.11}$$

which is actually one of the equations we would need to eventually solve for w_1 and w_2. This assumption implies that

$$y_p' = w_1 y_1' + w_2 y_2', \tag{3.12}$$

$$y_p'' = w_1' y_1' + w_1 y_1'' + w_2' y_2' + w_2 y_2''. \tag{3.13}$$

Substituting (3.10), (3.12) and (3.13) into (3.7) yields

$$a y_p''(t) + b y_p'(t) + c y_p(t) = f(t)$$

$$\Longrightarrow a(w_1 y_1'' + w_1' y_1' + w_2 y_2'' + w_2' y_2') + b(w_1 y_1' + w_2 y_2') + c(w_1 y_1 + w_2 y_2) = f(t),$$

$$\Longrightarrow w_1[a y_1'' + b y_1' + c y_1] + w_2[a y_2'' + b y_2' + c y_2] + a(w_1' y_1' + w_2' y_2') = f(t).$$

Since y_1 and y_2 are solutions to the homogeneous equation (3.8), the expressions in square brackets vanish and so

$$a(w_1' y_1' + w_2' y_2') = f(t)$$

$$w_1' y_1' + w_2' y_2' = \frac{f(t)}{a}. \tag{3.14}$$

Thus, equations (3.11) and (3.14) provides the two equations needed to be solved simultaneously for w_1 and w_2. We rewrite them below for convenience:

$$w_1' y_1 + w_2' y_2 = 0, \tag{3.15}$$

$$w_1' y_1' + w_2' y_2' = \frac{f(t)}{a}. \tag{3.16}$$

From (3.15) we have

$$w_1' = -\frac{y_2}{y_1} w_2', \tag{3.17}$$

and from (3.16) we get

$$\left(-\frac{y_2}{y_1} w_2'\right) y_1' + w_2' y_2' = \frac{f(t)}{a}$$

$$\Longrightarrow w_2' \left(y_2' - \frac{y_1' y_2}{y_1}\right) = \frac{f(t)}{a}$$

$$\Longrightarrow w_2'(y_1 y_2' - y_1' y_2) = \frac{f(t) y_1}{a}$$

$$\Longrightarrow w_2' = \frac{f(t) y_1}{a[y_1 y_2' - y_1' y_2]}. \tag{3.18}$$

From (3.17) we get

$$w_1' = -\left(\frac{y_2}{y_1}\right) \cdot \frac{f(t)y_1}{a[y_1y_2' - y_1'y_2]},$$
$$\implies w_1' = -\frac{f(t)y_2}{a[y_1y_2' - y_1'y_2]}. \tag{3.19}$$

It is important to note that the expression in square bracket, $W(y_1, y_2) = y_1y_2' - y_1'y_2 \neq 0$, since it is the Wronskian of the linearly independent solutions y_1 and y_2. Integrating (3.18) and (3.19) results in

$$w_1 = -\int \frac{f(t)y_2}{a[y_1y_2' - y_1'y_2]}dt, \quad \text{and} \quad w_2 = \int \frac{f(t)y_1}{a[y_1y_2' - y_1'y_2]}dt. \tag{3.20}$$

Hence, the particular solution is given by (3.10):

$$y_p = w_1(t)y_1 + w_2(t)y_2$$

where the variables $w_1(t)$ and $w_2(t)$ are given by (3.20).

Summary: Method of Variation of Parameters

In other to determine a particular solution y_p to the second order differential equation with constant coefficients:

$$ay''(t) + by'(t) + cy(t) = f(t),$$

execute the following steps:

(1) Determine the two linearly independent solutions y_1 and y_2 to the corresponding homogeneous equation $ay''(t) + by'(t) + cy(t) = 0$, and let

$$y_p = w_1(t)y_1(t) + w_2(t)y_2(t).$$

(2) Determine $w_1(t)$ and $w_2(t)$ by simultaneously solving equations (3.11) and (3.14):

$$\left.\begin{aligned} \boldsymbol{w_1'y_1 + w_2'y_2} &= \mathbf{0} \\ \boldsymbol{w_1'y_1' + w_2'y_2'} &= \boldsymbol{\tfrac{f(t)}{a}} \end{aligned}\right\} \tag{3.21}$$

to get $w_1(t)$ and $w_2(t)$ as in (3.20).

(3) Finally, substitute $w_1(t)$ and $w_2(t)$ into $y_p = w_1(t)y_1(t) + w_2(t)y_2(t)$ to get a particular solution.

Remark. It is not advisable to memorize the formulas for w_1 and w_2 in (3.20) since they can easily be derived from system (3.21).

Example 22. Solve the following equation

$$y'' + y = \sec(t)$$

Solution. Note that the method of undetermined coefficients cannot be used here because of the nonhomogeneity $f(t) = \sec(t)$. So we employ the method of variation of parameters. Considering the corresponding homogeneous equation $y'' + y = 0$, we find, after substituting $y = e^{rt}$, the characteristic equation

$$r^2 + 1 = 0, \quad \Longrightarrow r^2 = -1, \quad r = \pm i.$$

So the solution to the homogeneous equation is

$$y_h = c_1 \cos t + c_2 \sin t.$$

Let the particular solution be

$$y_p = w_1(t) \cos t + w_2(t) \sin t$$

and solve the system:

$$\begin{cases} w_1' \cos t + w_2' \sin t = 0, \\ w_1'(-\sin t) + w_2'(\cos t) = \frac{\sec t}{1}. \end{cases}$$

Multiplying the first equation by $\sin t$ and the second by $\cos t$ and summing them gives

$$w_2'(\sin^2 t + \cos^2 t) = \cos t \cdot \sec t = 1,$$

$$\Longrightarrow w_2' = 1, \quad \Longrightarrow w_2(t) = t,$$

since $\sin^2 t + \cos^2 t = 1$ and $\sec t = 1/\cos t$. From the first equation in the system, we get

$$w_1' = -\frac{\sin t}{\cos t} w_2' = -\tan t,$$

$$w_1 = -\int \tan t dt = -\int \frac{\sin t}{\cos t} dt = \ln|\cos t|.$$

In the integrations above, we let the constants of integration be zero because we're looking for just one particular solution. Substituting w_1 and w_2 into the form of the particular solution, we get

$$\boxed{y_p = \cos t \ln|\cos t| + t \sin t.}$$

The general solution becomes $y = y_h + y_p$:

$$\boxed{y = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t.}$$

3.3 Equations with Variable Coefficients

Second order differential equations with variable coefficients are of the form

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t), \tag{3.22}$$

where the coefficients are now functions of the independent variable t. Unlike equations with constant coefficients which are amenable to the method of undetermined coefficients or variation of paramaters, there are no straighforward methods for constructing explicit solutions in this case. However, there are some interesting aspects of variable-coefficient equations that we would like to study.

There is a theorem for the existence and uniqueness of solutions for variable-coefficient equations which we will state. We will also take a look at special equations with variable coefficients (known as Cauchy-Euler or Equidimensional equations) that can be solved using an approach similar to what was done for equations with constant coefficients. Finally, we will state the theorem on the method of "Reduction of Order" which states that, given one solution, say y_1, to the homogeneous variable coeffient equation $a(t)y''(t)+b(t)y'(t)+c(t)y(t) = 0$, one can find a second linearly independent solution, say y_2.

To proceed, we first write (3.22) in standard form by dividing through by $a(t)$ to get an equation of the form

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t). \tag{3.23}$$

Of course, this supposes that $a(t) \neq 0$ otherwise the whole structure of the equation changes from a second order to a first order equation. The following theorem guarantees the existence and uniqueness of solutions to equations with variable coefficients:

Theorem 2 (**Existence and Uniqueness of Solutions**)**.** Let the governing functions $p(t)$, $q(t)$, and $f(t)$ in the equation $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ be continuous on an interval I containing the point t_0. Then for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the interval I to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t); \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

The theorem may be used to determine the largest interval for which a unique solution to the differential equation exists, as demonstrated by the following example.

Example 23. Consider the initial value problem

$$(t-2)y''(t) + y'(t) + t\sqrt{t}y(t) = \ln t; \quad y(1) = 2, \quad y'(1) = -3.$$

Use Theorem 2 to determine the largest interval for which a unique solution exists.

Solution. Writing the equation in standard form gives

$$y''(t) + \frac{1}{t-2}y'(t) + \frac{t\sqrt{t}}{t-2}y(t) = \frac{\ln t}{t-2},$$

such that

$$p(t) = \frac{1}{t-2}, \quad q(t) = \frac{t\sqrt{t}}{t-2}, \quad f(t) = \frac{\ln t}{t-2}.$$

We see that the functions are all continuous on the intervals $0 < t < 2$ and $2 < t < \infty$. However, the interval $2 < t < \infty$ does not contain the initial point $t_0 = 1$ so we neglect it. Thus, the Theorem guarantees a unique solution in the interval $0 < t < 2$.

Remark. Note from the example above that $q(t)$ does not exist for $t < 0$ and does $f(t)$ exist for $t \leq 0$. Also note that the Theorem guarantees the existence of a unique solution but does not provide the solution nor the interval of existence of the solution.

Lemma 1 (**Linear Dependence of Solutions**)**.** If $y_1(t)$ and $y_2(t)$ are any two solutions to

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \tag{3.24}$$

on an interval I where $p(t)$ and $q(t)$ are continuous and if the Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

is zero at any point t of I, then y_1 and y_2 are linearly depedent on I.

Remark. Because we are interested in linearly independent solutions to differential equations, as expounded upon previously, Lemma 1 provides a simpler approach of determining if two solutions to a variable-coefficient equation are linearly dependent or independent. They're linearly independent if the Wronskian, $W(y_1, y_2) \neq 0$.

3.3.1 Representation of Solutions to IVPs

As in equations with constant coefficients, the following Theorem guarantees that two linearly independent solutions, y_1 and y_2, to equation (3.24) with given initial conditions can be combined to get a general solution.

Theorem 3. Suppose $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions to the IVP

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1, \tag{3.25}$$

on an interval I containing the point t_0, then unique constants c_1 and c_2 can always be found such that the general solution $y = c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions for any Y_0 and Y_1.

As stated in our introduction to this section, the biggest challenge with variable-coefficient equations is the lack of a general method for explicitly obtaining the solutions $y_1(t)$ and $y_(t)$ to (3.25); apart from some special cases. However, once these solutions are found, Theorem 3 assures us of a general solution that will always satisfy the initial conditions. Secondly, once these solutions have been obtained for the homogeneous equation (3.24), the method of variation of parameters (for variable-coefficient equations) can be used to solve the associated nonhomogeneous equation for a particular solutions as stated below.

Theorem 4 (Variation of Parameters). Suppose y_1 and y_2 are two linearly independent solutions to (3.24) on an interval I, then a particular solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t), \tag{3.26}$$

is given by

$$y_p = w_1(t)y_1(t) + w_2(t)y_2(t)$$

where w_1 and w_2 are determined from the equations

$$\left.\begin{aligned} w_1'y_1 + w_2'y_2 &= 0, \\ w_1'y_1' + w_2'y_2' &= f(t), \end{aligned}\right\} \tag{3.27}$$

with solution

$$w_1 = -\int \frac{f(t)y_2}{W(y_1, y_2)}dt, \quad \text{and} \quad w_2 = \int \frac{f(t)y_1}{W(y_1, y_2)}dt. \tag{3.28}$$

Remark. As adviced before, equations (3.28) can easily be derived from system (3.27) so it's not advisable to memorize them. Also keep in mind that, as opposed to equations with constant coefficients, system (3.27) is obtained from the normalized equation (3.26).

Let's do a few examples of finding the general solution, $y = y_h + y_p$, using the method of variation of parameters for which the linearly independent solutions y_1 and y_2 are given. After which we will tackle the special case of Cauchy-Euler equations for which y_1 and y_2 can be found explicitly.

Example 24. Find a general solution to the following IVPs using variation of parameters, given that y_1 and y_2 are solutions to the associated homogeneous equation.

(a) $ty'' - (t+1)y' + y = t^2,$

$$y_1 = e^2, \quad y_2 = t + 1.$$

(b) $ty'' + (5t-1)y' - 5y = t^2e^{-5t},$

$$y_1 = 5t - 1, \quad y_2 = e^{-5t}.$$

Solution. (a)
$ty'' - (t+1)y' + y = t^2, \quad y_1 = e^2, \quad y_2 = t + 1.$
It is easy to check that y_1 and y_2 are indeed solutions to the homogeneous equation $ty'' - (t+1)y' + y = 0$:

$$\begin{aligned} ty_1'' - (t+1)y_1' + y_1 &= te^t - (t+1)e^t + e^t \\ &= te^t - te^t - e^t + e^t \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} ty_2'' - (t+1)y_2' + y_2 &= t(0) - (t+1)(1) + (t+1) \\ &= 0 - t - 1 + t + 1 \\ &= 0. \end{aligned}$$

Writing the equation in standard form gives

$$y'' - \frac{(t+1)}{t}y' + \frac{1}{t}y = t, \quad t \neq 0.$$

Thus, $f(t) = t$ and

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = e^t - te^t - e^t = -te^t.$$

Thus, from (3.28), we have

$$w_1 = -\int \frac{f(t)y_2}{W(y_1, y_2)} dt = \int \frac{t(t+1)}{-te^t} dt = \int (t+1)e^{-t} dt.$$

Integrating by parts, let

$$u = t + 1, \quad dv = e^{-t} dt$$

$$\Longrightarrow du = dt, \quad \text{and} \quad v = \int e^{-t} dt = -e^{-t},$$

$$\Longrightarrow w_1(t) = -(t+1)e^{-t} + \int e^{-t} dt$$

$$\Longrightarrow w_1(t) = -te^{-t} - e^{-t} - e^{-t} = -te^{-t} - 2e^{-t} = -e^{-t}(t+2).$$

$$\Longrightarrow w_1(t) = -e^{-t}(t+2).$$

Also, we have $w_2(t)$:

$$w_2(t) = \int \frac{f(t)y_1}{W(y_1, y_2)} dt = \int \frac{te^t}{-te^t} dt = -t.$$

Again, it's redundant to keep the integration constants after integrating for w_1 and w_2. Finally, we get the particular solution:

$$\begin{aligned} y_p &= w_1(t)y_1(t) + w_2(t)y_2(t) \\ &= -e^{-t}(t+2) \cdot e^t - t(t+1) \\ &= -t - 2 - t^2 - t = -t^2 - 2t - 2. \end{aligned}$$

Hence, the general solution, $y = y_h + y_p$, is given by

$$\boxed{y = c_1 e^t + c_2(t+2) - (t^2 + 2t + 2)}.$$

So far, we have seen that given two linearly independent solutions y_1 and y_2 to the homogeneous equation of a variable-coefficient equation, we can use the method of variation of parameters to determine a particular solution and hence the general solution. This is because we have no technique of determining y_1 and y_2 in the general case of a variable-coefficient equation. However, given one of the solutions, say y_1 or y_2, the method of reduction of order can be used to find the other solution. We describe this next.

3.3.2 Reduction of Order

The method of reduction of order is used to construct a second, linearly independent solution, say y_2, from a known solution, say y_1. The following Theorem formalizes this procedure.

Theorem 5. Suppose $y_1(t)$ is a non-zero solution to the homogeneous differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \tag{3.29}$$

in an interval I. Then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2} dt, \tag{3.30}$$

is a second, linearly independent solution.

Proof. Since $y_h = cy_1(t)$ is a solution, we assume a second solution to be

$$y_2 = w(t)y_1(t)$$

$$\Longrightarrow y_2' = w'y_1 + wy_1'$$

$$\Longrightarrow y_2'' = w''y_1 + w'y_1' + w'y_1' + wy_1''$$

Substituting into (3.29) gives

$$(w''y_1 + w'y_1' + w'y_1' + wy_1'') + p(t)(w'y_1 + wy_1') + q(t)wy_1 = 0,$$

$$(wy_1'' + pwy_1' + qwy_1) + w''y_1 + (2w'y_1' + pw'y_1) = 0,$$

$$(y_1'' + py_1' + qy_1)w + w''y_1 + (2y_1' + py_1)w' = 0.$$

The expression in the first bracket vanishes since y_1 is a solution to (3.29). Thus

$$w''y_1 + (2y_1' + py_1)w' = 0. \tag{3.31}$$

Equation (3.31) can be reduced to a first order equation by letting

$$v = w'.$$

$$\Longrightarrow v'y_1 + (2y_1' + py_1)v = 0,$$

$$\Longrightarrow \int \frac{dv}{v} = -\int \frac{(2y_1' + py_1)}{y_1} dt,$$

$$\Longrightarrow \ln|v| = -\int \left(\frac{2y_1'}{y_1} + p\right) dt,$$

$$\Longrightarrow \ln|v| = -2\ln|y_1| - \int p(t)dt,$$

$$\Longrightarrow \ln|v| = \ln[y_1(t)]^{-2} - \int p(t)dt,$$

$$\Longrightarrow v = [y_1(t)]^{-2} \cdot e^{-\int p(t)dt} = \frac{e^{-\int p(t)dt}}{[y_1(t)]^2}$$

But $v = w'$, so we have

$$w(t) = \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2} dt.$$

Now $y_2(t) = y_1(t)w(t)$,

$$\therefore y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2} dt.$$

□

Example 25. Given a non-trivial solution y_1 of the following differential equation, use reduction of order to find a second linearly independent solution.

$$t^2y'' - 2ty' - 4y = 0, \quad y_1 = t^{-1}, \qquad t > 0.$$

Solution. Writing the equation in standard form gives

$$y'' - \frac{2}{t}y' - \frac{4}{t^2}y = 0,$$

so we have $p(t) = -2t^{-1}, \quad q(t) = -4t^{-2}$.

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{[y_1(t)]^2} dt = t^{-1} \int \frac{e^{-\int(-2t^{-1})dt}}{t^{-2}} dt,$$

$$= t^{-1} \int t^2 e^{2\ln t} dt = t^{-1} \int t^4 dt,$$

$$= t^{-1} \cdot \frac{t^5}{5} = \frac{1}{5}t^4.$$

Thus, the two linearly independent solutions to the equation are $y_1 = t^{-1}$ and $y_2 = t^4/5$.

We next take a look at special equations called Cauchy-Euler or Euler or Equidimensional equations for which it is possible for to determine both linearly independent solutions y_1 and y_2. In order words, they are special variable-coefficient equations that can be solved explicitly.

3.3.3 Cauchy-Euler (or Equidimensional) Equations

Definition 8. A Cauchy-Euler (or Equidimensional) equation is an equation of the form

$$at^2y''(t) + bty'(t) + cy(t) = f(t), \tag{3.32}$$

where $a(\neq 0)$, b and c are constants.

Remark. Writing the Equidimensional equation in standard form yields

$$y''(t) + \frac{bt}{at^2}y'(t) + \frac{c}{at^2}y(t) = \frac{f(t)}{at^2},$$

$$\implies y''(t) + \frac{b}{at}y'(t) + \frac{c}{at^2}y(t) = \frac{f(t)}{at^2}.$$

We see that the governing functions are continuous for $t \neq 0$. Thus, Theorem 2 guarantees a unique solution for $t < 0$ or $t > 0$, subject to other restrictions imposed by the function $f(t)$.

In what follows, we first construct a **general solution** y_h to the homogeneous equation

$$at^2y''(t) + bty'(t) + cy(t) = 0. \tag{3.33}$$

Later, using the method of **variation of parameters**, we would determine a particular solution y_p to the nonhomogeneous equation (3.32). Once y_h and y_p have been obtained, we can easily construct the general solution $y = y_h + y_p$.

Solution Approach to the homogeneous case

Assume a solution of the form

$$y(t) = t^r, \qquad t \neq 0, \tag{3.34}$$

where r is a constant. Then

$$y'(t) = rt^{r-1}, \tag{3.35}$$

$$\implies y''(t) = r(r-1)t^{r-2}. \tag{3.36}$$

Substituting (3.34)-(3.36) into (3.33) gives

$$at^2[r(r-1)t^{r-2}] + bt[rt^{r-1}] + ct^r = 0,$$

$$\Longrightarrow ar(r-1)t^r + brt^r + ct^r = 0,$$
$$\Longrightarrow [ar(r-1) + br + c]\, t^r = 0.$$

Since $t^r \neq 0$, we get the associated **characteristic equation**:

$$\boxed{\boldsymbol{ar(r-1) + br + c = 0,}} \tag{3.37}$$

such that

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}, \tag{3.38}$$

with discriminant

$$D = \sqrt{(b-a)^2 - 4ac}. \tag{3.39}$$

The solution to the characteristic equation (3.37) yields three different cases depending on the discrimiant: real distinct roots, repeated roots, and complex roots; as in the case for constant coefficient equations.

(a) For **real distinct** roots, r_1 and r_2, the solution of the homogeneous equation is given by

$$\boldsymbol{y_h = c_1 t^{r_1} + c_2 t^{r_2}.}$$

(b) For **repeated** roots, the solution of the homogeneous equation is

$$\boldsymbol{y_h = c_1 t^r + c_2 t^r \ln t, \qquad t > 0,}$$

and this can be verified by directly substituting the solution into the homogeneous equation.

(c) For a **complex** root, $r = \alpha + i\beta$ (or $r = \alpha - i\beta$), we can write

$$t^{\alpha+i\beta} = t^\alpha \cdot t^{i\beta} = t^\alpha \cdot e^{i\beta \ln t},$$

since $t = e^{\ln t}$. So we have

$$t^{\alpha+i\beta} = t^\alpha[\cos(\beta \ln t) + i \sin(\beta \ln t)],$$

after employing Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. Using the real and imaginary parts gives the two independent solutions

$$y_1 = t^\alpha \cos(\beta \ln t), \quad y_2 = t^\alpha \sin(\beta \ln t),$$

such that

$$\boldsymbol{y_h = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).}$$

Example 26. Determine the general solution to the following equation

$$at^2y'' + by' + cy = 0, \quad t > 0,$$

where

(a) $a = 1,\ b = 1,\ c = -1.$

(b) $a = 1,\ b = -1,\ c = 5.$

(c) $a = 1,\ b = 5,\ c = 4.$

Solution. (a) $t^2y'' + y' - y = 0$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

$$\Longrightarrow r = \frac{0 \pm \sqrt{4}}{2} = \pm 1.$$

So we have two distinct roots, $r_1 = 1,\ r_2 = -1$ Therefore

$$y_h = c_1 t + c_2 t^{-1}.$$

(b) $t^2y'' - y' + 5y = 0$. We have the roots:

$$r = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

These are complex roots, $\alpha \pm i\beta$, with $\alpha = 1$ and $\beta = 2$. Hence

$$y_h = c_1 t\cos(2\ln t) + c_2 t\sin(2\ln t).$$

(c) $t^2y'' + 5y' + 4y = 0$. We have the roots:

$$r = \frac{-4 \pm \sqrt{16 - 16}}{2} = -2.$$

So the roots are repeated, $r = 2, 2$, so we get

$$h_h = c_1 t^{-2} + c_2 t^{-2}\ln t.$$

Example 27. Solve the initial value problem

$$9t^2y'' + 15ty' + y = 0; \quad y(1) = 1, \quad y'(1) = -\frac{2}{3}.$$

Solution. The roots are given by

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 4(9)}}{18} = \frac{-6}{18} = -\frac{1}{3}$$

So we have repeated roots $r = -1/3$, and the general solution is given by

$$y = c_1 t^{-1/3} + c_2 t^{-1/3} \ln t.$$

Now,

$$y(1) = 1, \quad \Longrightarrow c_1 = 1.$$

$$y' = -\frac{1}{3}c_1 t^{-4/3} - \frac{1}{3}c_2 t^{-4/3} \ln t + c_2 t^{-1/3} \cdot \frac{1}{t}$$

$$y'(1) = -\frac{2}{3}, \Longrightarrow -\frac{1}{3} - c_2 = -\frac{2}{3}$$

$$\Longrightarrow c_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

$$\therefore \boxed{y = t^{-1/3} + \frac{1}{3}t^{-1/3} \ln t}.$$

Chapter 4

Series Solutions of Differential Equations

This method provides a means of computing accurate approximate solutions to differential equations near a point.

4.1 Introduction

In this section, we review a few concepts from Calculus needed to understand the approach of series solutions.

Definition 9. The Taylor Polynomial of degree n about the point x_0 approximating $f(x)$ at $x = x_0$ is given by

$$P_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\cdots+\frac{f^n(x_0)}{n!}(x-x_0)^n$$

$$\Longrightarrow \boxed{P_n(x) = \sum_{j=0}^{n} \frac{f^j(x_0)}{j!}(x-x_0)^j}.$$

Motivation

Find the first few Taylor Polynomials approximating the solution around $x_0 = 0$:

$$\begin{cases} y'' = 3y' + x^{7/3}y \\ y(0) = 10, \quad y'(0) = 5 \end{cases}$$

Solution.

$$y'' = 3y' + x^{7/3}y \tag{4.1}$$
$$y(0) = 10, \quad y'(0) = 5$$
$$P_n(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots + \frac{y^n(0)}{n!}x^2. \tag{4.2}$$

Now,

$$y''(0) = 3y'(0) + 0^{7/3}y(0) = 3(5) + 0 = 15.$$

From (4.1), we have

$$y''' = 3y'' + \frac{7}{3}x^{4/3}y + x^{7/3}y'$$

$$\Longrightarrow y'''(0) = 3y''(0) + \frac{7}{3}\cdot 0^{4/3}y(0) + 0^{7/3}\cdot y'(0) = 3(15) = 45$$

Similarly, we have

$$\begin{aligned} y^{(4)} &= 3y''' + \frac{28}{9}x^{1/3}y + \frac{7}{3}x^{4/3}y' + \frac{7}{3}x^{4/3}y' + x^{7/3}y'' \\ &= 3y''' + x^{7/3}y'' + \frac{14}{3}x^{4/3}y' + \frac{28}{9}x^{1/3}y \\ \Longrightarrow y^{(4)}(0) &= 3y'''(0) + 0^{7/3}\cdot y''(0) + \frac{14}{3}\cdot 0^{4/3}y'(0) + \frac{28}{9}\cdot 0^{1/3}y(0) \\ &= 3y'''(0) + 0 + 0 + 0 = 3(45) = 135. \end{aligned}$$

Also,

$$y^{(5)} = 3y^{(4)} + \frac{7}{3}x^{4/3}y'' + x^{7/3}y''' + \frac{56}{9}x^{1/3}y' + \frac{14}{3}x^{4/3}y'' + \boldsymbol{\frac{28}{27}x^{-2/3}y} + \frac{28}{9}x^{1/3}y'$$

Note that at $x = 0$ the fifth derivative $y^{(5)}$ does not exist because of the term $\boldsymbol{(28/27)x^{-2/3}y}$. So the Taylor Polynomial

$$P_4(x) = 10 + 5x + \frac{15}{2}x^2 + \frac{45}{6}x^3 + \frac{135}{24}x^4,$$

approximates the solution to the differential equation near $x = 0$.

Power Series

Definition 10. A Power Series about the point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{4.3}$$

where the a_n's are constants. Equation (4.3) converges at $x = c$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n$$

exists (as a finite number).

Theorem 6 (**Radius of Convergence**)**.** For each power series of the form (4.3), there is a number $\rho\,(0 \le \rho \le \infty)$, called the **radius of convergence** of the power series, such that (4.3) converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. If the series converges for all x, then $\rho = \infty$, and if it converges only at $x = x_0$ then $\rho = 0$.

Theorem 7 (**Ratio Test**)**.** Suppose the coefficients a_n are nonzero for large values of n and

$$\lim_{n\to\infty} \left| \frac{a_{n_1}}{a_n} \right| = L, \quad (0 \le L \le \infty)$$

then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

is $\rho = \dfrac{1}{L}$, with $\rho = \infty$ if $L = 0$ and $\rho = 0$ if $L = \infty$.

Example 28. Determine the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n+1}(x-3)^n$$

Solution.

$$a_n = \frac{(-2)^n}{n+1},$$

$$\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{n+2}\cdot\frac{(n+1)}{(-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2)^n(-2)}{(-2)^n}\cdot\frac{n+1}{n+2}\right|$$

$$= 2\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right| = 2\lim_{n\to\infty}\left|\frac{1+1/n}{1+2/n}\right| = 2 = L$$

$$\therefore \rho = \frac{1}{2}$$

Therefore the series converges for $|x-3| < \frac{1}{2}$.

Theorem 8. If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for all x in some open interval, then each coefficient a_n equals zero.

Remark. If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, then within the radius of convergence, we can differentiate and also integrate $f(x)$ so that

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1}, \quad |x-x_0| < \rho,$$

$$\int f(x)dx = \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-x_0)^{n+1} + C, \qquad |x-x_0| < \rho.$$

Analytic Function

Definition 11. A function f is said to be analytic at x_0 if, in an open interval about x_0, this function is the sum of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ that has a positive radius of convergence.

The following are examples of analytic functions with their corresponding power series representations:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{(2n+1)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

But $\ln x$ is analytic for $x > 0$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n}(x-1)^n.$$

4.2 Power Series Solutions to Linear Differential Equations

In this section, we try to find the power series solutions to linear differential equations with polynomial coefficients. Consider the equation

$$a_2(x)y'' + a_1(x)y' + a_0 y = 0. \tag{4.4}$$

Writing the equation in standard form gives

$$y'' + p(x)y' + q(x)y = 0, \tag{4.5}$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)}, \qquad q(x) = \frac{a_0(x)}{a_2(x)}.$$

Definition 12 (Ordinary and Singular Points). A point x_0 is called an **ordinary point** of (4.4) if $p = a_1/a_2$ and $q = a_0/a_2$ are analytic at x_0. If x_0 is not an ordinary point, it is called a **singular point** of the equation.

Example 29. Determine all the singular points of

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

Solution.

$$y'' + \frac{x}{x(1-x)}y' + \frac{\sin x}{x}y = 0.$$

Let

$$p(x) = \frac{x}{x(1-x)} = \frac{1}{1-x}, \qquad q(x) = \frac{\sin x}{x}.$$

Thus, $p(x)$ is analytic except at $x = 1$. The point $x = 0$ is called a **removable singularity** of $p(x)$ since we can cancel out x to get $p = 1/(1-x)$. Note that $q(x)$ is a ratio of analytic functions, so we write

$$q(x) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{4!} - \cdots$$

Therefore $q(x)$ is analytic for all x. Thus, the only singular point of the equation is $x = 1$.

Example 30. Find a power series solution about $x = 0$ to

$$y' + 2xy = 0. \tag{4.6}$$

Solution.

Remark. Note that we can solve the differential equation by separation of variables:

$$\frac{dy}{y} = -2xdx, \quad \Longrightarrow \ln|y| = -x^2 + C_1, \quad C_1 = \text{constant},$$

$$\therefore y = Ce^{-x^2}, \quad C = \text{constant},$$

For a power series solution, note that the coefficient of y, that is $2x$, is analytic everywhere, so $x = 0$ is an ordinary point. Thus, we expect a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{4.7}$$

$$\Longrightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots \tag{4.8}$$

Substituting (4.7)-(4.8) into (4.6) yields

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\Longrightarrow \sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0.$$

We can let both series have the same power of x by setting $k = n-1$ in the first equation and $k = n+1$ in the second to get

$$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k = 0.$$

Now, to be able to combine the two series, we let both of them start from the same value of k such that

$$a_1 + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k = 0,$$

$$\Longrightarrow a_1 + \sum_{k=1}^{\infty}\left[(k+1)a_{k+1} + 2a_{k-1}\right]x^k = 0.$$

Now, each coefficient must vanish. Thus

$$a_1 = 0$$

and we get the **recurrence relation** from the second term on the left-hand side:

$$(k+1)a_{k+1} + 2a_{k-1}, \quad k \geq 1$$

$$\Longrightarrow a_{k+1} = -\frac{2}{k+1}a_{k-1}, \quad k \geq 1$$

Substituting values of k into the recurrence relation gives the following:

$$\begin{aligned}
k = 1: &\quad a_2 = -\frac{2}{2}a_0 = -a_0 \\
k = 2: &\quad a_3 = -\frac{2}{3}a_1 = 0 \\
k = 3: &\quad a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0 \\
k = 4: &\quad a_5 = -\frac{2}{5}a_3 = 0 \\
k = 5: &\quad a_6 = -\frac{2}{6}a_4 = -\frac{1}{3}\cdot\frac{1}{2}a_0 = -\frac{1}{3!}a_0 \\
k = 6: &\quad a_7 = -\frac{2}{7}a_5 = 0 \\
k = 7: &\quad a_8 = -\frac{2}{8}a_6 = -\frac{1}{4}\cdot\frac{1}{3!}a_0 = \frac{1}{4!}a_0 \\
k = 8: &\quad a_9 = -\frac{2}{9}a_7 = 0
\end{aligned}$$

From the relation above, notice that

$$a_{2n} = \frac{(-1)^n}{n!}a_0, \quad n = 1, 2, \cdots$$

$$a_{2n+1} = 0, \quad n = 0, 1, 2, \cdots$$

which are the even and odd terms of (4.7) respectively. Substituting them into (4.7) gives

$$y(x) = a_0 - a_0x^2 + \frac{1}{2!}a_0x^4 + \cdots$$

$$\boxed{\therefore y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}}, \tag{4.9}$$

where a_0 is an arbitrary constant (as expected since the equation is first order).

Remark. (a) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Longrightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

This implies that

$$y(x) = a_0 e^{-x^2}$$

as found by the method of separation of variables.

(b) Applying the ratio test to (4.9) gives

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(-1)(-1)^n}{(n+1)\cdot n!} \cdot \frac{n!}{(-1)^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)}{n+1}\right| = 0 = L$$

$$\therefore \rho = \frac{1}{L} = \infty.$$

Thus (4.9) converges for all x.

(c) Let $a_0 = 1$. Then (4.9) yields

$$y(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$$

Figure XX shows the actual solution to the differential equation together with various partial sums of the power series solution. We observe that by using more terms we get better approximations to the solution around $x = 0$. However, partial sums diverge as $x \to \pm\infty$ since they are polynomials, but the actual solution approaches zero as $x \to \pm\infty$.

Example 31. Find a power series solution of the differential equation about $x = 2$:

$$y'' - y = 0. \tag{4.10}$$

Solution. We seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n \tag{4.11}$$

$$\implies y'(x) = \sum_{n=0}^{\infty} n a_n (x-2)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$\implies y''(x) = \sum_{n=1}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} \tag{4.12}$$

Substituting (4.11)-(4.12) into (4.10), we get

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

Letting $k = n-2$ in the first series and $k = n$ in the second gives

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} (x-2)^k - \sum_{k=0}^{\infty} a_k (x-2)^k = 0$$

$$\implies \sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - a_k\right] (x-2)^k = 0$$

The coefficient of each power vanishes, resulting in

$$(k+2)(k+1) a_{k+2} - a_k = 0$$

$$\implies a_{k+2} = \frac{1}{(k+2)(k+1)} a_k, \quad k \geq 0.$$

Substituting values of k into the recurrence relation above gives:

$$\begin{aligned}
k = 0: &\quad a_2 = \frac{1}{2\cdot 1}a_0 \\
k = 1: &\quad a_3 = \frac{1}{3\cdot 2\cdot 1}a_1 \\
k = 2: &\quad a_4 = \frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1}a_0 \\
k = 3: &\quad a_5 = \frac{1}{5\cdot 4}a_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1}a_1 \\
k = 4: &\quad a_6 = \frac{1}{6\cdot 5}a_4 = \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \\
\vdots = \vdots &\quad \vdots\vdots = \vdots
\end{aligned}$$

From the relations above, we can establish the following two recurrence relations:

$$a_{2n} = \frac{1}{(2n)!}a_0, \quad n = 1, 2, \cdots$$

$$a_{(2n+1)} = \frac{1}{(2n+1)!}a_1, \quad n = 1, 2, \cdots$$

From (4.11) we have

$$\begin{aligned}
y(x) &= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \cdots \\
&= \left[a_0 + a_2(x-2)^2 + a_4(x-2)^4 + \cdots\right] + \\
&\quad \left[a_1(x-2) + a_3(x-2)^3 + a_5(x-2)^5 + \cdots\right] \qquad (4.13) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}a_0(x-2)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}a_1(x-2)^{2n+1} \\
\therefore y(x) &= \sum_{n=0}^{\infty} \left[\frac{1}{(2n)!}(x-2)^{2n}a_0 + \frac{1}{(2n+1)!}(x-2)^{2n+1}a_1\right]
\end{aligned}$$

Remark. We first note again that the differential equation could have been solved much easily using our previous methods. Secondly, given initial conditions; say $y(2) = 1$ and $y'(2) = -1$, we can determine a_0 and a_1. From (4.13) we have

$$y(2) = a_0 = 1,$$

and

$$y'(x) = 2a_2(x-2) + \cdots + a_1 + 3a_3(x-2)^2 + \cdots$$

$$\Longrightarrow y'(2) = a_1 = -1.$$

Thus,

$$y(x) = \sum_{n=0}^{\infty} \left[\frac{1}{(2n)!}(x-2)^{2n} - \frac{1}{(2n+1)!}(x-2)^{2n+1} \right].$$

Example 32. Find the first few terms in a power series solution about $x = 0$ for a general solution to

$$(1 + x^2)y'' - y' + y = 0. \tag{4.14}$$

Solution. Note that $x = 0$ is an ordinary point of the equation so we look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.15}$$

$$\Longrightarrow y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{4.16}$$

$$\Longrightarrow y''(x) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \tag{4.17}$$

Substitute (4.15)-(4.17) into (4.14) to get

$$(1 + x^2) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Longrightarrow \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Longrightarrow 2a_2 + 6a_3 x - a_1 - 2a_2 x + a_0 + a_1 x +$$

$$\sum_{k=2}^{\infty} \left[(k+2)(k+1)a_{k+2} + k(k-1)a_k - (k+1)a_{k+1} + a_k\right] x^k = 0.$$

$$\Longrightarrow (2a_2 - a_1 + a_0) + (6a_3 - 2a_2 + a_1)x +$$

$$\sum_{k=2}^{\infty} \left[(k+2)(k+1)a_{k+2} + (k+1)a_{k+1} + (k^2 - k + 1)a_k\right] = 0.$$

Thus, we have

$$2a_2 - a_1 + a_0 = 0 \tag{4.18}$$
$$6a_3 - 2a_2 + a_1 = 0 \tag{4.19}$$
$$(k+2)(k+1)a_{k+2} + (k+1)a_{k+1} + (k^2 - k + 1)a_k = 0 \tag{4.20}$$

From (4.18) and (4.19) we get the relations

$$a_2 = \frac{a_1 - a_0}{2},$$

$$a_3 = \frac{2a_2 - a_1}{6} = \frac{a_1 - a_0 - a_1}{6} = -\frac{a_0}{6},$$

and (4.20) gives the recurrence relation

$$a_{k+2} = \frac{(k+1)a_{k+1} - (k^2 - k + 1)a_k}{(k+2)(k+1)}, \qquad k \geq 2.$$

So we have

$$\begin{aligned}
k = 2: \quad a_4 &= \frac{3a_3 - 3a_2}{4 \cdot 3} = \frac{3\left(\frac{-a_0}{6}\right) - 3\left(\frac{a_1 - a_0}{2}\right)}{12} \\
&= \frac{\frac{-a_0}{2} - \left(\frac{3a_1 - 3a_0}{2}\right)}{12} = \frac{-a_0 - 3a_1 + 3a_0}{24} \\
&= \frac{2a_0 - 3a_1}{24} \\
k = 3: \quad a_5 &= \frac{4a_4 - 7a_3}{5 \cdot 4} = \frac{4\left(\frac{2a_0 - 3a_1}{24}\right) - 7\left(\frac{-a_0}{6}\right)}{20} \\
&= \frac{2a_0 - 3a_1 + 7a_0}{120} = \frac{9a_0 - 3a_1}{120} = \frac{3a_0 - a_1}{40} \\
k = 4: \quad a_6 &= \frac{5a_5 - 13a_4}{6 \cdot 5} = \frac{5\left(\frac{3a_0 - a_1}{40}\right) - 13\left(\frac{2a_0 - 3a_1}{24}\right)}{30} \\
&= \frac{\left(\frac{3a_0 - a_1}{8}\right) - \left(\frac{26a_0 - 39a_1}{24}\right)}{30} \\
&= \frac{9a_0 - 3a_1 - 26a_0 + 39a_1}{720} \\
&= \frac{36a_1 - 17a_0}{720}
\end{aligned}$$

From (4.15) we have

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$\implies y(x) = a_0 + a_1 x + \left(\frac{a_1 - a_0}{2}\right) x^2 - \frac{a_0}{6} x^3 + \left(\frac{2a_0 - 3a_1}{24}\right) x^4 + \left(\frac{3a_0 - a_1}{40}\right) x^5 + \left(\frac{36a_1 - 17a_0}{720}\right) x^6 + \cdots \qquad (4.21)$$

Remark. Given initial conditions, we can determine a_0 and a_1, and (4.21) gives approximations to the solution of the differential equation about $x = 0$. But how useful are the partial sums; for example when $x = 0.3$ or $x = 2.4$? Note that we don't have a general form for a_n so we cannot use the well-known tests to find the radius of convergence. The following theorem helps us to determine the radius of convergence in cases like this.

Theorem 9. Suppose x_0 is an ordinary point for the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \qquad (4.22)$$

Then (4.22) has two linearly independent analytic solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \qquad (4.23)$$

Moreover, the radius of convergence of any power series solution of the form given by (4.23) is at least as large as the distance from x_0 to the nearest singular point (real or complex-valued) of equation (4.22).

Remark. (a) Recall that the distance between any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is given by

$$|z_1 z_2| = \sqrt{(a-c)^2 + (b-d)^2}.$$

(b) We are now in a position to address the concerns raised after the previous example as demonstrated in the example below.

Example 33. Find the minimum value for the radius of convergence of a power series solution about the point x_0:

$$(1 + x^2)y'' - y' + y = 0, \qquad x_0 = 0.$$

Solution. Writing the equation in standard form gives

$$y'' - \frac{1}{1+x^2} y' + \frac{1}{1+x^2} y = 0.$$

Singular points occur when

$$1 + x^2 = 0, \qquad x = \pm i.$$

Now, $x_0 = 0$ is an ordinary point and the distance from $x = 0$ to either $x = \pm i$ is 1. So the radius of convergence is at least 1. Thus, using the Theorem 9, the partial sums of the series solution will converge to the solution for $-1 < x < 1$.

Example 34. Find the minimum value for the radius of convergence of a power series solution about the point x_0:

(a) $2y'' + xy' + y = 0; \qquad x_0 = 0.$

(b) $(x^2 - 5x + 6)y'' - 3xy' - y = 0; \qquad x_0 = 0.$

(c) $(x + 1)y'' - 3xy' + 2y = 0; \qquad x_0 = 1.$

(d) $(1 + x + x^2)y'' - 3y' = 0; \qquad x_0 = 1.$

Solution. (a) $y'' + \frac{x}{2}y' + \frac{1}{2}y = 0, \qquad x_0 = 0.$

Now,

$$\Longrightarrow p(x) = \frac{1}{2}x, \qquad q(x) = \frac{1}{2},$$

are analytic with no singular points. So the distance between the ordinary point $x = 0$ and the nearest singular point is infinite. Thus, the radius of convergence is infinite.

(b) $(x^2 - 5x + 6)y'' - 3xy' - y = 0, \qquad x_0 = 0.$

$$y'' - \frac{3x}{(x^2 - 5x + 6)}y' - \frac{1}{(x^2 - 5x + 6)}y = 0; \qquad x = 0.$$

Singular points occur for

$$x^2 - 5x + 6 = (x - 2)(x - 3) = 0$$

$$\Longrightarrow x = 2 \quad \text{or} \quad x = 3.$$

Now $x = 0$ is an ordinary point. The distance from 0 to 2 is 2, and from 0 to 3 is 3. Since 2 is the nearest singular point to 0, Theorem 9 implies that the radius of convergence is at least 2.

(c) $(x+1)y'' - 3xy' + 2y = 0; \qquad x_0 = 1.$

$$y'' - \frac{3x}{x+1}y' + \frac{2}{x+1} = 0; \qquad x = 1.$$

The Singular point is $x = -1$. The radius of convergence is given by

$$\rho = \sqrt{(-1-1)^1} = 2.$$

(d) $(1+x+x^2)y'' - 3y' = 0; \qquad x_0 = 1.$

$$y'' - \frac{3}{(1+x+x^2)}y' = 0; \qquad x_0 = 1.$$

Singular points occur for

$$1 + x + x^2 = x^2 + x + 1 = 0$$

$$\Longrightarrow x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Now $x_0 = 1$ is an ordinary point and the distance, ρ, from $x = 1$ to each of the singular points is given by

$$\rho = \sqrt{\left(1 + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}}$$

$$\Longrightarrow \rho = \sqrt{\frac{12}{4}} = \sqrt{3}.$$

So the radius of convergence is $\rho = \sqrt{3}$.

4.2.1 Expanding About $x = 0$

It is easier to expand power series about $x = 0$. A shift in variable enables every power series to be expanded about $x = 0$ as shown below.

Example 35. Find the first few terms in a power series exapansion about $x = 1$ for a general solution to

$$2y'' + xy' + y = 0.$$

Determine the radius of convergence of the series.

Solution.

$$2y'' + xy' + y = 0. \tag{4.24}$$

$x = 1$ is an ordinary point. So we expect a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n. \tag{4.25}$$

The radius of convergence is infinite. To shift from $x_0 = 1$ to say $t_0 = 0$, let

$$t = x - 1 \quad \Longrightarrow x = t + 1$$

Let $Y(t) = y(t+1)$. Applying the Chain Rule:

$$\frac{dY}{dt} = \frac{dy(t+1)}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx},$$

since $dx/dt = 1$. Thus, we have

$$\frac{dy}{dx} = \frac{dY}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2}.$$

So equation (4.24) becomes

$$2\frac{d^2Y}{dt^2} + (t+1)\frac{dY}{dt} + Y = 0,$$

and we seek solutions of the form

$$Y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

We can now proceed as usual to get the solution as

$$Y(t) = a_0 \left(1 - \frac{1}{4}t^2 + \frac{1}{24}t^3 + \cdots\right) + a_1 \left(t - \frac{1}{4}t^2 - \frac{1}{8}t^3 + \cdots\right).$$

Setting $t = x - 1$, yields the solution with the correct variables as

$$y(x) = a_0 \left\{1 - \frac{1}{4}(x-1)^2 + \frac{1}{24}(x-1)^3 + \cdots\right\} + a_1 \left\{(x-1) - \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^3 + \cdots\right\}$$

4.2.2 Series Solutions about Regular Singular Points

Definition 13. A singular point x_0 of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{4.26}$$

is said to be a **regular singular point** if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an **irregular singular point**.

Example 36. Classify the singular points of the equation

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0.$$

Solution.

$$y''(x) + \frac{(x+1)}{(x^2-1)^2} y'(x) - \frac{1}{(x^2-1)^2} y(x) = 0.$$

Thus,

$$p(x) = \frac{(x+1)}{(x^2-1)^2} = \frac{(x+1)}{[(x-1)(x+1)]^2} = \frac{1}{(x-1)^2(x+1)}$$

$$q(x) = \frac{-1}{(x^2-1)^2} = \frac{-1}{(x-1)^2(x+1)^2}$$

Thus, the singular points are

$$x = 1 \quad \text{and} \quad x = -1.$$

For $x = 1$, we have

$$(x-1)p(x) = \frac{1}{(x-1)(x+1)}$$

which is not analytic at $x = 1$. So $x = 1$ is an irregular singular point. (Note that we don't need to proceed to analyze $(x - 1)^2 q(x)$).
For $x = -1$, we have

$$(x+1)p(x) = \frac{1}{(x-1)^2},$$

which is analytic at $x = -1$. Now

$$(x+1)^2 q(x) = \frac{-1}{(x-1)^2},$$

which is also analytic at $x = -1$. So $x = -1$ is a regular singular point.

4.2.3 Method of Frobenius

This is a method used to find a series solution to

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{4.27}$$

about a regular singular point, say x_0. The idea is that, since $y = x^r$ is the form of a solution to Cauchy-Euler equations ($ax^2y'' + bxy' + cy = 0$, $x > 0$), we expect, at a regular singular point $x = 0$, a solution of the form

$$y(x) = W(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n, \quad x > 0$$

$$\implies W(r, x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \tag{4.28}$$

with $a_0 \neq 0$. Now

$$W' = \sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1}, \tag{4.29}$$

$$\implies W'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2}, \tag{4.30}$$

Also, since $xp(x)$ and $x^2q(x)$ are analytic, we can expand them as

$$xp(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$\implies p(x) = \sum_{n=0}^{\infty} a_n x^{n-1}, \tag{4.31}$$

and

$$q(x) = \sum_{n=0}^{\infty} a_n x^{n-2}. \tag{4.32}$$

Substituting (4.28)-(4.32) into (4.27) we get

$$\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} p_n x^{n-1}\right)\left(\sum_{n=0}^{\infty} (n + r)a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} q_n x^{n-2}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

Expanding and grouping like terms results in the equation

$$[r(r-1)+p_0r+q_0]\,a_0x^{r-2}+\cdots=0$$

where x^{r-2} is the lowest power in this case. Equating each coefficient to zero results in the indicial equation:

$$r(r-1)p_0r+q_0=0, \quad a_0\neq 0,$$

as defined formerly below.

Definition 14 (**Indicial Equation**). If x_0 is a regular singular point of $y''+p(x)y'+q(x)y=0$, then the **indicial equation** for x_0 is

$$\boldsymbol{r(r-1)+p_0r+q_0=0},$$

where

$$p_0 := \lim_{x\to x_0}(x-x_0)p(x) \quad \text{and} \quad q_0 := \lim_{x\to x_0}(x-x_0)^2q(x).$$

The roots of the indicial equation are called **exponents** or **indices** of the singularity x_0.

Example 37. Find the indicial equation and exponents at the singularity $x=-1$ of

$$(x^2-1)^2y''(x)+(x+1)y'(x)-y(x)=0.$$

Solution. We analyzed this equation in example 36, where we found that $x=-1$ is a regular singular point, and

$$p(x)=\frac{1}{(x-1)^2(x+1)} \quad \text{and} \quad q(x)=\frac{-1}{(x-1)^2(x+1)^2}.$$

$$p_0=\lim_{x\to -1}(x+1)p(x)=\lim_{x\to -1}\frac{1}{(x-1)^2}=\frac{1}{4}$$

$$q_0=\lim_{x\to -1}(x+1)^2q(x)=\lim_{x\to -1}\frac{-1}{(x-1)^2}=-\frac{1}{4}$$

So the indicial equation $r(r-1)+p_0r+q_0=0$ becomes

$$r(r-1)+\frac{1}{4}r-\frac{1}{4}=0,$$

$$\implies 4r(r-1)+r-1=0, \quad \implies 4r^2-3r-1=0$$

$$\implies 4r^2 - 4r + r - 1 = 0, \quad \implies 4r(r-1) + (r-1) = 0$$
$$\implies (r-1)(4r+1) = 0.$$

So the exponents are:

$$r = 1, \quad -\frac{1}{4}.$$

The Frobenius Theorem stated next enables us to find one series solution of a variable-coefficient equation.

Theorem 10 (**Frobenius Theorem**). If x_0 is a regular singular point of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{4.33}$$

then there exists at least one series solution of the form

$$W(r, x) = (x - x_0) \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r},$$

where $r = r_1$ is the larger root of the associated indicial equation. Moreover, this series converges for all x such that $0 < x - x_0 < \rho$, where ρ is the distance from x_0 to the nearest other singular point (real or complex) of (4.33).

Example 38. Find the series expansion about the regular singular point $x = 0$ for a solution to

$$(x+2)x^2 y''(x) - xy'(x) + (1+x)y(x) = 0, \quad x > 0. \tag{4.34}$$

Solution. Writing the equation in standard form gives

$$y''(x) - \frac{x}{(x+2)x^2} y'(x) + \frac{(1+x)}{(x+2)x^2} y(x) = 0$$

So we have

$$p(x) = \frac{-x}{(x+2)x^2} = \frac{-1}{(x+2)x}.$$
$$q(x) = \frac{1+x}{(x+2)x}.$$

Thus,

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} \frac{-1}{x+2} = -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{1+x}{x+2} = \frac{1}{2}$$

The indicial equation $r(r-1) + p_0 r + q_0 = 0$ becomes

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = 0,$$

$$2r^2 - 2r - r + 1 = 0, \quad 2r^2 - 3r + 1 = 0$$

$$\Longrightarrow (r-1)(2r-1) = 0$$

So the exponents are

$$r = 1, \quad \text{and} \quad r = \frac{1}{2}.$$

So we seek a solution of the form

$$W(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

To obtain the constants a_n, we apply Frobenius Theorem and use the larger root $r = 1$ to get one solution $y_1(x) = W(1,x)$ such that

$$W(1,x) = \sum_{n=0}^{\infty} a_n x^{n+1} \tag{4.35}$$

$$\Longrightarrow W' = \sum_{n=0}^{\infty} (n+1) a_n x^n$$

$$\Longrightarrow W'' = \sum_{n=0}^{\infty} (n+1) n a_n x^{n-1}$$

Substitute the above expressions into (4.34) to get

$$(x+2)x^2 \sum_{n=0}^{\infty} (n+1) n a_n x^{n-1} - x \sum_{n=0}^{\infty} (n+1) a_n x^n + (1+x) \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

This may be written as

$$\sum_{k=2}^{\infty} \left[(k-1)(k-2) + 1\right] a_{k-2} x^k + \sum_{k=1}^{\infty} \left[2k(k-1) - k + 1\right] a_{k-1} x^k = 0$$

Combining the summations results in

$$(k^2 - 3k + 3)a_{k-2} + (2k-1)(k-1)a_{k-1} = 0$$
$$\implies a_{k-1} = -\frac{k^2 - 3k + 3}{(2k-1)(k-1)}a_{k-2}, \quad k \geq 2.$$

Thus, we get

$$a_1 = -\frac{1}{3}a_0$$
$$a_2 = \frac{1}{10}a_0$$
$$a_3 = -\frac{1}{30}a_0$$
$$\vdots = \vdots$$

From (4.35) we have

$$W(1,x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$
$$\implies W(1,x) = a_0x - \frac{1}{3}a_0x^2 + \frac{1}{10}a_0x^3 - \frac{1}{30}a_0x^4 + \cdots$$
$$\therefore y_1(x) = a_0x\left(1 - \frac{1}{3}x + \frac{1}{10}x^2 - \frac{1}{30}x^3 + \cdots\right); \quad x > 0$$

4.2.4 Form of a Second Linearly Independent Solution

Theorem 11. Let x_0 be a regular singular point for

$$y'' + p(x)y' + q(x)y = 0$$

and let r_1 and r_2 be the roots of the associated indicial equation, where $\mathbb{R}(r_1) \geq \mathbb{R}(r_2)$.

(a) If $r_1 - r_2$ is not an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+r_1}, \quad a_0 \neq 0, \tag{4.36}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2}, \quad b_0 \neq 0. \tag{4.37}$$

(b) If $r_1 = r_2$, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \quad a_0 \neq 0, \tag{4.38}$$

$$y_2(x) = y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0. \tag{4.39}$$

(b) If $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \quad a_0 \neq 0, \tag{4.40}$$

$$y_2(x) = C y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0. \tag{4.41}$$

where C is a constant that could be zero.

Chapter 5

Laplace Transforms

Definition 15. Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F or $\mathcal{L}\{f\}$ defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)dt. \tag{5.1}$$

The domain of $F(s)$ is all the values of s for which the integral in (5.1) exists.

Remark. (1) The integral in (5.1) is an improper integral so to integrate we effect

$$\int_0^\infty e^{-st} f(t)dt = \lim_{N\to\infty} \int_0^N e^{-st} f(t)dt$$

whenever the limit exists.

(2) The transform is used to convert constant coefficient differential equations from the $t-$domain to (simpler) algebraic equations in the $x-$domain.

(3) Consider the equation

$$ay'' + by' + cy = f(t) \tag{5.2}$$

Laplace transforms are more useful if $f(t)$ is not a continuous function. Besides, Laplace transforms are more simpler to use in solving certain more complicated nonhomogeneous differential equations.

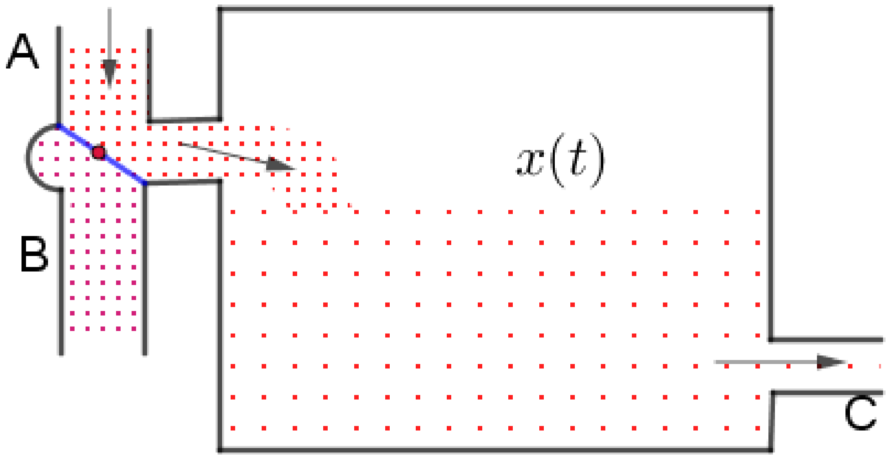

Figure 5.1: Schematic of the mixing problem

MOTIVATION: Mixing Problem

The rate of change of the amount of salt in a tank, $x(t)$, is governed by the differential equation

$$\frac{dx}{dt} + \frac{3}{500}x = f(t), \quad x(0) = 30,$$

where

$$f(t) = \begin{cases} 2.4\text{km/min}, & 0 < t < 10, \quad \text{Valve A} \\ 1.2\text{km/min}, & t > 10, \quad \text{Valve B} \end{cases} \tag{5.3}$$

Using the Laplace transform to solve the mixing problem is much easier than our previous methods.

Example 39. Determine the Laplace transform of

(a) $f(t) = 1, \quad t \geq 1.$

(b) $f(t) = e^{at}, \quad a = \text{constant}$

(c) $f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & t > 2. \end{cases}$

Solution. (a)

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{N\to\infty} \int_0^N e^{-st} \cdot 1dt$$

$$= \lim_{N\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^N = \lim_{N\to\infty}\left[-\frac{1}{s}e^{-sN}+\frac{1}{s}\right]$$

$$\therefore F(s) = \frac{1}{s}, \quad \text{for} \quad s > 0.$$

If $s \leq 0$ the integral diverges, so

$$F(s) = \frac{1}{s} \quad \forall s > 0.$$

(b)

$$F(s) = \int_0^\infty e^{-st}e^{at}dt = \lim_{N\to\infty}\int_0^N e^{-(s-a)t}dt$$

$$= \lim_{N\to\infty}\left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^N = \lim_{N\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)N}+\frac{1}{(s-a)}\right]$$

$$\therefore F(s) = \frac{1}{s-a}, \quad \text{for} \quad s-a>0 \quad \text{or} \quad s>a.$$

If $s \leq a$ the integral diverges, so the domain of $F(s)$ is all $s > a$.

$$\therefore \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \forall s > a.$$

(c) $f(t) = \begin{cases} 0, & 0 < t < 2, \\ t, & t > 2. \end{cases}$

$$F(s) = \int_0^\infty e^{-st}f(t)dt = \int_0^2 e^{-st}\cdot 0dt + \int_2^\infty e^{-st}dt$$

$$= \lim_{N\to\infty}\int_2^N te^{-st}dt$$

We integrate by parts. Let

$$u = t, \qquad dv = e^{-st}dt$$

$$\Longrightarrow du = dt, \qquad v = -\frac{1}{s}e^{-st}$$

$$\Longrightarrow F(s) = \lim_{N\to\infty}\left\{\left[-\frac{1}{s}te^{-st}\right]_2^N + \int_2^N \frac{1}{s}e^{-st}dt\right\}$$

$$= \lim_{N\to\infty}\left\{-\frac{1}{s}Ne^{-sN}+\frac{2}{s}e^{-2s}+\left[-\frac{1}{s^2}e^{-st}\right]_2^N\right\}$$

$$= \lim_{N\to\infty} \left\{ -\frac{1}{s}Ne^{-sN} + \frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-sN} + \frac{1}{s^2}e^{-2s} \right\}$$

After taking the limit, we get

$$F(s) = \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}, \quad s > 0.$$

$$\therefore F(s) = e^{-2s}\left(\frac{2s+1}{s^2}\right), \quad s > 0.$$

5.0.5 Table of Laplace Transforms

Table 5.1 shows some Laplace transforms.

1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n, $n = 1, 2, \cdots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n$, $n = 1, 2, \cdots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

Table 5.1: A brief table of Laplace transforms

Example 40. Use the Table of Laplace transforms to compute the Laplace transform of

(a) $f(t) = te^{3t}$

(b) $f(t) = \cos 2t$

(c) $f(t) = e^{2t}\cos(3t)$

(d) $f(t) = t^2$

(e) $f(t) = e^{-3t}\sin 2t$

(f) $f(t) = t^2 e^{4t}$

Solution. Let $\mathcal{L}\{f(t)\} = F(s)$.

(a) $f(t) = te^{3t}$.
We use the relation

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a,$$

with $n = 1$ and $a = 3$. Thus

$$F(s) = \frac{1}{(s-3)^2}.$$

(b) $f(t) = \cos 2t$. We apply

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2 + b^2}, \quad s > 0$$

with $b = 2$. Thus

$$F(s) = \frac{s}{s^2 + 4}.$$

(c) $f(t) = e^{2t}\cos(3t)$.
We use

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a$$

with $a = 2$ and $b = 3$. So we get

$$F(s) = \frac{s-2}{(s-2)^2 + 9}, \quad s > 2.$$

5.0.6 Properties of Laplace Transforms

1. $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ $c =$ constant.
3. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s - a)$
4. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
5. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
6. $\mathcal{L}\{f^n\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$.
7. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$.

Example 41. Determine the Lapace transform of the following expressions

(a) $3t^2 - e^{2t}$

(b) $e^{-2t}\sin 2t + e^{3t}t^2$

Solution. (a) $3t^2 - e^{2t}$

$$\begin{aligned}\mathcal{L}\{3t^2 - e^{2t}\} &= \mathcal{L}\{3t^2\} + \mathcal{L}\{e^{2t}\} = 3\mathcal{L}\{t^2\} + \mathcal{L}\{e^{2t}\} \\ &= 3\left(\frac{2!}{s^3}\right) + \frac{1}{s-2}, \quad s > 0 \\ &= \frac{6}{s^3} + \frac{1}{s-2}, \quad s > 0.\end{aligned}$$

(b) $e^{-2t}\sin 2t + e^{3t}t^2$. Using $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$, we have

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} \quad \Longrightarrow \mathcal{L}\{e^{-2t}\sin 2t\} = \frac{2}{(s+2)^2+4}$$

Also, employing $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$ gives

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}, \quad \Longrightarrow \mathcal{L}\{t^2 e^{3t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{2}{(s-3)^3}.$$

Example 42. Determine

(a) $\mathcal{L}\{t\cos bt\}$

(b) $\mathcal{L}\{t^2\cos bt\}$

Solution. (a) $\mathcal{L}\{t\cos bt\}$. Now

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

$$\Longrightarrow \mathcal{L}\{t\cos bt\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$

$$= -\left[\frac{(s^2+b^2)-s(2s)}{(s^2+b^2)^2}\right] = -\left[\frac{s^2-2s^2+b^2}{(s^2+b^2)^2}\right]$$

$$= \frac{s^2-b^2}{(s^2+b^2)^2}.$$

(b) $\mathcal{L}\{t^2\cos bt\}$. Using the results in (a), it can be shown that

$$\mathcal{L}\{t^2\cos bt\} = \mathcal{L}\{t\cdot t\cos bt\} = \frac{2s^3-6sb^2}{(s^2+b^2)^3}.$$

5.0.7 Inverse Laplace Transforms

Definition 16. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies $\mathcal{L}\{f\} = F$, then $f(t)$ is the inverse Laplace transform of $F(s)$:

$$f(t) = \mathcal{L}^{-1}\{F\}$$

Properties of Inverse Laplace Transforms

1. $\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$
1. $\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}, \quad c = \text{contant}.$

Motivation

Given the IVP:

$$\begin{cases} y''(t) - y(t) = 0 \\ y(0) = 0, \quad y'(0) = 1. \end{cases} \tag{5.4}$$

Let $\mathcal{L}\{y\} = Y(s)$. Then (5.4) becomes, after taking the Laplace transform,

$$\begin{aligned} \mathcal{L}\{y''\} - \mathcal{L}\{y\} &= 0 \\ \Longrightarrow \left[s^2Y(s) - sy(0) - y'(0)\right] - Y(s) &= 0 \\ \Longrightarrow s^2Y(s) - 1 - Y(s) &= 0 \\ \Longrightarrow Y(s)[s^2 - 1] &= 1 \\ \Longrightarrow Y(s) &= \frac{1}{s^2 - 1}. \end{aligned}$$

But we want the solution of the IVP to be $y(t)$. To get this, we need to find the inverse transform of $Y(s)$. Let

$$\begin{aligned} Y(s) = \frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)} &\equiv \frac{A}{s-1} + \frac{B}{s+1} \\ \Longrightarrow 1 &\equiv A(s+1) + B(s-1) \\ s = 1, \quad \Longrightarrow A &= \frac{1}{2} \\ s = -1, \quad \Longrightarrow B &= -\frac{1}{2} \\ \therefore Y(s) &= \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1} \end{aligned}$$

Thus,

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s-1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\}$$

From the Table of Laplace transforms, we get

$$y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \frac{e^t - e^{-t}}{2}$$

$$\therefore \boxed{y(t) = \sinh(t).}$$

Remark. Note that the motivational example above could have been solved much more easily with our previous methods for homogeneous differential equations with constant coefficients. However, the example demonstrates some of the capabilities of the Laplace transform approach. And as mentioned earlier, it is a much more powerful approach, for instance, in cases where the initial conditions are discontinuous as we'll encounter later.

Example 43. Determine $\mathcal{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^2+4}$

(b) $F(s) = \dfrac{2}{s^3}$

(c) $F(s) = \dfrac{4}{s^2+9}$

(d) $F(s) = \dfrac{s-1}{s^2-2s+5}$

Solution. Determine $\mathcal{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^2+4}$
Note that

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} \Longrightarrow \mathcal{L}^{-1}\{\frac{b}{s^2+b^2}\} = \sin bt.$$

Let $b = 2$, $\Longrightarrow \mathcal{L}^{-1}\{F\} = \sin 2t$.

(b) $F(s) = \dfrac{2}{s^3}$

$$t^n \to \frac{n!}{s^{n+1}}, \quad \Longrightarrow t^2 \to \frac{2!}{s^3}$$
$$\Longrightarrow \mathcal{L}^{-1}\{F\} = t^2.$$

(c) $F(s) = \dfrac{4}{s^2+9}$

$$F(s) = \frac{4}{s^2+9} = \frac{4}{3} \cdot \frac{3}{s^2+9}$$
$$\Longrightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \frac{4}{3}\sin 3t.$$

(d) $F(s) = \dfrac{s-1}{s^2-2s+5}$

$$F(s) = \frac{s-1}{s^2-2s+5} = \frac{s-1}{(s-1)^2+4}.$$ By completing the square.

Note that

$$\cos bt = \frac{s}{s^2+b^2}, \quad \text{and} \quad e^{at}\cos bt = \frac{s-a}{(s-a)^2+b^2}.$$

Thus, if we let $a = 1$ and $b = 2$, we get

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^t\cos 2t.$$

Remark. To compute $\mathcal{L}^{-1}$ of rational functions, review and apply the techniques of partial fractions as demonstrated by the following examples.

Example 44. Determine $\mathcal{L}^{-1}\{F\}$.

(a) $F(s) = \dfrac{7s-1}{(s+1)(s+2)(s-3)}$. Non-repeated linear factors

(b) $F(s) = \dfrac{s^2+9s+2}{(s-1)^2(s+3)}$. Repeated linear factors

(c) $F(s) = \dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}$. Quadratic factors

Solution. Determine $\mathcal{L}^{-1}\{F\}$.

(a) $F(s) = \dfrac{7s-1}{(s+1)(s+2)(s-3)}$.

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$7s-1 \equiv A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

$$s=-1: \quad -8 = -4A \quad \Longrightarrow A = 2$$

$$s=-2: \quad -15 = 5B \quad \Longrightarrow B = -3$$

$$s=3: \quad 20 = 20C \quad \Longrightarrow C = 1.$$

$$\Longrightarrow \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{2}{s+1} + \frac{-3}{s+2} + \frac{1}{s-3}$$

$$\therefore \mathcal{L}^{-1}\{F\} = 2e^{-t} - 3e^{-2t} + e^{3t}.$$

(b) $F(s) = \dfrac{s^2+9s+2}{(s-1)^2(s+3)}$.

$$F(s) = \frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$
$$\implies s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$
$$s=1: \quad 12 = 4B \quad \implies B = 3$$
$$s=-3: \quad -16 = 16C \quad \implies C = -1$$
$$s=1: \quad 2 = -3A + 9 - 1 = -3A + 8$$
$$\implies -6 = -3A \quad \implies A = 2.$$
$$\implies F(s) = \frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}$$
$$\therefore \mathcal{L}^{-1}\{F\} = 2e^t + 3te^t - e^{-3t}.$$

(c) $F(s) = \dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}$.

By completing the square, we have $s^2-2s+5 = (s-1)^2+4$. Thus

$$F(s) = \frac{2s^2+10s}{[(s-1)^2+4](s+1)} = \frac{2s^2+10s}{[(s-1)^2+2^2](s+1)}$$
$$\equiv \frac{A(s-1)+2B}{(s-1)^2+4} + \frac{C}{s+1}$$
$$\implies 2s^2+10s = [A(s-1)+2B](s+1) + C[(s-1)^2+4]$$
$$s=1: \quad 12 = 4B+4C \quad \implies 3 = B + C$$
$$s=-1: \quad -8 = 8C \quad \implies C = -1$$
$$\implies 3 = B - 1 \quad \implies B = 4.$$
$$s=0: \quad 0 = -A + 8 - 5 \quad \implies A = 3.$$
$$\implies F(s) = \frac{3(s-1)+8}{(s-1)^2+4} - \frac{1}{s+1}$$
$$\implies F(s) = 3\frac{(s-1)}{(s-1)^2+2^2} + 4\frac{2}{(s-1)^2+2^2} - \frac{1}{s+1}$$
$$\therefore \mathcal{L}^{-1}\{F(s)\} = 3e^t\cos 2t + 4e^t \sin 2t - e^{-t}.$$

EXERCISE 2. Determine $\mathcal{L}^{-1}\{F\}$.

1) $F(s) = \dfrac{2s+8}{(s-1)^2+4}$

2) $F(s) = \dfrac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

5.0.8 Solving IVPs Using Laplace Transforms

Solution Approach

(a) Take the Laplace transform of both sides of the differential equation.

(b) Get an equation for the Laplace transform of the solution, say $y(t)$, and solve for it, say $Y(s)$.

(c) Find the inverse Laplace transform of $Y(s)$ to get the solution, say $y(t)$.

Example 45. Use Laplace transforms to solve the following IVPs.

(a) $y'' - 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 4.$

(b) $y'' + 6y' + 9y = 0; \quad y(0) = -1, \quad y'(0) = 6.$

Solution. (a) $y'' - 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 4.$
Taking the Laplace transform of the equation gives

$$\begin{aligned} \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 0 \\ \implies [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 5Y(s) &= 0. \\ \implies s^2Y(s) - 2s - 4 - 2sY(s) + 4 + 5Y(s) &= 0 \\ \implies Y(s)[s^2 - 2s + 5] - 2s &= 0 \\ \implies Y(s) &= \frac{2s}{s^2 - 2s + 5}. \end{aligned}$$

Thus, we have

$$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Now,

$$Y(s) = \frac{2s}{s^2 - 2s + 5} = \frac{2(s-1) + 2}{(s-1)^2 + 4} = 2\frac{s-1}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4}$$

$$\implies y(t) = 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2 + 4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2 + 4}\right\}$$

$$\therefore \boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t.}$$

Check: We may verify that our solution is correct using our previous technique. The roots of the characteristic equation can be found from

$$r^2 - 2r + 5 = 0.$$

$$\Longrightarrow r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$
$$\Longrightarrow y(t) = C_1 e^t \cos 2t + C_2 e^t \sin 2t$$

We next apply the initial conditions. Now

$$y' = C_1 e^t \cos 2t - 2C_1 e^t \sin 2t + C_2 e^t \sin 2t + 2C_2 e^t \cos 2t$$

$$y(0) = 2 \quad \Longrightarrow 2 = C_1$$
$$y'(0) = 4 \quad \Longrightarrow 4 = C_1 + 2C_2 \quad \Longrightarrow C_2 = 1$$
$$\therefore y(t) = 2e^t \cos 2t + e^t \sin 2t,$$

which is the same as the solution we got using Laplace transforms.

(b) $y'' + 6y' + 9y = 0; \quad y(0) = -1, \quad y'(0) = 6.$
Taking the Laplace transform of the equation gives

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0$$

$$\Longrightarrow [s^2Y(s) - sy(0) - y'(0)] + 6[sY(s) - y(0)] + 9Y(s) = 0.$$
$$\Longrightarrow s^2Y(s) + s - 6 + 6sY(s) + 6 + 9Y(s) = 0$$
$$\Longrightarrow Y(s)[s^2 + 6s + 9] + s = 0$$
$$\Longrightarrow Y(s) = \frac{-s}{s^2 + 6s + 9} = \frac{-s}{(s+3)^2}.$$
$$\Longrightarrow Y(s) = \frac{-(s-3)+3}{(s+3)^2} = -\frac{1}{s+3} + \frac{3}{(s+3)^2}$$
$$\Longrightarrow \mathcal{L}^{-1}\{Y(s)\} = -e^{-3t} + 3te^{-3t}$$
$$\therefore \boxed{y(t) = -e^{-3t} + 3te^{-3t}}.$$

The next set of examples reveal some of the advantages of the Laplace transform approach compared to the other techniques.

Example 46. Solve for the Laplace transform, $Y(s)$, of $y(t)$.

(a) $y'' + 4y = g(t), \quad y(0) = -1, \quad y'(0) = 0,$
where

$$g(t) = \begin{cases} t, & t < 2 \\ 5, & t > 2. \end{cases}$$

(b) $y'' - 3y' + 2y = \cos t$
$y(0) = 0, \quad y'(0) = -1.$

Solution. (a) Let the Laplace transform of $g(t)$ be $G(s)$. Taking the Laplace transform of the equation results in

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g\}$$

$$\Longrightarrow [s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = G(s). \tag{5.5}$$

Now, from definition of the Laplace transform, we have

$$G(s) = \int_0^2 te^{-st}dt + \int_2^\infty 5e^{-st}dt$$

Let

$$I_1 = \int_0^2 te^{-st}dt$$

Integrating by parts, we let $u = t, \quad dv = e^{-st}dt$

$$\Longrightarrow du = dt, \quad v = -\frac{1}{s}e^{-st}$$

$$\Longrightarrow I_1 = -\frac{t}{s}e^{-st} + \frac{1}{s}\int_0^2 e^{-st}dt$$

$$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^2$$

$$= \left(-\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s}\right) - \left(-\frac{1}{s^2}\right)$$

$$\Longrightarrow I_1 = -\left(\frac{2s+1}{s^2}\right)e^{-2s} + \frac{1}{s^2}.$$

Let

$$I_2 = \int_2^\infty 5e^{-st}dt = \lim_{N\to\infty}\int_2^N 5e^{-st}dt$$

$$= \lim_{N\to\infty}\left[-\frac{5}{s}e^{-st}\right]_2^N = \lim_{N\to\infty}\left[-\frac{5}{s}e^{-Ns} + \frac{5}{s}e^{-2s}\right]$$

$$\Longrightarrow I_2 = \frac{5}{s}e^{-2s}, \quad s > 0.$$

Thus,

$$G(s) = I_1 + I_2 = \frac{1}{s^2} - \left(\frac{2s+1}{s^2}\right)e^{-2s} + \frac{5}{s}e^{-2s}.$$

From (5.5), we have

$$s^2Y(s) + s + 4Y(s) = G(s)$$

$$\Longrightarrow Y(s)[s^2+4] = \frac{1}{s^2} - \frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{5}{s}e^{-2s} - s$$

$$\Longrightarrow Y(s) = \frac{1}{s^2(s^2+4)} - \frac{2}{s(s^2+4)}e^{-2s} - \frac{1}{s^2(s^2+4)}e^{-2s} + \frac{5}{s(s^2+4)}e^{-2s} - \frac{s}{(s^2+4)}$$

$$\Longrightarrow Y(s) = \frac{1}{s^2(s^2+4)} + \frac{3}{s(s^2+4)}e^{-2s} - \frac{1}{s^2(s^2+4)}e^{-2s} - \frac{s}{(s^2+4)} \tag{5.6}$$

$$\therefore Y(s) = \frac{1 + (3s-1)e^{-2s} - s^3}{s^2(s^2+4)}. \tag{5.7}$$

This question does not ask for the actual solution, y(t). But to determine the actual solution, you could find the inverse Laplace transform of (5.7) or (5.6).

(b) This is much easier, try it.

5.0.9 IVPs with Non-zero Initial Conditions

We use the following example to illustrate how to solve IVPs with non-zero initial conditions.

Example 47. Solve the IVP using Laplace transforms:

$$z''(t) + 5z'(t) - 6z(t) = 21e^{t-1} \tag{5.8}$$

$$z(1) = -1, \quad z'(1) = 9. \tag{5.9}$$

Solution. The idea is to shift the initial condition to start from $t = 0$. To do that, let

$$y(t) = z(t+1),$$

where the 1 in $t + 1$ is because the initial value is 1.

$$\Longrightarrow y'(t) = z'(t+1), \quad \text{and} \quad y''(t) = z''(t+1).$$

In equation (5.8), replace t by $t + 1$ to get

$$z''(t+1) + 5z'(t+1) - 6z(t+1) = 21e^t.$$

So we get a new IVP:

$$y'' + 5y' - 6y = 21e^t \tag{5.10}$$
$$y(0) = z(1) = -1, \quad \text{and} \quad y'(0) = z'(1) = 9.$$

We can now solve as usual by taking the Laplace transform of (5.10) to get

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] - 6Y(s) = \frac{21}{s-1}$$
$$\implies [s^2Y(s) + s - 9] + 5sY(s) + 5 - 6Y(s) = \frac{21}{s-1}$$
$$\implies Y(s)[s^2 + 5s - 6] + s - 4 = \frac{21}{s-1}$$
$$\implies Y(s)(s^2 + 5s - 6) = \frac{21}{s-1} + 4 - s$$
$$\implies Y(s) = \frac{21}{(s-1)(s^2+5s-6)} + \frac{4-s}{(s^2+5s-6)}$$
$$= \frac{21}{(s-1)(s^2+5s-6)} + \frac{-(s-1)+3}{(s-1)(s+6)}$$
$$= \frac{21}{(s-1)(s^2+5s-6)} - \frac{1}{(s+6)} + \frac{3}{(s-1)(s+6)}$$
$$\implies Y(s) = \frac{21 + 3(s-1)}{(s-1)^2(s+6)} - \frac{1}{(s+6)}$$
$$= \frac{3(s+6)}{(s-1)^2(s+6)} - \frac{1}{(s+6)}$$
$$= \frac{3}{(s-1)^2} - \frac{1}{(s+6)}$$

Taking the inverse Laplace transform gives

$$y(t) = 3te^t - e^{-6}.$$
$$\implies z(t+1) = 3te^t - e^{-6t}$$

We can now replace t by $t-1$ to get

$$\boxed{z(t) = 3(t-1)e^{t-1} - e^{-6(t-1)}.}$$

5.0.10 Transforms of Discontinuous Functions

Unit Step Function

Definition 17. The unit step function or Heaviside function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

See Figure 5.2.

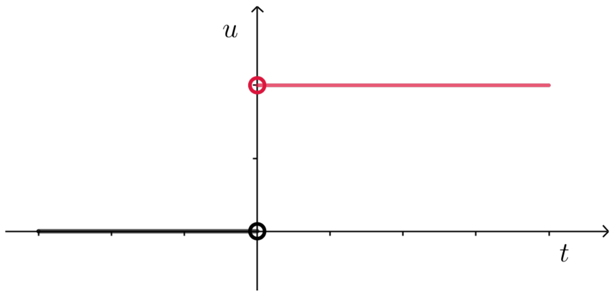

Figure 5.2: The unit step function

Remark. (a) The jump can be moved to a different location, say $t = a$:

$$u(t-a) = \begin{cases} 0, & t-a < 0 \\ 1, & t-a > 0 \end{cases} = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

(b) The height of the jump can also be changed by multiplying by a constant, say M:

$$Mu(t-a) = \begin{cases} 0, & t < a \\ M, & t > a \end{cases}$$

Figure 5.3 illustrates some of these properties.

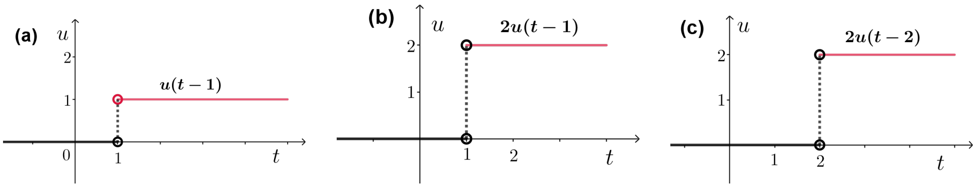

Figure 5.3:

Example 48. Write the given function, $f(t)$, in terms of unit step functions.

(a) $f(t) = \begin{cases} 3, & t < 2 \\ 1, & 2 < t < 5. \end{cases}$

(b) $f(t) = \begin{cases} 3, & t < 2 \\ 1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ \frac{t^2}{10}, & t > 8 \end{cases}$

Solution. (a) $f(t) = \begin{cases} 3, & t < 2 \\ 1, & 2 < t < 5. \end{cases}$

The function is sketched in Figure 5.4. We see that it is equal to 3 until t reaches 2, then it jumps by -2 units to the value 1. Thus, it can be written as

$$f(t) = 3 - 2u(t - 2).$$

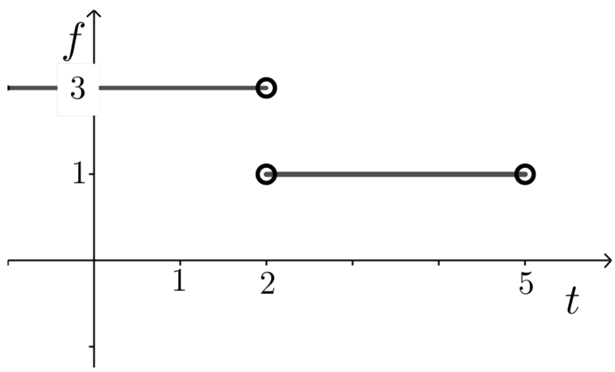

Figure 5.4:

(b) $f(t) = \begin{cases} 3, & t < 2 \\ 1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ \frac{t^2}{10}, & t > 8 \end{cases}$

The function is sketched in Figure 5.5, and is can be written as

$$f(t) = 3 - 2u(t - 2) + (t - 1)u(t - 5) + \left(\frac{t^2}{10} - t\right) u(t - 8).$$

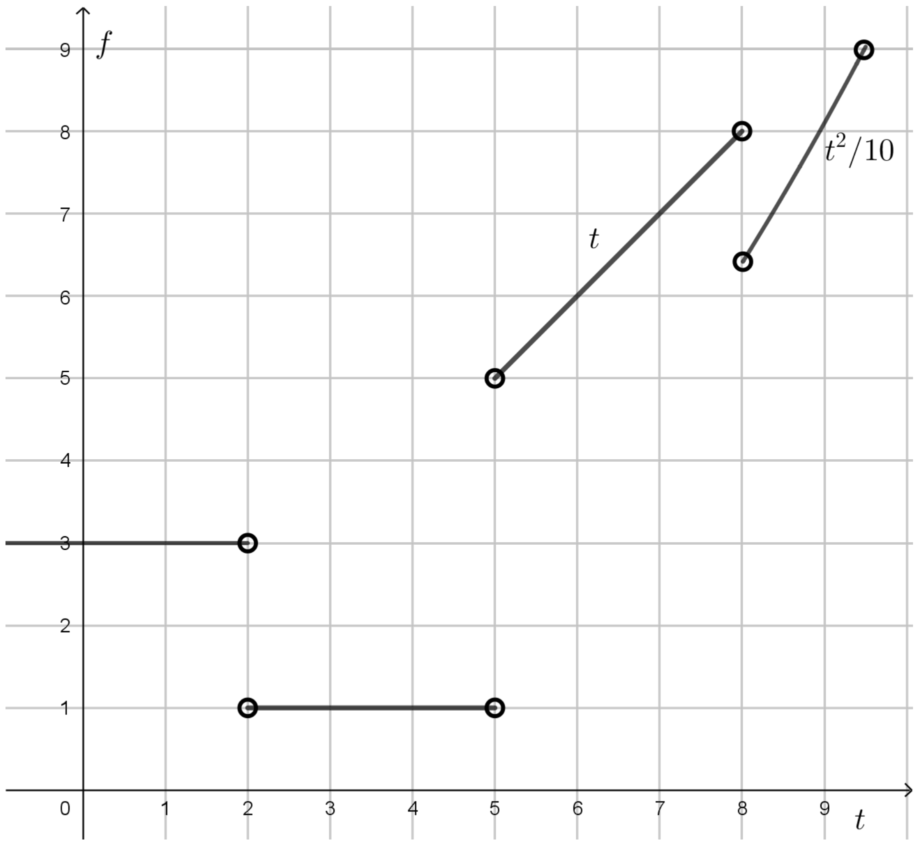

Figure 5.5:

5.0.11 Properties

(1) $\mathcal{L}\{u(t-a)\}(s) = \dfrac{e^{-as}}{s}, \quad s > 0.$

(2) $\mathcal{L}^{-1}\left\{\dfrac{e^{-as}}{s}\right\} = u(t-a)$

(3) $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$

(4) $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a)$

(5) $\mathcal{L}\left\{g(t)u(t-a)\right\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s).$

Proof. (3) $\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$

But

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

$$\Longrightarrow \mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_a^\infty e^{-st} f(t-a)dt$$

Let $v = t - a$, $\Longrightarrow dv = dt$.

$$\Longrightarrow \mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-s(v+a)} f(v)dv$$

$$= e^{-as} \int_0^\infty e^{-sv} f(v)dv$$

$$\therefore \mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s).$$

(1) Let $f = 1$ in property (3). Since $\mathcal{L}\{1\} = 1/s$, we get

$$\mathcal{L}\{u(t-a)\}(t) = \frac{e^{-as}}{s}.$$

(5) From property (3) let $g(t) = f(t-a)$. Then $f(t) = g(t+a)$. Thus

$$\mathcal{L}\left\{g(t)u(t-a)\right\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s).$$

□

Example 49. Determine the Laplace transform of

(a) $(t-1)^2 u(t-1)$

(b) $t^2 u(t-2)$

(c) $u(t-1) - u(t-4)$.

Solution. (a) $(t-1)^2 u(t-1)$

From property (3), we have

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}(s) = e^{-as}F(s)$$

Let

$$f(t-a) = (t-1)^2 \quad \text{and} \quad a = 1.$$

$$\Longrightarrow f(t) = t^2 \quad \Longrightarrow F(s) = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\Longrightarrow \mathcal{L}\{(t-1)^2 u(t-1)\} = e^{-s} \cdot \frac{2}{s^3} = \frac{2e^{-s}}{s^3}.$$

(b) $t^2u(t-2)$
From property (5), we have $\mathcal{L}\left\{g(t)u(t-a)\right\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$.

$$g(t) = t^2, \quad a = 2.$$

$$\implies g(t+a) = (t+2)^2 = e^2 + 4t + 4$$

$$\implies \mathcal{L}\{g(t+a)\} = \mathcal{L}\{g(t+2)\} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

$$\implies \mathcal{L}\{t^2u(t-2)\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).$$

(c) Let $q = u(t-1) - u(t-4)$.

$$\mathcal{L}\{q\} = \frac{e^{-s}}{s} - \frac{e^{-4s}}{s}.$$

Example 50. Determine the inverse Laplace transform of

(a) $q = \dfrac{e^{-3s}}{s^2}$

(b) $q = \dfrac{e^{-2s}}{s-1}$

(c) $q = \dfrac{e^{-2s} - 3e^{-4s}}{s+2}$

(d) $q = \dfrac{se^{-3s}}{s^2+4s+5}$

Solution. (a) $q = \dfrac{e^{-3s}}{s^2}$

$$q = \frac{e^{-3s}}{s^2} = e^{-3s} \cdot \frac{1}{s^2}$$

Use the property

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a)$$

$$F(s) = \frac{1}{s^2} \quad \implies f(t) = t, \quad a = 3$$

$$\implies f(t-3) = (t-3)$$

$$\implies \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\} = (t-3)u(t-3).$$

(b) $q = \dfrac{e^{-2s}}{s-1}$
Use the property

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a)$$

$$F(s) = \frac{1}{s-1} \quad \Longrightarrow f(t) = e^t, \quad a = 2$$

$$f(t-2) = e^{t-2}$$

$$\Longrightarrow \mathcal{L}^{-1}\{q\} = e^{t-2}u(t-2).$$

(c) $q = \dfrac{e^{-2s} - 3e^{-4s}}{s+2}$

$$\Longrightarrow q = e^{-2s} \cdot \frac{1}{s+2} - 3e^{-4s} \cdot \frac{1}{s+2}$$

Use the property

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a)$$

For

$$e^{-2s} \cdot \frac{1}{s+2}, \quad F(s) = \frac{1}{s+2}, \quad a = 2$$

$$\Longrightarrow f(t) = e^{-2t} \quad \Longrightarrow f(t-2) = e^{-2(t-2)}$$

$$\Longrightarrow f(t-2) = e^{-2t+4}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s+2}\right\} = e^{-2t+4}u(t-2).$$

For

$$e^{-4s} \cdot \frac{1}{s+2}, \quad F(s) = \frac{1}{s+2}, \quad a = 4$$

$$\Longrightarrow f(t-4) = e^{-2(t-4)} = e^{-2t+8}$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{e^{-4s} \cdot \frac{1}{s+2}\right\} = e^{-2t+8}u(t-4).$$

$$\therefore \mathcal{L}^{-1}\{q\} = e^{-2t+4}u(t-2) - 3e^{-2t+8}u(t-4).$$

(d) $q = \dfrac{se^{-3s}}{s^2+4s+5}$

$$\Longrightarrow q = e^{-3s} \cdot \frac{s}{s^2+4s+5}$$

But

$$\frac{s}{s^2+4s+5} = \frac{(s+2)-2}{(s+2)^2+1} = \frac{(s+2)}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1}$$

$$\Longrightarrow q = e^{-3s} \cdot \frac{(s+2)}{(s+2)^2+1} - 2e^{-3s} \cdot \frac{1}{(s+2)^2+1}$$

Use the property

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a)$$

Let

$$F_1(s) = \frac{(s+2)}{(s+2)^2+1}$$

$$\Longrightarrow f_1(t) = e^{-2t}\cos t \quad \Longrightarrow f_1(t-a) = e^{-2(t-3)}\cos(t-3)$$

Let

$$F_2(s) = \frac{1}{(s+2)^2+1}$$

$$\Longrightarrow f_2(t) = e^{-2t}\sin t \quad \Longrightarrow f_2(t-3) = e^{-2(t-3)}\sin(t-3)$$

$$\Longrightarrow \mathcal{L}^{-1}\{q\} = e^{-2(t-3)}\cos(t-3)u(t-3) - 2e^{-2(t-3)}\sin(t-3)u(t-3)$$

$$\therefore \boxed{\mathcal{L}^{-1}\{q\} = e^{-2(t-3)}\left[\cos(t-3) - 2\sin(t-3)\right]u(t-3).}$$

Example 51. Solve the IVP using Laplace transforms.

(a) $y'' + y = u(t-3), \quad y(0) = 0, \quad y'(0) = 1.$

(b) $y'' + 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 2,$
where

$$g(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 < t < 5, \\ 1, & t > 5. \end{cases}$$

Solution. (a) $y'' + y = u(t-3), \quad y(0) = 0, \quad y'(0) = 1.$
Taking the Laplace transform of the equation:

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u(t-3)\}$$

$$\Longrightarrow \left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \frac{e^{-3s}}{s}$$

$$\Longrightarrow s^2Y - 1 + Y = \frac{e^{-3s}}{s}$$

$$\Longrightarrow Y(s)(s^2+1) = \frac{e^{-3s}}{s} + 1$$

$$\Longrightarrow Y(s) = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{s^2+1} = e^{-3s}\cdot\frac{1}{s(s^2+1)} + \frac{1}{s^2+1}$$

Note that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.$$

Now,

$$\frac{1}{s^2+1} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$
$$\Longrightarrow 1 = A(s^2+1) + (Bs+C)s$$

$$\begin{aligned}
&s = 0: \quad 1 = A \\
&s = 1: \quad 1 = 2A + B + C \\
&\quad \Longrightarrow B + C = -1 \\
&s = -1: \quad 1 = 2 - (-B + C) = 2 + B - C \\
&\quad \Longrightarrow B - C = -1 \\
&\quad \Longrightarrow 2B = -2 \quad \Longrightarrow B = -1 \\
&\quad \Longrightarrow C = -1 + 1 = 0
\end{aligned}$$

Thus,

$$\frac{1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$
$$\Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 1 - \cos t$$
$$\Longrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$\therefore \boxed{y(t) = [1 - \cos(t-3)]u(t-3) + \sin t.}$$

(b)

$$\begin{aligned}
y'' + 5y' + 6y &= g(t), \\
y(0) = 0, \quad y'(0) &= 2,
\end{aligned} \tag{5.11}$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 < t < 5, \\ 1, & t > 5, \end{cases}$$

is sketched in Figure 5.6. $g(t)$ can be written as

$$g(t) = tu(t-1) + (1-t)u(t-5).$$

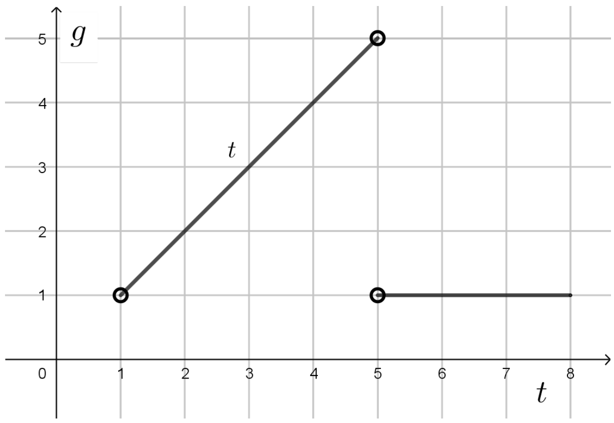

Figure 5.6:

Let the Laplace transform of $g(t)$ be $G(s)$.

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{tu(t-1)\} + \mathcal{L}\{(1-t)u(t-5)\}$$

For $\mathcal{L}\{tu(t-1)\}$, we use the property

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$

$$f(t) = t, \quad a = 1$$

$$\Longrightarrow \mathcal{L}\{f(t+a)\} = \mathcal{L}\{(t+1)\} = \frac{1}{s^2} + \frac{1}{s}$$

$$\Longrightarrow \mathcal{L}\{tu(t-1)\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

For $\mathcal{L}\{(1-t)u(t-5)\}$, we again use the property

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}, \quad a = 5.$$

Now $f(t) = 1 - t \quad \Longrightarrow f(t+a) = 1 - (t+5) = -4 - t.$

$$\Longrightarrow \mathcal{L}\{f(t+a)\} = \mathcal{L}\{-4-t\} = \left(-\frac{4}{s} - \frac{1}{s^2}\right)$$

$$\Longrightarrow \mathcal{L}\{(1-t)u(t-5)\} = e^{-5s}\left(-\frac{4}{s} - \frac{1}{s^2}\right).$$

$$\therefore G(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - 4\frac{e^{-5s}}{s} - \frac{1}{s^2}e^{-5s}.$$

Now, from the differential equation (5.11):

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 6Y(s) = G(s).$$

$$\Longrightarrow (s^2Y - 2) + 5sY + 6Y = G(s)$$

$$\Longrightarrow Y(s)(s^2 + 5s + 6) = G(s) + 2$$

$$\Longrightarrow Y(s) = \frac{G(s)}{s^2 + 5s + 6} + \frac{2}{s^2 + 5s + 6}$$

$$s^2 + 5s + 6 = (s + 2)(s + 3)$$

Thus,

$$\begin{aligned} Y(s) = e^{-s}\frac{(s+1)}{s^2(s+2)(s+3)} - 4e^{-5s}\frac{1}{s(s+2)(s+3)} + \\ \frac{2}{(s+2)(s+3)} - e^{-5s}\frac{1}{s^2(s+2)(s+3)} \\ Y(s) = e^{-s}\frac{(s+1)}{s^2(s+2)(s+3)} + \frac{2}{(s+2)(s+3)} - \\ e^{-5s}\left[\frac{4}{s(s+2)(s+3)} + \frac{1}{s^2(s+2)(s+3)}\right] \end{aligned}$$

$$Y(s) = e^{-s}\frac{(s+1)}{s^2(s+2)(s+3)} + \frac{2}{(s+2)(s+3)} - e^{-5s}\left[\frac{4s+1}{s^2(s+2)(s+3)}\right] \tag{5.12}$$

We next simplify each term in (5.12) using partial fractions and find their inverse.

$$Y_1(s) = \frac{(s+1)}{s^2(s+2)(s+3)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$\Longrightarrow s+1 = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2)$$

$$s = 0 : \quad 1 = 6B \quad \Longrightarrow B = \frac{1}{6}$$

$$s = -2 : \quad -1 = 4C \quad \Longrightarrow C = -\frac{1}{4}$$

$$s = -3 : \quad -2 = -9D \quad \Longrightarrow D = \frac{2}{9}$$

$$s = 1 : \quad 2 = 12A + 12\left(\frac{1}{6}\right) + 4\left(-\frac{1}{4}\right) + 3\left(\frac{2}{9}\right)$$

$$\Longrightarrow 2 = 12A + 2 - 1 + \frac{2}{3}$$

$$\Longrightarrow 12A = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Longrightarrow A = \frac{1}{36}.$$

Thus,

$$\frac{(s+1)}{s^2(s+2)(s+3)} = \frac{1}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{1}{s+2}\right) + \frac{2}{9}\left(\frac{1}{s+3}\right)$$

$$\Longrightarrow \mathcal{L}^{-1}\{Y_1(s)\} = \frac{1}{36} + \frac{1}{6}t - \frac{1}{4}e^{-2t} + \frac{2}{9}e^{-3t}$$

Thus,

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{(s+1)}{s^2(s+2)(s+3)}\right\} = \left[\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right]u(t-1) \tag{5.13}$$

From (5.12), we let

$$\frac{2}{(s+2)(s+3)} \equiv \frac{A}{s+2} + \frac{B}{s+3}$$

$$\Longrightarrow 2 = A(s+3) + B(s+2)$$

$$s = -2: \quad 2 = A$$

$$s = -3: \quad 2 = -B \quad \Longrightarrow B = -2$$

$$\Longrightarrow \frac{2}{(s+2)(s+3)} = \frac{2}{s+2} - \frac{2}{s+3}$$

$$\Longrightarrow \mathcal{L}\left\{\frac{2}{(s+2)(s+3)}\right\} = 2e^{-2t} - 2e^{-3t} \tag{5.14}$$

From (5.12), we let

$$\frac{4s+1}{s^2(s+2)(s+3)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$$

$$\Longrightarrow 4s+1 = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2)$$

$$s = 0: \quad 1 = 6B \quad \Longrightarrow B = \frac{1}{6}$$

$$s = -2: \quad -7 = 4C \quad \Longrightarrow C = -\frac{7}{4}$$

$$s = -3: \quad -11 = -9D \quad \Longrightarrow D = \frac{11}{9}$$

$$s = 1: \quad 5 = 12A + \left(\frac{1}{6}\right)(3)(4) - \left(-\frac{7}{4}\right)(4) + \left(\frac{11}{9}\right)(3)$$

$$\Longrightarrow 5 = 12A + 2 - 7 + \frac{11}{3} = -\frac{4}{3}$$

$$\Longrightarrow 12A = 5 + \frac{4}{3} = \frac{19}{3}$$

$$\Longrightarrow A = \frac{19}{36}$$

So,

$$\frac{4s+1}{s^2(s+2)(s+3)} = \frac{19}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) - \frac{7}{4}\left(\frac{1}{s+2}\right) + \frac{11}{9}\left(\frac{1}{s+3}\right)$$

$$\Longrightarrow \mathcal{L}^{-1}\{\frac{4s+1}{s^2(s+2)(s+3)}\} = \frac{19}{36} + \frac{1}{6}t - \frac{7}{4}e^{-2t} + \frac{11}{9}e^{-3t}$$

Thus,

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{4s+1}{s^2(s+2)(s+3)}\right\} = \left[\frac{19}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}\right]u(t-5) \tag{5.15}$$

From (5.12) we have

$$y(t) = \left\{e^{-s}\frac{(s+1)}{s^2(s+2)(s+3)}\right\} + \left\{\frac{2}{(s+2)(s+3)}\right\} - \left\{e^{-5s}\left[\frac{4s+1}{s^2(s+2)(s+3)}\right]\right\} \tag{5.16}$$

Substituting (5.13)-(5.15) into (5.16), we finally get

$$y(t) = \left[\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right]u(t-1) -$$
$$\left[\frac{19}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}\right]u(t-5) + e^{-2t} - 2e^{-3t}.$$

EXERCISE 3. Solve the IVP using Laplace transforms

$$y'' + y = u(t-2) - u(t-4)$$

$$y(0) = 1, \quad y'(0) = 0.$$

Solution. Taking the Laplace transform of the equation, we get

$$[s^2Y - sy(0) - y'(0)] + Y = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$\Longrightarrow s^2Y - s + Y = \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}$$

$$\Longrightarrow Y(s)(s^2+1) = e^{-2s}\frac{1}{s} - e^{-4s}\frac{1}{s} + s$$

$$Y(s) = e^{-2s}\frac{1}{s(s^2+1)} - e^{-4s}\frac{1}{s(s^2+1)} + \frac{s}{(s^2+1)} \tag{5.17}$$

Let

$$\frac{1}{s(s^2+1)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\Longrightarrow 1 = A(s^2+1) + (Bs+C)s$$

$$s = 0: \quad 1 = A$$

$$s = 1: \quad 1 = 2 + B + C \quad \Longrightarrow B + C = -1$$

$$s = -1: \quad 1 = 2 - (-B + C) \quad \Longrightarrow B - C = -1$$

$$\Longrightarrow 2B = -2 \quad \Longrightarrow B = -1$$

$$\Longrightarrow C = -1 + 1 = 0$$

$$\Longrightarrow \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

$$\Longrightarrow \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s(s^2+1)}\right\} = [1 - \cos(t-2)]u(t-2),$$

and

$$\mathcal{L}^{-1}\left\{e^{-4s}\frac{1}{s(s^2+1)}\right\} = [1 - \cos(t-4)]u(t-4).$$

Also

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

Hence, from (5.17), we get

$$y(t) = [1 - \cos(t-2)]u(t-2) - [1 - \cos(t-4)]u(t-4) + \cos t.$$

Chapter 6

Fourier Series

6.1 Introduction