Homework 8:
Fourier Analysis of Dynamic Systems

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Problem 1: Multiplication in the Time Domain

In civil engineering systems (namely, physical systems), the Fourier Transform is used to convert
dynamical systems from the time domain to the frequency domain so that convolution of signals
in the time domain simplify to multiplication in the frequency domain. In electrical engineering,
often signals are multiplied in the time domain. For example, consider two signals, \( g(t) \) and \( h(t) \),
that are multiplied in the time domain: \( y(t) = g(t)h(t) \). Show that the signal \( y(t) \) in the frequency
domain is the convolution of the Fourier Spectra of \( g(t) \) and \( h(t) \): \( Y(\omega) = \int G(\omega)H(\omega - u)du \).

Problem 2: Compression of Signal

The following saw-tooth signal is a periodic signal of period, \( T_p \). We will explore an alternative
representation of the signal using Fourier Series.

(a) Write an alternative representation of the signal using Fourier Series.

(b) Plot from \( t = -2T_p \) to \( 2T_p \) the first term of the Fourier Series. Superimpose on the plot
the original signal.

(c) Plot from \( t = -2T_p \) to \( 2T_p \) the sum of the first two terms of the Fourier Series. Superimpose on the plot the original signal.
Plot from $t = -2T_p$ to $2T_p$ the sum of the first three terms of the Fourier Series. Superimpose on the plot the original signal.

Plot from $t = -2T_p$ to $2T_p$ the sum of the first four terms of the Fourier Series. Superimpose on the plot the original signal.

The signal’s representation via the Fourier Series is an infinite sum. However, the coefficients of the sinusoidal terms of the Fourier Series represents a compressed (sparse) representation of the original signal. If only the first $N$ terms of the Fourier Series are used, the signal will not be an exact representation. Plot the root mean square error per period of the signal as a function of $N$.

**Problem 3: Fourier Transform**

Write and plot the Fourier representation, $X(\omega)$, of the following signals:

![Signal diagrams](image)

**Problem 4: Dynamic Analysis**

The one-story structure shown above is defined by its mass, $m$, stiffness, $k$, and damping, $c$, is exposed to a lateral load, $f(t)$. We are not certain of the precise lateral load but we do know its spectral density which is assumed to be $S_f(\omega) = S_\alpha$. 

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(a) Determine its output spectral density function, $S_x(\omega)$.
(b) Provide an analytical expression for the mean square response, $E[x^2]$ of the structure.

**Problem 5: Signal Analysis**

Let $x(t)$ be a periodic signal with fundamental period $T$ and Fourier series $a_k$. Derive the Fourier series coefficients of each of the following signals in terms of $a_k$

(a) $x(t - t_0) + x(t + t_0)$
(b) $Re[x(t)]$
(c) $\frac{d^2}{dt^2}x(t)$
(d) $x(3t - 1)$ [Hint: first determine the period of $x(3t - 1)$]