Problem 1: Numerical Integration for System Modeling

A dynamical system is defined in continuous-time as: \( \dot{y} + 4y = \dot{u} \)

(a) Using a backward-difference approximation for system derivatives, derive a discrete-time system equivalent.

(b) We are interested in modeling the response of the system to an impulse and step excitation. In continuous-time, \( u(t) = \delta(t) \) and \( u(t) = 1(t) \) correspond to the impulse and step excitations, respectively. In discrete-time, these system inputs can be modeled as \( u(0) = 1/T \) and \( u(k) = 1 \), respectively.
   i. Find the closed form solution (impulse and step) to the dynamic system in continuous-time
   ii. Find the closed form solution (impulse and step) to the dynamic system in discrete-time

(c) Plot the continuous and discrete-time solutions (on the same plot overlayed) for \( T = 0.1 \text{sec} \) and \( T = 0.01 \text{sec} \)

Problem 2: Inverse Z-Transform of System Response

Consider a system exposed to a dynamic load, \( f(t) \), undergoing a dynamic response, \( x(t) \). Assume system identification yields a transfer function for the structure for a given applied load as:

\[
H(z) = \frac{X(z)}{F(z)} = \frac{z^2}{z^2 - 1.25z + 0.25}
\]

Determine the system response, \( x(k) \), to an impulse, \( f(t) = \delta(t) \), in discrete-time by:

(a) Long division
(b) Partial fraction expansion
(c) Plot the impulse response of the system using \texttt{dimpulse.m} (in MATLAB)
Problem 3: Modeling Discrete-Time Sequences

You are provided with a discrete-time sequence of \( e(k) = 1 \) when \( k = 0, 1, 2, 3 \) and \( e(k) = 0 \) elsewhere. You will model the system by a trapezoidal integration scheme:

\[
u(k) = u(k-1) + \frac{T}{2}(e(k) + e(k-1))\]

(a) Find \( E(z) \) via the Z-transform
(b) Find \( H(z) \), the transfer function of the trapezoidal integration scheme
(c) Find \( U(z) \)
(d) By long division, find \( u(k) \)
(e) Simulate the system in MATLAB and verify the previous answer for \( u(k) \)

Problem 4: FFT Modeling for Dynamic Response of Structures

A simple structure initially at rest with a total structural weight of 120 kips, stiffness of 10 kips/in and damping ratio of 0.01 is excited by a lateral pulse load with intensity \( f_o \) from \( t = 0 \) to \( T_o \) where \( f_o = 1 \) kips and \( T_o = 0.4 \) sec.

(a) Derive the exact displacement history of the structure. In MATLAB, plot the response of the structure from 0 to 2 sec with an interval spacing of \( dt = 0.05 \) sec.
(b) Using Fourier methods, determine the response of the structure over the same 2 second duration using a time step of \( f \ dt = 0.05 \) sec. You will have a chance to conduct the analysis in the frequency (Fourier) domain using a set number of points, \( N \). Naturally, \( N \) should be 40 (=2/0.05) which you will use to conduct the analysis. Plot the results in MATLAB using the same duration of 2 sec and overlay your plot with the exact answer determined above.
(c) Redo the analysis but use \( N = 64 \). In this case, what is the implied duration between periodic repetitions during which the pulse force is not acting (and during which the free vibration response from its last application is hoped to become damped out)? Plot the results in MATLAB using the same duration of 2 sec and overlay your plot with the exact answer and the answer when \( N = 40 \).
(d) Redo the analysis but use \( N = 128 \). Again, plot the results in MATLAB using the same duration of 2 sec and overlay your plot with the exact answer, \( N = 40 \) and \( N = 64 \). How do the results compare to the exact solution when \( N \) increases?
(e) A potential drawback of FFT analysis is that, in its standard form, it provides a steady-state analysis of a periodically applied force. It may therefore fail to match the true initial conditions of the transient response, because it contains the free vibration response to previous force applications. To prove this issue, let us consider a case where \( N = 64 \) but we increase damping from 0.01 to 0.2. Under higher damping, we expect transients to
die out very quickly. Determine the exact response to the structure when the damping ratio is 0.2. Next, determine the response by Fourier methods when $N = 64$. Plot the Fourier-based response to the exact response.

(f) One solution to the transients is to “zero pad” the excitation but this approach is limited in its application and represents an increase in computational effort. An alternative is to analytically handle the transients through superposition. Using the structure when damping has a ratio of 0.01 (i.e., the original structure) and when $N = 64$, determine initial displacement and velocity of the structure artificially created by the Fourier analysis. These initial conditions will likely violate the fact that the structure was initially at rest. Through superposition, determine the response of the structure to these non-rest initial conditions and analytically remove them from the Fourier results. Plot the corrected time history of the structure and overlay the theoretical answer.