Homework 2:
SISO System Models in Continuous-Time

Jerome P. Lynch, Ph.D.
Department of Civil and Environmental Engineering
University of Michigan

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Problem 1: System Classification

Please classify the following systems as: i) dynamic versus static, ii) linear versus nonlinear, and iii) time varying versus time invariant:

(a) \( y(t) = 5 \)
(b) \( y(t) = e^{u(t)} \)
(c) \( y(t) = au(t) + b \)
(d) \( y(t) = \int_0^t e^{-\sigma} u(t - \sigma) d\sigma \)

Problem 2: Building Environmental Systems

Buildings are controlled with environmental control systems (e.g., HVAC systems). With recent interest focused in the profession on more sustainable buildings that can be environmentally controlled using less energy, civil engineers are modeling the heat transfer dynamics of buildings. Assume the outside ambient temperature of a building is \( T_a \) while the interior temperature is \( T_i \). The envelope of the building will conduct heat through based on the thermal resistance of the envelope, \( R_e \). Interior heaters can warm the space based on heat flux, \( \Phi_h \); the heaters have area \( A_h \). On a sunny day, solar radiation with energy flux, \( \Phi_s \), can warm the building interior by shining through windows of area, \( A_w \). Note that the heat capacity of the interior space of the building is \( C_i \).

(a) Develop a dynamic model representing the continuous-time dynamics of heat transfer in a simple one-room building with windows of area, \( A_w \); also, the heater has an area, \( A_h \). [Hint: the system inputs will be the flux sources and ambient temperature while the system output is the internal temperature, \( T_i \)]

(b) What is the order of the dynamic model?

(c) Write an equivalent circuit that has the same dynamic model. [Hint: flux can be treated as a current source while temperatures are equivalent to voltage (potential)]
Problem 3: Dynamic Structural Systems

As discussed in class, structures exposed to a dynamic load, \( f(t) \), undergoing a dynamic response, \( x(t) \), can be modeled as a second-order continuous-time dynamical system. Assume the structure can be modeled as a single input-single output (SISO) system defined by mass, damping (i.e., equivalent viscous) and stiffness (i.e., \( m \), \( c \), and \( k \)) as shown in the figure. Here, we can safely assume the mass and stiffness are \( m = 17,280 \text{ kg} \) and \( k = 2,730 \text{kN/m} \), respectively.

(a) Write the SISO dynamic system model as a differential equation using the dynamic equilibrium method.

(b) Write the SISO dynamic system model using system operator, \( L(p) \), notation.

(c) Find the structure’s response in closed form when \( x(0) = 0.1 \) and \( \dot{x}(0) = 0 \) for the following damping values \( (c) \). For each, also plot the response for \( t \in [0, 20] \) and comment on the nature of the response including the behavior of the response amplitude and oscillatory behavior (if any).

i. \( c = 0 \text{Ns/m} \);
ii. \( c = 86,000 \text{Ns/m} \)
iii. \( c = 434,294 \text{Ns/m} \)
iv. \( c = 600,000 \text{Ns/m} \)

Problem 4: Flow through a Tank

This problem will provide you with experience in linearizing an otherwise nonlinear system (a common approach to system analysis). Consider an incompressible fluid with density, \( \rho \), pumped into a tank with mass flow rate, \( w_{\text{in}} \). The tank has a cross-sectional area of \( A \) and the height, \( h \), of the water in the tank can be observed. At the bottom of the tank is a pipe that allows water to flow out with mass flow rate, \( w_{\text{out}} \). Based on basic fluid mechanics, the flow out of the tank is dependent on the properties of the outflow piping including the pipe geometry and pipe friction. The most general form to modeling this mass flow rate is to consider the change in pressure at the pipe inlet and outlet, \( \Delta p \):
\[
 w_{\text{out}} = R^{-1}(\Delta p)^{1/\alpha}
\]
where \( R \) and \( \alpha \) are constants that are functions of the outlet piping. The ambient pressure outside of the tank is \( p_a \).
(a) Write the SISO dynamic system model as a differential equation with $w_{in}$ as the system input and height of the fluid in the tank, $h$, as the system output given all of the system parameters. [Hint: begin by defining system behavior over a time period $\Delta t$ later taking the limit as $\Delta t$ approaches zero.]

(b) Is the system a linear time invariant system (LTI)?

(c) Consider the outflow piping in more detail. If the pipe is long and smooth that favors laminar flow, then $\alpha$ is equal to 1 ($Re < 1000$). In contrast, if the pipe is short and rough, the flow is turbulent, $\alpha$ is equal to 2 ($Re > 10^5$). Under what condition is the system linear?

(d) Assume the outlet pipe is short and rough resulting in a large Reynolds number ($Re > 10^5$). Also, assume the tank begins with a water elevation of $h_o$. Write the system equation describing the dynamics of the system.

(e) Even though the system in nonlinear, linearize the system around the equilibrium point, $h_o$. [Hint: to do so, you will need to consider the dynamical system not with the output, $h$, but rather, $\Delta h$]

(f) Write the SISO dynamic system model using system operator, $L(p)$, notation.