CLASS #22: DISCRETE FOURIER TRANSFORM

OBJECTIVES:
1. SUMMARIZE VARIOUS TRANSFORMS
2. DERIVE DISCRETE FOURIER TRANSFORM (D.F.T.)
3. DESCRIBE APPLICATION OF D.F.T.

OVERVIEW OF TRANSFORM METHODS

- TRANSFORM FROM TIME (t or k) TO FREQUENCY (ω, s, z)

CONTINUOUS - TIME

\[ F(s) = \int_{0}^{\infty} f(t) e^{-st} dt \]  
\[ s = \pm iw \]  
\[ f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \]  
\[ s = \pm i\omega \]

DISCRETE - TIME

\[ F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \]  
\[ k \rightarrow z \]

DISCRETE FOURIER TRANSFORM

\[ z = \cos\theta + i\sin\theta \]  
\[ z = e^{-i\theta} \]

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\[ X(k) = \{ x_1, x_2, \ldots, x_N \} \]  

**DISCRETE-TIME SAMPLES OF \( x(t) \)**

- **FINITE SIZE** "\( N \)"
- \( N \) IS TOTAL \# OF SAMPLES
- \( T \) SAMPLE PERIOD
- \( NT \) IS TOTAL TIME DURATION

**ASSUME PERIODICITY OF SIGNAL \( \rightarrow \) APPLY FOURIER SERIES**

**FOURIER SERIES OF \( x(t) \):**

\[
X_c(t) = a_0 + 2 \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k t}{NT} + b_k \sin \frac{2\pi k t}{NT} \right)
\]

\[
a_k = \frac{1}{NT} \int_{0}^{NT} X_c(t) \cos \frac{2\pi k t}{NT} \, dt
\]

\[
b_k = \frac{1}{NT} \int_{0}^{NT} X_c(t) \sin \frac{2\pi k t}{NT} \, dt
\]

\[
a_0 = \frac{1}{NT} \int_{0}^{NT} X_c(t) \, dt
\]

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Complex Form:

\[ X_k = a_k - ib_k \]

\[ = \frac{1}{NT} \int_{0}^{NT} x(t) \left[ \cos \frac{2\pi kt}{NT} - i \sin \frac{2\pi kt}{NT} \right] dt \]

\[ = \frac{1}{NT} \int_{0}^{NT} x(t) e^{-i \left( \frac{2\pi kt}{NT} \right)} dt \]

- We don't know \( x(t) \) & we only have \( x(t) \)
- Approximate Fourier Series Integral

\[ X_k = \frac{1}{NT} \sum_{r=0}^{N-1} x(r) e^{-i \left( \frac{2\pi kr}{NT} \right)} \]

\[ X_k = \frac{1}{N} \sum_{r=0}^{N-1} x(r) e^{-i \left( \frac{2\pi r k}{N} \right)} \]

Discrete Fourier Transform

- \( r = \text{Index of Time Quanta} \)
- \( k = \text{Index of Freq. Quanta} \)

- In MATLAB: \( \gg x = \text{fft}(x) \)
- FFT is for Fast Fourier Transform ("Fast" Algorithm) for D.F.T.

Inverse Discrete Fourier Transform:

\[ x(r) = \sum_{k=0}^{N-1} X_k e^{i \left( \frac{2\pi kr}{N} \right)} \]

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3. Frequency Scale

\[ X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-i \frac{2\pi kn}{N}} \quad \iff \quad X(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-i\omega t} dt \]

\[ \omega_k = \frac{2\pi k}{TN} \]

4. Symmetry Properties

Consider: \( k = l \) \[ X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-i \left( \frac{2\pi nl}{N} \right)} \]

\( k = -l \) \[ X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{i \left( \frac{2\pi nl}{N} \right)} \]

\( k = 1 \) \[ X_1 = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-i \left( \frac{2\pi n}{N} \right)} \]

\( k = N-1 \) \[ X_{N-1} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-i \left( \frac{2\pi (N-1)n}{N} \right)} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-i \left( \frac{2\pi n}{N} \right)} \]

Conjugate Pairs:

\( X_1^* = X_1 \)

\( X_{N-1}^* = X_1 \)
(5) **Nyquist Frequency**

- For all intensive purposes, information in $X_k$ is unique for $k = 0 \rightarrow \frac{N}{2} - 1$

  $W = 0$

  $W = \frac{2\pi}{NT} \left( \frac{N}{2} - 1 \right)$

  $\approx \frac{\pi}{T}$

- $W_{\text{sampling}} = \frac{2\pi}{T}$

  $W = \frac{\pi}{T} = \text{Nyquist Frequency} = \frac{1}{2} W_{\text{sample}}$

- Can uniquely resolve signal harmonics from 0 (DC) to Nyquist frequency

- Are going from "N" points in time to $N/2$ in freq?
  - No, b/c $X_k$ contains amplitude & phase
  - So $N/2$ unique $X_k$ has $N$ information

- Sample at least 2x highest harmonic if not 3x to 5x

**Example:**

If $X(t)$ is continuous

Now say $X(t)$ is sampled at $W_s$

Lose $W_4$ and higher harmonics

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Relation to Z-Transform

\[ Z: F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} \]

- Let \( z \) be on unit circle:
  \[ z = e^{-i \frac{2\pi k}{N}} \]

\[ F(z) = \sum_{n=0}^{N-1} f(n) e^{-i \frac{2\pi n k}{N}} \] (D.F.T.)

- So, DFT is Z-Transform on unit circle
- Explains why we repeat \( x_k \) over and over

As \( k \to \infty \), we keep "looping" on unit circle

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