CLASS #20: DISCRETE-FOURIER TRANSFORM

OBJECTIVES: 1. SUMMARIZE VARIOUS TRANSFORMS
2. DERIVE DISCRETE FOURIER TRANSFORM (D.F.T.)
3. DESCRIBE APPLICATION OF D.F.T.

1. OVERVIEW OF TRANSFORM METHODS
   - TRANSFORM FROM TIME (t or k) TO FREQUENCY (ω, s, z)

   CONTINUOUS - TIME
   \[ F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt \]  
   \[ s = \pm \omega i \]
   \[ f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \]

   DISCRETE - TIME
   \[ F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \]  
   \[ k \rightarrow z \]

   DISCRETE FOURIER TRANSFORM
   \[ k \rightarrow \omega \]

   LHP [STABILITY REGION]
   \[ Re(s) \]
   \[ Im(s) \]

   SLICE THROUGH \( F(s) \) AT \( s = \pm \omega i \)

   UNIT CIRCLE [STABILITY REGION]
   \[ z = \cos \theta + i \sin \theta \]
   \[ = e^{-i\theta} \]
2 DISCRETE FOURIER TRANSFORM

\[ X(k) = \{ x_1, x_2, \ldots, x_n \} \quad \text{DISCRETE-TIME SAMPLES of } X_c(t) \]

FINITE SIZE "N"

- \( N \) IS TOTAL # OF SAMPLES
- \( T \) SAMPLING PERIOD
- \( NT \) IS TOTAL TIME DURATION

ASSUME PERIODICITY OF SIGNAL → APPLY FOURIER SERIES

FOURIER SERIES OF \( X_c(t) \):

\[ X_c(t) = a_0 + 2 \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k t}{NT} + b_k \sin \frac{2\pi k t}{NT} \right) \]

\[ b_k = \frac{1}{NT} \int_{0}^{NT} X_c(t) \sin \frac{2\pi k t}{NT} \, dt \]

\[ a_k = \frac{1}{NT} \int_{0}^{NT} X_c(t) \cos \frac{2\pi k t}{NT} \, dt \]

\[ a_0 = \frac{1}{NT} \int_{0}^{NT} X_c(t) \, dt \]
In complex form:

\[ X_k = a_k - ib_k \]

\[ = \frac{1}{NT} \int_0^{NT} X_e(t) \left[ \cos \frac{2\pi k t}{NT} - i \sin \frac{2\pi k t}{NT} \right] dt \]

\[ = \frac{1}{NT} \int_0^{NT} X_e(t) e^{-i \left( \frac{2\pi k t}{NT} \right)} dt \]

- We don't know \( X_e(t) \) & we only have \( X(t) \)
- Approximate Fourier Series Integral

\[ X_k = \frac{1}{NT} \sum_{r=0}^{N-1} X(r) e^{-i \left( \frac{2\pi k r T}{NT} \right)} \]

**Discrete Fourier Transform**

- \( r \) = Index of time quanta
- \( k \) = Index of freq. quanta

- In MATLAB:  \( \rightarrow X = \text{fft}(X) \)
- FFT is for fast Fourier transform ("fast" algorithm for D.F.T.)

- Inverse Discrete Fourier Transform:

\[ X(t) = \sum_{k=0}^{N-1} X_k e^{i \left( \frac{2\pi k r}{N} \right)} \]
3. Frequency Scale

\[ X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi nk}{N}} \leftrightarrow X(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-j\omega t} \, dt \]

\[ \omega_k = \frac{2\pi k}{NT} \]

4. Symmetry Properties

Consider: \( k = l \) \( \rightarrow \) \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi kl}{N}} \)

\( k = -l \) \( \rightarrow \) \( X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{j \frac{2\pi kl}{N}} \)

\( k = 1 \) \( \rightarrow \) \( X_1 = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi n}{N}} \)

\( k = N-1 \) \( \rightarrow \) \[ X_{N-1} = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi (N-1)n}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{j \frac{2\pi n}{N}} \]

\( X_1 = X_1^* \)

\( X_{N-1} = X_1^* \)
- **Nyquist Frequency**

For all intensive purposes, information in \( X_k \) is unique for \( k = 0 \rightarrow \frac{N}{2} - 1 \)

\[
W = \frac{2\pi}{NT} (\frac{N}{2} - 1) = \frac{\pi}{T} - \frac{2\pi}{NT} = \frac{(\pi - 2\pi/N)}{T} \approx \frac{\pi}{T}
\]

- \( W_{\text{Sampling}} = \frac{2\pi}{T} \)

\( W = \frac{\pi}{T} = \text{Nyquist Frequency} = \frac{1}{2} W_{\text{Sample}} \)

- Can uniquely resolve signal harmonics from 0 (DC) to Nyquist frequency

- Are going from "N" points in time to \( N/2 \) in freq?
  - No, b/c \( X_k \) contains amplitude \& phase
  - So \( \frac{N}{2} \) unique \( X_k \) has \( N \) information

- Sample at least 2\( \times \) highest harmonic if not 3\( \times \) to 5\( \times \)

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**Example:**

If \( X(t) \) is continuous

\[
X(w)
\]

Now say \( X(t) \) is sampled at \( W_s \)

\[
\frac{-W_s}{2} \rightarrow 0 \rightarrow \frac{W_s}{2} \rightarrow W_s
\]

\( W_s \), lose \( W_4 \) and higher harmonics
6. **RELATION TO Z-TRANSFORM**

\[ F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n} \]

- Let \( z \) be on unit circle:
  \[ z = e^{-i \frac{2\pi k}{N}} \]

\[ F(z) = \sum_{n=-\infty}^{\infty} f(n) e^{-i \frac{2\pi n k}{N}} \quad \text{(D.F.T.)} \]

- So, DFT is Z-transform on unit circle
- Explains why we repeat \( x_k \) over and over

As \( k \to \infty \), we keep "looping" on unit circle