CLASS #18: CORRELATION ANALYSIS OF RANDOM SIGNALS

OBJECTIVES: 1. DEFINE CORRELATION  
2. DERIVE REGRESSION ANALYSIS  
3. RELATE CORRELATION TO CHARACTERISTICS OF RANDOM SIGNAL

REGRESSION BETWEEN 2 RANDOM VARIABLES:

VISUAL OBSERVATIONS: "UNCORRELATED"  "CORRELATED"

CASE 1: \( E[x] = E[y] = 0 \)

FIND \( y = mx \) THAT BEST FITS DATA:

\[ e = y - mx \]

(MINIMIZE)

SAME AS, MINIMIZE:

\[ E[e^2] = E[(y - mx)^2] \]

\[ E[e^2] = E[y^2 - 2mxy + m^2x^2] \]

\[ \frac{dE[e^2]}{dm} = -2E[xy] + 2mE[x^2] = 0 \]

\[ \therefore m_{opt} = \frac{E[xy]}{E[x^2]} = \frac{E[xy]}{\sigma_x^2} \]
\[ y = m_{\text{opt}} x \]
\[ y = \frac{E[x y]}{\sigma_x^2} \cdot x \cdot \frac{1}{\sigma_y} \]

\[ \frac{y - m_y}{\sigma_y} = \frac{E[(x - m_x)(y - m_y)]}{\sigma_x \sigma_y} \cdot \frac{x - m_x}{\sigma_x} \]

**CASE 2**: \( E[x] = m_x \neq 0 \); \( E[y] = m_y \neq 0 \)

\[ \rho_{xy} = \text{CORRELATION COEFFICIENT} \]

\[ \rho_{xy} = 1 \]

"PERFECT LINEAR FIT"

\[ \rho_{xy} = -1 \]
**Auto Correlation**

Consider random signal, $x$

$$R_{xx}(\tau) = E[ x(t) x(t + \tau) ]$$

- **If Stationary**: $m_x$, $\sigma_x \neq f(t)$
  
  $E[x] = m_x$
  
  $E[x(t+\tau)] = m_x$
  
  $E[(x-m_x)^2] = \sigma_x^2$
  
  $E[(x(t+\tau)-m_x)^2] = \sigma_x^2$

- **Consider** $\rho_{xx}$:

  $$\rho_{xx} = \frac{E[(x-m_x)(x(t+\tau)-m_x)]}{\sigma_x^2} = \frac{E[x(t)x(t+\tau) - m_x x(t+\tau) - m_x x(t) + m_x^2]}{\sigma_x^2}$$

  $$= \frac{R_{xx} - m_x E[x(t+\tau)] - m_x E[x(t)] + E[m_x^2]}{\sigma_x^2}$$

  $$\rho_{xx} = \frac{R_{xx}(\tau) - m_x^2}{\sigma_x^2}$$
- Hence \( R_{xx}(t) = \rho_{xx} \sigma_x^2 + m_x^2 \)
  \[ -1 \leq \rho_{xx} \leq 1 \]
  \[ m_x^2 - \sigma_x^2 \leq R_{xx}(t) \leq \sigma_x^2 + m_x^2 \]

- How about \( R_{xx}(0) \)
  \[ R_{xx}(0) = E[x^2] = \sigma_x^2 + m_x^2 \]

- How about \( R_{xx}(\pm \infty) \)
  - Random process gets uncorrelated as \( t \to \infty \)
  - \( \rho \to 0 \)
  \[ R_{xx}(\pm \infty) = m_x^2 \]

Symmetric about \( t = 0 \)
3. Cross Correlation:

- Consider $X$ & $Y$ as random processes:

\[
R_{xy}(\tau) = E[ X(t) \gamma(t+\tau)] \\
R_{yy}(\tau) = E[ \gamma(t) \times (t+\tau)]
\]

- If stationary:

\[
R_{xy}(\tau) = R_{yx}(-\tau)
\]

- Relate $\rho_{xy}$ to $R_{xy}$:

\[
\rho_{xy}(\tau) = \frac{E[ (X(t) - m_x)(\gamma(t+\tau)-m_y)]}{\sigma_x \sigma_y}
\]

\[
\rho_{xy}\sigma_x \sigma_y = E[ X(t) \gamma(t+\tau) - m_x \gamma(t+\tau) - m_y X(t) + m_x m_y]
\]

\[
\rho_{xy}\sigma_x \sigma_y = R_{xy}(\tau) - m_x m_y - m_y m_x + m_x m_y
\]

\[
R_{xy}(\tau) = \rho_{xy}(\tau)\sigma_x \sigma_y + m_x m_y
\]

\[
R_{yx}(\tau) = \rho_{yx}(\tau)\sigma_x \sigma_y + m_x m_y
\]
- $R_{xy}(T)$ is **maximum** at $T_{\text{max}}$


$$R_{xy}(T)_{\text{max}} = \rho_{xy} \sigma_x \sigma_y + m_x m_y$$

- **At** $T \to \infty$, $\rho_{xy} \to 0$ 

  \[
  \therefore R_{xy} \to m_x m_y
  \]

![Graph showing symmetric about $T_{\text{max}}$]

**($\mathbf{4}$) Covariance:**

**Variance:** \[ \text{Var}(X) = \sigma_x^2 \]

**Covariance:** \[ \text{Cov}(X,Y) = \rho_{xy} \sigma_x \sigma_y \]

**Covariance Matrix:** 

\[
\begin{bmatrix}
\sigma_x^2 & \rho_{xy} \sigma_x \sigma_y \\
\rho_{xy} \sigma_x \sigma_y & \sigma_y^2
\end{bmatrix}
\]

- **If** $X$ & $Y$ **are independent**, $\rho_{xy} = 0$