CLASS #4: Solving Single Variable Continuous-Time Systems

OBJECTIVES:
1. Review Continuous-Time Systems & Operators
2. Define Solution Types
3. Present Examples

1. Review of C.T. Single Variable Systems

\[
\begin{array}{c|c|c}
\text{Input} & \text{System} & \text{Output} \\
\hline
u(t) & L(p) & y(t) \\
\end{array}
\]

- Convert System into Differential Equation (Model)
- Highest-order derivative on output defines system order

EXAMPLE: Consider tank in water treatment system:

Tank is: 100 KL capacity
1 mg of salt dissolved in 60 KL

Intake: incoming brine solution is flowing at 500 L/min, 100 kg/L of salt

Outtake: 300 L/min

What is salt concentration in tank?

i) Variables:
   \( t = \text{time (min)} \)
   \( X = \text{salt in tank (salt, kg)} \)
   \( Q = \text{flow rate (L/min)} \)
   \( C = \text{concentration (kg/L)} \)
   \( V = \text{volume of solution (L)} \)

ii) \( \Delta X = Q_i \cdot C_i \Delta t - Q_o \cdot C_o \Delta t \)

\[
\lim_{\Delta t \to 0} \frac{\Delta X}{\Delta t} = Q_i C_i - Q_o C_o \\
\frac{dx}{dt} = \frac{X}{V} = \frac{X}{V_0 + (Q_i - Q_o)t}
\]

\[
\frac{dx}{dt} = \left( \frac{500}{60 \times 10^3} \right) \left( 100 \times 10^3 \right) - \left( \frac{300}{200 \times 10^3} \right) \frac{x}{60 \times 10^3 + (200 \times 10^3)t}
\]

\[
\frac{dx}{dt} = 50 \times 10^3 \frac{\text{kg min}}{\text{min}} - 1.5 \frac{x}{300 + t}
\]

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\[
\frac{dx}{dt} + \frac{1.5}{300+t} x = 50 \times 10^3
\]

1st ORDER DIFF. EQUATION

ii) CHARACTERIZE: LINEAR? YES
TIME INVARIANT? NO \[
\frac{1.5}{300+t} = a_0 = f(t)
\]

2) HOMOGENEOUS SOLUTION:
- CONSIDER Nth ORDER LTI SYSTEM (M=0)
\[
\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = u
\]
- TWO SOLUTION TYPES: HOMOGENEOUS (u=0) & PARTICULAR
- \[
L(p) = \frac{1}{p^n + a_{n-1}p^{n-1} + \ldots + a_0} \leftarrow \text{CHARACTERISTIC EQ.}
\]

A) SOLUTION STRATEGY:
- ASSUME: \( y(t) = e^{\lambda t} \lambda \) IS CONSTANT
\[
\frac{d^n y}{dt^n} = \lambda^n e^{\lambda t}
\]
- HENCE:
\[
\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = 0
\]
\[
\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0 = 0
\]
IDENTICAL TO CHARACTERISTIC EQ.
- SOLVE FOR N-ROOTS OF CHARACTER. EQ. (\( \lambda \) OR P)

THEOREM: IF \( z_1 = e^{\lambda_1 t} \), \ldots, \( z_n = e^{\lambda_n t} \) ARE ALL SOLUTIONS TO LINEAR HOMOGENEOUS DIFF. EQ, THEN ANY LINEAR COMBINATION IS A SOLUTION
\[
\therefore y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}
\]
- REQUIRES N-BOUNDARY CONDITIONS TO FIND \( c_1, \ldots, c_n \)
(USUALLY AFTER PARTICULAR SOLUTION IS FOUND)

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8) Roots to Characteristic Equation:
   i) \( \lambda_n \in \mathbb{R} \) and distinct \( \rightarrow y_1(t) = e^{\lambda t} \)
   
   ii) \( \lambda_n \in \mathbb{R} \) but repeated \( \rightarrow \) For Each Repeated Root, Repeated \( m \) times, \( y(t) = e^{\lambda t} + te^{\lambda t} + \ldots + t^{m-1} e^{\lambda t} \)
   
   iii) \( \lambda_n \in \mathbb{C} \) as conjugate pairs
       \( \lambda_n = a + bj \) where \( j = \sqrt{-1} \)
       \( y_2(t) = e^{\lambda t} = e^{a + bj}t = e^{at}e^{bjt} \)

   Employ Euler's Identity: \( e^{aji} = \cos a + j\sin a \)

   \[ y_2(t) = e^{at} \left( \cos bt + j\sin bt \right) \]

   - Envelope Shaping
   - Harmonic Response

   - Treat Imaginary Component w/ B.C. (Remove)

   iv) \( \lambda_n \in \mathbb{C} \) but repeated \( \rightarrow \) For Each Repeated Root Repeated \( m \) times,
       \( y_2(t) = e^{\lambda t} + te^{\lambda t} + \ldots + t^{m-1} e^{\lambda t} \)
3. **Total Solution**:

\[ Y_{\text{total}}(t) = Y_{\text{mono}}(t) + Y_p(t) \]

- **Solution to** \[ a_d \frac{d^n Y}{dt^n} + a_{n-1} \frac{d^{n-1} Y}{dt^{n-1}} + \ldots + a_0 Y = u(t) \]

- \( Y_p(t) \) is System Response due to Forcing Function, \( u(t) \), and is independent of Initial Conditions (i.e., Forced System Response)

- \( Y_{\text{total}} \) is Based on Superposition Principle

4. **Example**:

Consider DC Grid Line: What is Voltage Across Capacitor?

\[ V = 100 \text{V} \]
\[ V_c(0) = 5 \text{V} \]

\[ 100 \text{V} \]
\[ \overline{\text{+}} \]
\[ V_c(t) \]
\[ \overline{\text{-}} \]
\[ 0.2 \mu \text{F} \]

i) **Model**:

Input is \( V \) Output is \( V_c(t) \)

**KVL**:

\[ 100V - Ri - V_c = 0 \]

\[ i_c(t) = \frac{100 - V_c}{R} \]

\[ V_c = C \frac{dV_c}{dt} \]

\[ \frac{100 - V_c}{R} = C \frac{dV_c}{dt} \]

\[ 100 = RC \frac{dV_c}{dt} + V_c \]

\[ 100 = 0.2 \left( \frac{dV_c}{dt} + V_c \right) \]

\[ L(p) = \frac{100}{0.2 p + 1} \]

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ii) Homogeneous Eq:

\[ 0.2 \frac{dV_c}{dt} + V_c = 0 \]

Characteristic Eq:

\[ 0.2 \rho + 1 = 0 \]

\[ \rho = -5 \]

\[ V_c = Ae^{-5t} \]

iii) Particular Solution:

\[ 0.2 \frac{dV_c}{dt} + V_c = 100 \]

\[ V_c(0) = 100 \]

\[ \frac{dV_c}{dt} = 0 \]

\[ 0.2(0) + 100 = 100 \]

iv) Total Solution:

\[ V_c(t) = 100 + Ae^{-5t} \]

\[ V_c(0) = 5 = 100 + Ae^0 \]

\[ \therefore A = -95 \]

\[ V_c(t) = 100 - 95e^{-5t} \]

\[ V_c(t) \text{ at } t = 1 \text{ sec} \]

\[ V_c(t) = 99.36 \text{ V} \]

- Really Fast Rise Time

\[ \tau = \frac{1}{RC} = 5 \text{ (Higher-The Faster Response)} \]

"Time Constant"