Modal Responses:

Dynamic Response, \( r(t) \) (internal forces, base shear, etc.) can be computed mode by mode

\[
    r(t) = \sum_{i=1}^{n} r_{i}(t) = \sum_{i=1}^{n} r_{i}^{st} A_{i}(t)
\]

- \( r_{i}^{st} \) is "static" response of mode \( i \) due to load pattern \( S_{i} \)
- \( A_{i}(t) \) is pseudo acceleration \( A_{i} = \omega_{i} D_{i} \)

Peak Modal Response:

\[
    r_{i}^{max} = r_{i}^{st} \cdot A_{i}^{max} = r_{i}^{st} \cdot S_{a}(\xi_{i}, T_{i})
\]

(c) T.P. Lynch 2012
Response spectrum for El Centro ground motion
$\zeta = 0, 2, 5, 10, \text{ and } 20\%$.

Figure 6.6.4 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 2001.
But, how would you combine $R_{\text{max}}$ if $R_{\text{max}}$ occur at different times?!!

3 Ways to combine modal contributions:

1. **Absolute Sum Rule:**
   \[ R_{\text{max}} \leq \sum_{i=1}^{n} |R_{i,\text{max}}| \]
   - Peak responses assumed to occur at same time → Worst-case Scenario
   - Conservative upper bound

2. **Square-Root-Sum-Square (SRSS) Rule**
   \[ R_{\text{max}} \approx \sqrt{\sum_{i=1}^{n} (R_{i,\text{max}})^2} \]
   - Approach works well when modes are well separated
   (Modal responses are not correlated)

3. **Complete Quadratic Combination (CQC) Rule**
   \[ R_{\text{max}} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} R_{i,\text{max}, R_{j,\text{max}}}} \]
   where $\rho_{ij}$ is a correlation coefficient between 2 modes

(C) Jerome P. Lynch, 2012
\[
\rho_{ij} = \frac{8 \sqrt{\xi_i \xi_j \left( \frac{\eta_i}{\xi_i} + \beta_{ij} \frac{\eta_j}{\xi_j} \right)} \beta_{ij}^{3/2}}{(1 - \beta_{ij}^2)^2 + 4 \xi_i \xi_j \beta_{ij} (1 + \beta_{ij}^2)}
\]

where \( \beta_{ij} = \frac{w_i}{w_j} \)

- Empirically Derived

\[\text{SRSS is special case of CQC} \]

where assume

\[
P = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

(assume modes very far apart).
Correlation coefficient

Figure 13.7.1 from *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, by Anil K. Chopra, Prentice-Hall, 2001.
Example: A five story shear building

\[ m_i = m = 100 \text{kips/g} \; ; \; k_i = k = 31.54 \text{kips/in} \; ; \; h_i = h = 12 \text{ft} \]

\[ \text{Floor Mass} \quad \text{Story Stiffness} \]

\[
[ M ] = m \begin{bmatrix}
1 & & & & \\
& 1 & & & \\
& & 1 & & \\
& & & 1 & \\
& & & & 1
\end{bmatrix} \quad ; \quad [ K ] = k \begin{bmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & -1 & 2 & -1
\end{bmatrix}
\]

The natural frequencies, modes and modal properties are computed as follows:

\[
\omega_1 = 0.285 \sqrt{\frac{k}{m}} \; ; \quad \omega_2 = 0.831 \sqrt{\frac{k}{m}} \; ; \quad \omega_3 = 1.310 \sqrt{\frac{k}{m}} \\
\omega_4 = 1.682 \sqrt{\frac{k}{m}} \; ; \quad \omega_5 = 1.919 \sqrt{\frac{k}{m}}
\]

\[
T_1 = 2.0 \text{sec} \; ; \quad T_2 = 0.6852 \text{sec} \; ; \quad T_3 = 0.4346 \text{sec} \; ; \\
T_4 = 0.3383 \text{sec} \; ; \quad T_5 = 0.2966 \text{sec}
\]

modal vectors

\[
m_i^* = 1.0 \quad \Gamma_1 = 1.067 \; ; \; \Gamma_2 = -0.336 \; ; \; \Gamma_3 = 0.177 \; ; \; \Gamma_4 = -0.099 \; ; \; \Gamma_5 = 0.045
\]
Earthquake Response Spectrum

(El Centro Ground Motion, f = 5%)
Using the response spectrum as shown (assume $\xi = 5\%$), the floor displacements can be computed as:

$$\{u\}_i = r_i \{\phi\}_i D_i$$

$$\max U = \{u\}_i = 1.067 \begin{bmatrix} 0.334 \\ 0.641 \\ 0.895 \\ 1.078 \\ 1.173 \end{bmatrix} \begin{bmatrix} 1.916 \\ 3.677 \\ 5.139 \\ 6.188 \\ 6.731 \end{bmatrix} \text{ in.} \quad T = 2 \text{ sec}$$

The elastic forces can be computed as:

$$\{f\}_i = r_i [M] \{\phi\}_i A_i$$

$$\{f\}_1 = 1.067 \left(\frac{100}{g}\right) \begin{bmatrix} 0.334 \\ 0.641 \\ 0.895 \\ 1.078 \\ 1.173 \end{bmatrix} \begin{bmatrix} 4.899 \\ 9.401 \\ 13.141 \\ 15.817 \\ 17.211 \end{bmatrix} \text{ kips} \quad T = 2 \text{ sec}$$

The results of the peak displacements and static lateral forces per each mode are shown in the figure. Based on the modal responses, the base shear $V_b$, base moment $M_b$, and the displacement $u_5$ of the top floor per each mode can be calculated as shown in the figure. (NOTE: Alternatively, one can calculate response such as the base shear and base moment using the general equation: $r_i = r_i^g S_o(\xi, T_i)$.)
Modal combinations: For example, calculation of base shear

- Absolute sum rule:
  
  \[ V_{b,\text{max}} \leq \sum_{i=1}^{n} |V_{b_i,\text{max}}| \]

  \[ = 60.469 + 24.533 + 9.867 + 2.943 + 0.595 = 98.407 \text{kips} \]

- Square-Root-Sum-Square (SRSS) rule:
  
  \[ V_{\text{max}} \approx \sqrt{\sum_{i=1}^{n} (V_{b_i,\text{max}})^2} \]

  \[ = \sqrt{60.469^2 + 24.533^2 + 9.867^2 + 2.943^2 + 0.595^2} = 66.066 \text{kips} \]

- Complete Quadratic Combination (CQC) rule:
  
  \[ V_{\text{max}} \approx \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} V_{b_i,\text{max}} V_{b_j,\text{max}}} \]

The calculations of the natural frequency ratio \( \beta_{ij} \) and the correlation coefficients \( \rho_{ij} \) are shown as

Natural frequency ratio \( \beta_{ij} \)

<table>
<thead>
<tr>
<th>Mode (i,j)</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
<th>j=4</th>
<th>j=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>0.343</td>
<td>0.217</td>
<td>0.169</td>
<td>0.148</td>
</tr>
<tr>
<td>2</td>
<td>2.919</td>
<td>1.000</td>
<td>0.634</td>
<td>0.494</td>
<td>0.433</td>
</tr>
<tr>
<td>3</td>
<td>4.602</td>
<td>1.576</td>
<td>1.000</td>
<td>0.778</td>
<td>0.683</td>
</tr>
<tr>
<td>4</td>
<td>5.911</td>
<td>2.025</td>
<td>1.285</td>
<td>1.000</td>
<td>0.877</td>
</tr>
<tr>
<td>5</td>
<td>6.742</td>
<td>2.310</td>
<td>1.465</td>
<td>1.141</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Correlation coefficients \( \rho_{ij} \)

<table>
<thead>
<tr>
<th>Mode (i,j)</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
<th>j=4</th>
<th>j=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>0.007</td>
<td>0.003</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>2</td>
<td>0.007</td>
<td>1.000</td>
<td>0.044</td>
<td>0.018</td>
<td>0.012</td>
</tr>
<tr>
<td>3</td>
<td>0.003</td>
<td>0.044</td>
<td>1.000</td>
<td>0.136</td>
<td>0.062</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>0.018</td>
<td>0.136</td>
<td>1.000</td>
<td>0.365</td>
</tr>
<tr>
<td>5</td>
<td>0.001</td>
<td>0.012</td>
<td>0.062</td>
<td>0.365</td>
<td>1.000</td>
</tr>
</tbody>
</table>

CORRELATION IS RELATIVELY SMALL
Individual $V_i$ term in CQC rule

<table>
<thead>
<tr>
<th>Mode (i,j)</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
<th>j=4</th>
<th>j=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3656.476</td>
<td>10.172</td>
<td>1.615</td>
<td>0.306</td>
<td>0.049</td>
</tr>
<tr>
<td>2</td>
<td>10.172</td>
<td>601.844</td>
<td>10.687</td>
<td>1.284</td>
<td>0.178</td>
</tr>
<tr>
<td>3</td>
<td>1.615</td>
<td>10.687</td>
<td>97.354</td>
<td>3.943</td>
<td>0.365</td>
</tr>
<tr>
<td>4</td>
<td>0.306</td>
<td>1.284</td>
<td>3.943</td>
<td>8.658</td>
<td>0.639</td>
</tr>
<tr>
<td>5</td>
<td>0.049</td>
<td>0.178</td>
<td>0.365</td>
<td>0.639</td>
<td>0.354</td>
</tr>
</tbody>
</table>

Total base shear = 66.507 kips.

Comparison of response history analysis and response spectrum analysis

<table>
<thead>
<tr>
<th></th>
<th>$V'_i$ (kips)</th>
<th>$V_5$ (kips)</th>
<th>$M_b$ (kip-ft)</th>
<th>$u_5$ in</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSSUM</td>
<td>98.407</td>
<td>56.608</td>
<td>3018.8</td>
<td>7.971</td>
</tr>
<tr>
<td>SRSS</td>
<td>66.066</td>
<td>30.074</td>
<td>2575.6</td>
<td>6.800</td>
</tr>
<tr>
<td>CQC</td>
<td>66.507</td>
<td>29.338</td>
<td>2572.7</td>
<td>6.793</td>
</tr>
<tr>
<td>RHA</td>
<td>73.273</td>
<td>35.217</td>
<td>2593.2</td>
<td>6.847</td>
</tr>
</tbody>
</table>
Natural vibration modes of five-story shear building

Figure 12.8.2 from Dynamics of Structures: Theory and Applications to Earthquake Engineering, by Anil K. Chopra, Prentice-Hall, 2001.
Modal expansion of excitation vectors $s_a$ and $s_b$

$$s_a = s_1 + s_2 + s_3 + s_4 + s_5$$

$$s_b = s_1 + s_2 + s_3 + s_4 + s_5$$

Figure 12.8.3 from Dynamics of Structures: Theory and Applications to Earthquake Engineering, by Anil K. Chopra, Prentice-Hall, 2001.
Conceptual explanation of modal analysis

<table>
<thead>
<tr>
<th>Mode</th>
<th>Static Analysis of Structure</th>
<th>Dynamic Analysis of SDF System</th>
<th>Modal Contribution to Dynamic Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>Forces $s_n$</td>
<td>Unit mass $D_n(t)$</td>
<td>$r_n(t) = r_n^{st} [\omega_n^2 D_n(t)]$</td>
</tr>
</tbody>
</table>

Modal Responses: $n = 1, 2, \ldots N$

$$r_n(t) = r_n^{st} [\omega_n^2 D_n(t)]$$

Total Response

$$r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{st} [\omega_n^2 D_n(t)]$$