1. **Impulsive Load:**

Impulsive Load occurs over very short duration and contain one impulse wave-form.

For Example:
- Blast Load

\[ p(t) \]

Forced Behavior: \( t \in (0, t_1) \)

Free Vibrations: \( t \in (t_1, \infty) \)

**Strategy:**

Directly Solve

Free Vibration

\( u(t_1), \quad \ddot{u}(t_1) \)

2. **Rectangular Load:**

\[ p(t) = \begin{cases} 
  p_0 & 0 \leq t \leq t_1 \\
  0 & t > t_1 
\end{cases} \]
Consider undamped system (SDOF \( c = 0 \))

Phase I:
\[
MU + Ku = P_0
\]

\[
U_c = A \cos \omega_n t + B \sin \omega_n t
\]

\[
U_p = \frac{P_0}{K} \quad \text{(By inspection)}
\]

\[
\therefore U(t) = \frac{P_0}{K} + A \cos \omega_n t + B \sin \omega_n t
\]

Assume at rest initial conditions \((U(0), \dot{U}(0) = 0)\)

\[
U(t) = \frac{P_0}{K} + A \cos \omega_n t + B \sin \omega_n t \rightarrow 0 = \frac{P_0}{K} + A \rightarrow A = -\frac{P_0}{K}
\]

\[
\dot{U}(t) = -A \omega_n \sin \omega_n t + B \omega_n \cos \omega_n t \rightarrow 0 = 0 + B \omega_n \rightarrow B = 0
\]

\[
U(t) = \frac{P_0}{K} - \frac{P_0}{K} \cos \omega_n t
\]

At \( t = t_1 \):

\[
U(t_1) = \frac{P_0}{K} - \frac{P_0}{K} \cos \omega_n t_1
\]

\[
\dot{U}(t_1) = -\frac{P_0}{K} \omega_n \sin \omega_n t_1
\]
**Phase 2:**

\[ m\ddot{u} + ku = 0 \]

Define \( T = t - t_i \), so that at \( t = t_i \), \( T = 0 \)

**Free Vibrations**

\[ U(T) = A\sin w_n T + B\cos w_n T \]

\[ \frac{\dot{U}(t_i)}{w_n} \sin w_n T + \dot{U}(t_i)\cos w_n T \]

\[ U(t) = \frac{\dot{U}(t_i)}{w_n} \sin w_n(t-t_i) + U(t_i)\cos w_n(t-t_i) \]

**Response Spectra:**

What is worse case (max) response?
- Of greatest concern in design
- Increase over "static" response \( U_{st} = \frac{P_o}{K} \)

**Phase I:**

\[ U(t) = \frac{P_o}{K} (1 - \cos w_n t) \]

\[ D = \frac{U(t)}{U_{st}} = 1 - \cos w_n t \]

If \( t_i \geq \frac{T_o}{2} \)

\[ D_{max} = 1 - \cos \frac{\pi}{2} = 2 \]

\[ U_{max} = 2 \times U_{st} \]

**Phase II:**

\[ U(t) = \frac{\ddot{U}(t_i)}{w_n} \sin w_n(t-t_i) + U(t_i)\cos w_n(t-t_i) \]

Amplitude: \( \rho = \sqrt{\left(\frac{\ddot{U}(t_i)}{w_n}\right)^2 + U(t_i)^2} \)

\[ U(t_i) = \frac{P_o}{K} (1 - \cos w_n t_i) \]

\[ \rho = \frac{P_o}{K}\sqrt{2 - 2\cos w_n t_i} = \frac{P_o}{K}\sqrt{2 \left(2\sin^2 \frac{w_n t_i}{2}\right)} \]

\[ \rho = 2 \frac{P_o}{K}\sin \frac{w_n t_i}{2} \]

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Response to rectangular pulse forces

$t_d/T_n = 1/8$

$t_d/T_n = 1/4$

$t_d/T_n = 1/2$

$t_d/T_n = 1$

$t_d/T_n = 1.25$

$t_d/T_n = 1.5$

$t_d/T_n = 1.75$

$t_d/T_n = 2$

Figure 4.7.2 from Dynamics of Structures: Theory and Applications to Earthquake Engineering, by Anil K. Chopra, Prentice-Hall, 2001.
For rectangular loads, conservatively do static analysis with \( D = 2 \).

\[
D = \frac{U(t)_{\text{max}}}{U_{\text{st}}} \quad \text{max} \quad U_{\text{st}}
\]

\[
D_{\text{max}} = 2 \sin \frac{\omega t}{2}
\]

\[\Rightarrow U_{\text{max}} = D \cdot U_{\text{st}}\]

"Displacement response spectrum" (Remember, assumed no damping so conservative)

\( t_{1}/T_n = \frac{1}{2} \)

---

4. Other impulse loads

Triangular (for blast loads)

\[
D^2
\]

---

5. Infinitely short impulses

Let \( t_i \to 0 \)

As \( t_i \to 0 \), phase 1 disappears and only phase 2 persists

(c) Jerome P. Lynch, 2012
FIGURE 2.6 Typical responses of one-degree elastic systems. (a) Rectangular pulse; (b) suddenly applied triangular pulse; (c) symmetrical triangular pulse; (d) constant force with finite rise time.

FIGURE 2.7 Maximum response of one-degree elastic systems (undamped) subjected to rectangular and triangular load pulses having zero rise time. (U.S. Army Corps of Engineers, 10)
PHASE 2: \[ U(t) = \frac{\dot{u}(t)}{w_n} \sin w_n (t-t_i) + u(t_i) \cos w_n (t-t_i) \]

\[ = \frac{\dot{u}(t_i)}{w_n} \sin w_n t + u(t_i) \cos w_n t \]

at \( t=t_i \), \[ u(t) = \frac{P_0}{k} (1 - \cos w_n t_i) = 0 \]

\[ \dot{u}(t) = \frac{P_0}{k} w_n \sin w_n t, = \frac{P_0}{k} w_n t_i \]

\[ \therefore U(t) = \frac{P_0}{k} w_n t_i, \sin w_n t \]

\[ P_0 \]

\[ t \]

\[ T_n \]
Convolution:

Assume Inf. small impulse is at $t^e$

\[ u(t) = \frac{p(t) \delta (t-t^e)}{m \omega_n} \sin \omega_n (t-t^e) \]

If linear system $\rightarrow$ superposition

Inf. sum of impulsive loads
\[ U(t) = \int_{0}^{t} \frac{P(\tau)}{M\omega_n} \sin \omega_n (t-\tau) \, d\tau \]

**Duhamel's Integral (Convolution)**

- Closed Form Analysis Response Function

\[ h(t-\tau) = \frac{1}{M\omega_n} \sin \omega_n (t-\tau) \]

**Include Initial Conditions!**

**Undamped:**

\[ U(t) = \frac{U_0}{\omega_n} \sin \omega t + u(0) \cos \omega t + \int_{0}^{t} P(\tau) \left( \frac{1}{M\omega_n} \sin \omega_n (t-\tau) \right) \, d\tau \]

**Damped:**

\[ U(t) = (u(0) \cos \omega t + \frac{u(0) + \sin u(0)}{\omega_d} \sin \omega t) e^{-\frac{t}{\omega_d}} \]

\[ + \int_{0}^{t} P(\tau) \left( \frac{1}{M\omega_d} e^{-\frac{\omega_d (t-\tau)}{2}} \sin \omega_d (t-\tau) \right) \, d\tau \]