On the use of Operator Representations for Angular Momentum

The functional form of the Schrödinger equation for a spherically symmetric potential governs the angular dependence of the state function \( \Psi(\theta, \phi) \) with an operator of the form:

\[
L^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \text{or} \quad L^2 = \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]

The operators for angular momentum can be written out in this form, either in x-y-z coordinates or in spherical coordinates. \( \hat{L} \) is the usual notation for orbital angular momentum where one can identify a trajectory. We will use the symbol \( \mathbf{J} \) to denote angular momentum that encompasses both orbital and spin degrees of freedom.

\[
J_z \Psi_{jm} = \hbar m \Psi_{jm} \quad J^2 \Psi_{jm} = \hbar^2 (j+1) \Psi_{jm} \quad J^2 \equiv J_x^2 + J_y^2 + J_z^2
\]

25.1 Basic Operators \( J^2, J_z, J_+, \) and \( J_- \)

Abstract Operators For Angular Momentum are based on Commutation Relations

The commutation relationship that qualifies an operator \( J \) to be considered in the general category of angular momentum are given by \([J_i, J_j] = i\hbar J_k\), with cyclic permutation of indices; explicitly these are:

\[
\begin{align*}
[J_x, J_y] &= i\hbar J_z \\
[J_z, J_x] &= i\hbar J_y \\
[J_y, J_z] &= i\hbar J_x
\end{align*}
\]

We show that \( J_\pm \equiv \pm J_x \pm iJ_y \) acts as a raising and lowering operator

Given \( J_z \Psi_{jm} = m \hbar \Psi_{jm} \), (which can be found by solving the azimuthal equation directly), we show here that the commutation relations (above) permit us to define raising and lowering operators: We begin by writing out the commutator \([J_z, J_+]\):

\[
[J_z, J_+] = J_z (J_x + iJ_y) - (J_x + iJ_y) J_z
\]

\[
= (J_z J_x - J_x J_z) + i(J_z J_y - J_y J_z)
\]

\[
= [J_z, J_x] + i [J_z, J_y] = i\hbar J_y + i(-i\hbar J_x) = \hbar (J_x + iJ_y)
\]

so

\[
[J_z, J_+] = \hbar J_+
\]

and therefore

\[
[J_z, J_+] \Psi^m_j = \hbar J_+ \Psi^m_j
\]

We now apply our derived relationship \([J_z, J_+] = \hbar J_+\) to the function \( \Psi^m_j \) explicitly:

This derivation for \( J_+ \equiv J_x + iJ_y \) has its parallel for the lowering operator \( J_- \equiv J_x - iJ_y \),

\[
J_z J_+ \Psi^m_j - J_+ J_z \Psi^m_j = \hbar J_+ \Psi^m_j \quad \rightarrow \quad J_z J_+ \Psi^m_j - \hbar J_+ \Psi^m_j = \frac{\hbar}{J_+} \Psi^m_j
\]

\[
J_z J_+ \Psi^m_j = \hbar J_+ \Psi^m_j + \hbar m J_+ \Psi^m_j
\]

\[
J_z (J_+ \Psi^m_j) = (m+1)\hbar (J_+ \Psi^m_j) \quad \text{so} \quad (J_+ \Psi^m_j) \quad \text{is an eigenfunction of} \quad J_z
\]

with eigenvalue \((m+1)\hbar\)
Since $J^2$ and $J_z$ commute, it follows that $J_z \psi_{jm}$ is an eigenfunction of both $J^2$ and $J_z$. The operator $J^2$, defined in terms of the $J_k$'s, commutes with any of them:

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

so that the operation $J_z$ acting on a function such as $J_z \psi_{jm}$ will not change the eigenvalue $[J(J+1)]$ of the operator $J^2$. (NOTE: $[J^2, J_z] = 0$ and $[J^2, J_{\pm}] = 0$ does NOT imply $[J_z, J_{\pm}] = 0$)

These operators (conceptually related to the creation and destruction operators that we defined earlier for the harmonic oscillator) have commutation relations as follows:

A state can be defined as having simultaneous eigenvalues of $J^2$ and one (but only one) component $J_z$. It is conventional to choose $J_z$ as the component operator that commutes with $J^2$. (You are making a choice of $J_z$ when, for example, you say to your colleague, "let's let the direction of the magnetic field define the z-axis in this experiment")

25.2 Generating a set of eigenfunctions/eigenvalues

**Eigenvalues of $J_z$ when operating on $J_{\pm} \psi_{j,m}$**

Let's see what the ladder operators $J_+$ or $J_-$ will do to the eigenvalues of $J_z$ when it acts on $\psi_{j,m}$:

$$[J_z, J_{\pm}] = \pm \hbar J_{\pm}$$

expands out to

$$J_z J_{\pm} \psi_{j,m} = J_{\pm} J_z \psi_{j,m} = \pm \hbar J_{\pm} \psi_{j,m}$$

where we've used the $J_z \psi_{j,m} = m\hbar \psi_{j,m}$ relation. From this we have that $(J_{\pm} \psi_{j,m})$ is an eigenfunction of $J_z$ with an $m$-value one step up (or down) from the original.

$$J_z (J_{\pm} \psi_{j,m}) = (m \pm \hbar) (J_{\pm} \psi_{j,m})$$

**Upper/lower limits to quantum number m**

It follows from the definitions that

$$J_+ [J \psi_{jm}] = J \alpha \psi_{j,m+1} = \beta \cdot \psi_{j,m+2}, \text{ and } J_- [J \psi_{jm}] = J \alpha' \psi_{j,m-1} = \beta' \cdot \psi_{j,m-2}$$

where $\alpha$ and $\beta$ are scalars (yet to be determined). Therefore if we have just one of the eigenfunctions we can use the raising and lowering operators to generate the entire set of eigenfunctions that have the same angular momentum quantum number $j$ but with a different projection quantum numbers $m$.

If we begin with $\psi_{jm}$ using the most negative value of $m$, and then operate with $J_+$, we get other eigenfunctions with the range of $m$ values: $-m, (-m + 1), (-m + 2), \ldots + m$.

We find that the lowest allowed value of $m$ for the function $\psi_{jm}$ is $m = -j$, and highest allowed value is $m = +j$. 
25.3 Proportionality between \([J_\pm \psi_{jm}], \psi_{jm}, \text{and } [J_\pm \psi_{jm}]\)

we adopt (temporarily) the notation: \(|J_\pm \psi_{jm}| = C_j^n|\psi_{j,m+1}|\), where \(C_j^n\) is a scalar chosen so that the functions are normalized:

\[
\langle J_+ \psi_{j,m}|J_+ \psi_{j,m}\rangle = |C_j^n|^2 \langle \psi_{j,m+1}|\psi_{j,m+1}\rangle \to |C_j^n|^2 = \frac{\langle J_+ \psi_{j,m}|J_+ \psi_{j,m}\rangle}{\langle \psi_{j,m+1}|\psi_{j,m+1}\rangle}
\]

We now work on the numerator of this expression.

now \(J_-\) and \(J_+\) are hermitian conjugates, so \(\langle J_- \psi_{j,m}|J_+ \psi_{j,m}\rangle = \langle \psi_{j,m}|J_- \psi_{j,m}\rangle\)

and we make use of this because we can evaluate \(J_- J_\pm \psi_{j,m}\) with use of other relationships:

\[
J_- J_\pm = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i(J_x J_y - J_y J_x) \quad J_- J_\pm \psi_{j,m} = [J^2 - J_z^2 - \hbar J_z]\psi_{j,m}
\]

so we have

\[
\langle J_+ \psi_{j,m}|J_+ \psi_{j,m}\rangle = \langle \psi_{j,m}|J_- \psi_{j,m}\rangle = \langle \psi_{j,m}|\frac{J^2 - J_z^2 - \hbar J_z}{j(j+1)-m(m+1)\hbar^2}\psi_{j,m}\rangle
\]

and our normalization for the raising operator is

\[
|C_j^n|^2 = \frac{\langle J_+ \psi_{j,m}|J_+ \psi_{j,m}\rangle}{\langle \psi_{j,m+1}|\psi_{j,m+1}\rangle} = \frac{j(j+1)-m(m+1)\hbar^2}{j(j+1)-m(m+1)\hbar^2}\frac{\langle \psi_{j,m}|\psi_{j,m}\rangle}{\langle \psi_{j,m+1}|\psi_{j,m+1}\rangle}
\]

if the \(\langle \psi_{j,m}|\psi_{j,m}\rangle\) and \(\langle \psi_{j,m+1}|\psi_{j,m+1}\rangle\) are with unity normalization, as usual, then

\[
J_+ \psi_{jm} = \hbar \sqrt{j(j+1)-m(m+1)} \cdot \psi_{j,m+1}
\]

and (it can be shown)

\[
J_- \psi_{jm} = \hbar \sqrt{j(j+1)-m(m-1)} \cdot \psi_{j,m-1}
\]
25.4 Eigenvalues of $J^2$

All states within the same J-manifold have the same eigenvalues of $J^2$

If $\psi_{j,m}$ is an eigenfunction of $J^2$ with eigenvalue taken (for the moment) to be $\hbar^2K^2$ we write

$$J^2\psi_{jm} = \hbar^2K^2\psi_{jm}$$

(we don’t yet know the value of $K$)

Now we use the raising operator $J_+$. Since we know that $[J^2, J_\pm] = 0$, we get

$$J_+(J^2\psi_{jm}) = J^2(J_+\psi_{jm}) = \hbar^2K^2(J_+\psi_{jm})$$

so $J_+\psi_{jm}$ is also an eigenfunction of $J^2$ with eigenvalue $\hbar^2K^2$. (A similar argument could be made using the lowering operator $J_-$.) We conclude that the eigenvalue that results when operator $J^2$ is applied to $\psi_{jm}$ is the same, irrespective of the value of the azimuthal quantum number $m$.

Eigenvalues of the $J^2$ operator

We construct the operator pair $J_\pm J_\mp$

$$J_\pm J_\mp = (J_x \pm iJ_y)(J_x \mp iJ_y) = J^2_x + J^2_y \mp i\left(J_x J_y - J_y J_x\right)$$

$$= J^2 - J_z^2 + i\hbar J_z$$

We apply the operator to $\psi_{jm}$

$$J_\pm J_\mp \psi_{jm} = [J^2 - J_z^2 \pm \hbar J_z]\psi_{jm}$$

and re-arrange:

$$J^2\psi_{jm} = [J_\pm J_\mp + J_z^2 \pm \hbar J_z]\psi_{jm}$$

for ease of reading, we temporarily designate the extreme values of $m$ by the capital letter $M$.

We have already pointed out that applying a raising (lowering) operator to a state with $m=\pm J$ will give a null result: $J_\pm \psi_{j,\pm M} = 0$, so the equation just above simplifies to:

$$J^2\psi_{j,\pm M} = [J_z^2 \pm \hbar J_z]\psi_{j,\pm M} = [M_z^2 \pm \hbar(M)\psi_{j,\pm M}$$

$$= j(j+1)\hbar^2$$

Conclusion: the eigenvalue of operator $J^2$ is $j(j+1)$,

$$J^2\psi_{j,m} = j(j+1)\hbar \cdot \psi_{j,m}$$