4. FUNCTIONAL DESCRIPTION OF INTERFERENCE EXPERIMENTS

The general principles of quantum interference, both for particles and for photons, can be understood in terms of the symbolic description of Sec. 3, but we need a more detailed theory in order to calculate numbers that apply to actual experiments. In the previous section we discussed the example of electron interference because this is such a novel aspect of our subject, and because our primary, overall goal is to develop a theory of particle behavior that explains the properties of atoms and their aggregates. In this section we will develop the more detailed theory using optical interference as our example because the formalism and notation is somewhat more familiar from classical electromagnetic (and acoustic) theory. The results will be generally useful because the electric field E(x,t) used to predict photon behavior has a close correspondence, *mutatis muntandis,* to the Ψ(x,t) functions used to describe particles.

4.1 $|E|^2$ AS A MEASURE OF PHOTON DENSITY

Most discernible effects on a material are proportional to the work W done by the external agency on the material. And (on a macroscopic scale and to first order at least)

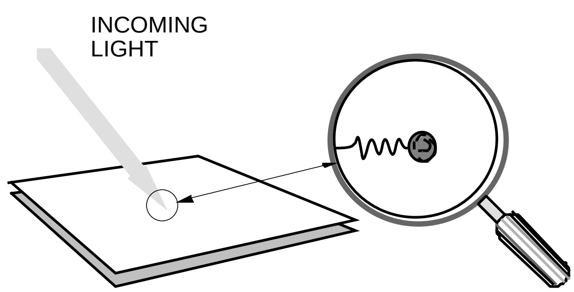

Fig 4.1 Magnified view showing elements of surface bound with linear forces

most materials respond in a linear way to an applied force. Thus, if the microscopic constituents of the material are bound with forces of the form **F** = -k**x**, the constituents undergo peak displacements X, where X^2 is proportional to the work done:

$$W = \int_0^X \vec{F} \cdot d\vec{x} \quad \longrightarrow \quad W = \int_0^X (kx)dx = \tfrac{1}{2}kX^2 \tag{4.1}$$

The force is proportional to the field strength, so in this linear material we find:

$$\left.\begin{aligned} F &= -kx \\ W &= \frac{1}{2}kX^2 \end{aligned}\right\} \text{ for linear materials, and if forces are electric: } \left\{\begin{aligned} F &= qE \\ W &= \frac{k^2}{2q}E^2 \end{aligned}\right. \tag{4.2}$$

So the work done on the detector material (comprised of objects of charge *q* confined with force constant *k*) is proportional to the square of the applied electric field, $|\mathbf{E}(x,t)|^2$:

$$W = \frac{q^2}{2k}\,\left|\vec{E}\right|^2 \tag{4.3}$$

We often use a photon representation for the electromagnetic field, where (at a given frequency) the photon density is proportional to the energy density and thus to $|\mathbf{E}(x,t)|^2$, where **E** is governed by Maxwell's equations. There is an obvious parallel to the representation of particle probability density by $|\Psi(x,t)|^2$, where Ψ is governed by Schrödinger's equation.

4.2 E(x,t) AS PHOTON PROBABILITY AMPLITUDE: INTERFERENCE

Interference may occur when there are multiple paths to the event in which a photon comes from a source to a given location. Consider the case in which the net field at such a location is the sum of two individual fields:

$$\underline{\underline{\vec{E}}} = \vec{E}_1 + \vec{E}_2 \longrightarrow \left|\underline{\underline{\vec{E}}}\right|^2 = E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2 \tag{4.4}$$

The individual fields each have their own magnitude and phase, so it is convenient to use complex numbers to represent them. $\vec{E}_n = E_n \ e^{i\phi_n}$. The square of the net field is easily shown to be:

$$\left|\underline{\underline{\vec{E}}}\right|^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos(\phi_2 - \phi_1) \tag{4.5}$$

In experimental work it frequently arises that two sources generate fields E_1 and E_2 that are of constant magnitude but that have a fluctuating phase relationship. In that case, the time average expectation value for $|E(\text{total})|^2$ is

$$\underbrace{\left\langle \left|\underline{\underline{\vec{E}}}\right|^2 \right\rangle}_{\textit{Time Average}} = E_1^2 + E_2^2 + 2E_1E_2 \underbrace{\langle \cos(\phi_2 - \phi_1) \rangle}_{\textit{Time Average}} \tag{4.6}$$

a value which is seen to depend not only on phase angles but also on the length of time over which averages are taken.

----- Field $\mathbf{E}_1$, by itself would produce an effect proportional to $(\mathbf{E}_1)^2$;

-------field $\mathbf{E}_2$, by itself, would have an effect proportional to $(\mathbf{E}_2)^2$.

The contribution of the cross term, a scalar with magnitude $2\mathbf{E}_1 \cdot \mathbf{E}_2$, depends on the time average of the relative phase of the two fields. If the phases are correlated (in which case the fields are said to be *coherent*) then the cross term can be large and can be either positive or negative. On the other hand, if the phases are random over the time of the measurement, the interference term has an average value of zero.

If we are in a **strong** field situation, the expressions for the net field can be interpreted as the relative size of an effect produced by a radiation field comprised of many photons.

If we are in a **weak** field situation (the weak flux case for photons), we interpret $|E_{TOTAL}|^2$ as a relative probability that a photon produced any effect at all.

4.3 PHASES ARE ALL-IMPORTANT

We have written the individual fields as $\mathbf{E}_k = E_k e^{i\varphi_k}$ and we see that the observational results of experiments depend on phase differences ($\varphi_k - \varphi_j$) between fields, perhaps even on time averages of such differences. So it is important to understand the origins of phase angles.

The phase φ_k of field $\mathbf{E}_k$ as observed at location x has several contributions:

$$\vec{E}_k = E_k \quad e^{i\phi_k}$$

$$where \quad \phi_k(x,t) = \underbrace{-\omega_k t}_{\substack{periodic\\field\\oscillations}} + \underbrace{\Delta_k}_{\substack{path\\differences}} + \underbrace{\delta_k(t)}_{\substack{random\\temporal\\fluctuations}} \tag{4.7}$$

ω : The phase depends on time in an expected way (ωt) from the regular oscillation of the field at angular frequency ω

Δ: The relative phase depends on difference in effective path length (Δ_K (x)) between the k-th source and the detector. By "effective" we mean that the difference can be either an actual length difference or an apparent length difference as, for example, when the refractive index of the intervening medium changes. These differences may be controllable parameters of the experiment (e.g. slit widths, detector position) or they may arise as noise and drift (e.g. vibration, thermal expansion). Differences can arise from density fluctuations (accidental or deliberate) if the propagation is not in vacuum.

δ: The phase may have noise and thus undergo random temporal fluctuations. The contribution from this effect, denoted $\delta_k(t)$, is a number that increases, randomly, with $\sqrt{t}$.

If the phases are stable over the time of observation, then interference effects can be pronounced. The value of $<|E|^2>$ could be as large as $|E_1 + E_2|$ or as small as $|E_1 - E_2|$; indeed if the fields are of equal magnitude the expectation value could vanish entirely for observers at those locations where the fields were opposite in phase.

If the phases are random with respect to each other, then the time average of $\cos(\varphi_2 - \varphi_1)$ approaches zero and the cross term does not contribute to the expectation value: we are left with the arithmetic sum of what each of the sources would make by itself .... in that case there is no interference effect whatever.

4.4 EXTENSION TO MORE THAN TWO COHERENT SOURCES

Suppose we have a situation in which radiation from N sources propagates, each over its own path, to a single detector at location x. The net electric field **E** at that detector is a sum of N individual vector terms, each with a magnitude $E_k(x,t)$ and a phase $\varphi(x,t)$. (to reduce algebraic clutter we often just write these as E and φ). Then:

$$\left|\vec{E}\right|^2 = \left|\sum_{k=1}^{N} E_k \; e^{i\phi_k}\right|^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} \vec{E}_j \cdot \vec{E}_k = \sum_{j=1}^{N}\sum_{k=1}^{N} E_j E_k e^{-i\left(\phi_j - \phi_k\right)} \qquad (4.8)$$

It is instructive to write this out as an N×N array: (we use N=4 for illustration)

$$\left|\vec{E}\right|^2 = \sum_{j=1}^{4}\sum_{k=1}^{4} \vec{E}_j \cdot \vec{E}_k = \begin{array}{cccc} \vec{E}_1\cdot\vec{E}_1 & +\vec{E}_1\cdot\vec{E}_2 & +\vec{E}_1\cdot\vec{E}_3 & +\vec{E}_1\cdot\vec{E}_4 \\ +\vec{E}_2\cdot\vec{E}_1 & +\vec{E}_2\cdot\vec{E}_2 & +\vec{E}_2\cdot\vec{E}_3 & +\vec{E}_2\cdot\vec{E}_4 \\ +\vec{E}_3\cdot\vec{E}_1 & +\vec{E}_3\cdot\vec{E}_2 & +\vec{E}_3\cdot\vec{E}_3 & +\vec{E}_3\cdot\vec{E}_4 \\ +\vec{E}_4\cdot\vec{E}_1 & +\vec{E}_4\cdot\vec{E}_2 & +\vec{E}_4\cdot\vec{E}_3 & +\vec{E}_4\cdot\vec{E}_4 \end{array} \qquad (4.9)$$

Writing the sum as an N×N array gives visual emphasis to properties that are important not only in this calculation but in much of our work to follow.

1. The diagonal terms in this array, N in number, are intrinsically positive and real; and they always contribute to $|\mathbf{E}(x,t)|^2$
2. The sum of any off-diagonal term with its mirror term, ($\mathbf{E}_j \bullet \mathbf{E}_k + \mathbf{E}_k \bullet \mathbf{E}_j$), is a real number *that may be positive or negative*.
3. The algebraic sign of ($\mathbf{E}_j \bullet \mathbf{E}_k + \mathbf{E}_k \bullet \mathbf{E}_j$) depends on $\cos(\varphi_j - \varphi_k)$, so if the phases are random the terms being added in the off-diagonal sum will have random signs and their average contribution to the net **E**-field will approach zero.
4. The off-diagonal terms in the array are $N(N-1) \sim N^2$ in number, so if the phases of the $\mathbf{E}_k$ are relatively stable, the sum of the off-diagonal terms can make substantial contributions to $|E(\text{net})|^2$.
5. Indeed if all the individual fields $\mathbf{E}_k$ have the same magnitude and if their phases are such that all $\mathbf{E}_j \bullet \mathbf{E}_k$ terms make a maximum positive contribution, then the coherent off-diagonal terms overpower the diagonal terms (the classical contribution) by a factor of (N-1).

The coherent contributions of off-diagonal terms, so dramatic when N is large, are basic to the operation of dichroic mirrors, diffraction gratings, radar antennas, lasers, and countless other systems that involve photons.

We shall see that the formalism presented here adapts directly to calculations of probabilities for particle behavior, even (or perhaps one should say especially) in the weak flux limit.

4.5 FIELDS WITH BOTH COHERENT AND INCOHERENT CONTRIBUTIONS

We now expand the discussion to include the common situation in which an array of coherent sources (N_1 in number) is operating in a background of radiation from multiple incoherent sources (N_2 in number) . We regard each individual contribution to the net **E** field at the detector as being one of two kinds:

> $\mathbf{E}_k$: We assume that the field has N_1 contributions that have a **regular** (non-random, coherent) phase relation with one another;
> we denote these as $\mathbf{E}_k$, or $E_k \times e^{i\varphi_k}$.
>
> $\mathbf{F}_j$: We assume that the field also has N_2 contributions that have an **irregular** (random, incoherent) phase relationship with one another and with the E_k;
> we denote these incoherent contributions as $\mathbf{F}_j$: , or $F_j \times e^{i\varphi_j}$:

So the net field in a simple case would be $\boldsymbol{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{F}_4 + \mathbf{F}_5$

E and **F** are both electric fields; we use the different symbols to provide visual distinction between coherent and incoherent terms in the algebraic expressions that follow.

As in the previous examples, the net electric field E at that detector is a sum of N individual vector terms, and the effect of that net field is proportional to $|\mathbf{E}|^2$

$$\left|\vec{E}\right|^2 = \left(\sum \vec{E}_k + \sum \vec{F}_j\right)\cdot\left(\sum \vec{E}_m + \sum \vec{F}_n\right) = \left|\sum_{N_1} E_k \; e^{i\phi_k} + \sum_{N_2} F_j \; e^{i\phi_j} + \right|^2 \qquad (4.10)$$

This expression (4.10) for $|\mathbf{E}|^2$ is a sum of vector products of individual field components.. The sum can be understood in more detail by grouping those products according to phase relationships:

$$\left|\vec{E}\right|^2 = \underbrace{\overbrace{\sum_{N_1} E_k^2}^{I} + \overbrace{\sum_{N_2} F_j^2}^{II}}_{\substack{\text{diagonal terms}\\ \text{always contribute}}} + \underbrace{\overbrace{\sum_{j\neq k}^{N_1} E_k \; E_j \; e^{i(\phi_k-\phi_j)}}^{III}}_{\substack{\text{coherent products}\\ \text{very important if } N_1 >> 1}}$$

$$+ \underbrace{\overbrace{\sum_{j\neq k} E_k \; F_j \; e^{i(\phi_k-\phi_j)}}^{IV} + \overbrace{\sum_{j\neq k} F_k \; E_j \; e^{i(\phi_k-\phi_j)}}^{V} + \overbrace{\sum_{j\neq k} F_j \; F_k \; e^{i(\phi_k-\phi_j)}}^{VI}}_{\substack{\text{net contribution of these is small}\\ \text{because of jitter in phases of the } F_k}} \qquad (4.11)$$

Sums (I) and (II) together comprise $N_1 + N_2 = N$ terms that are not dependent on phase in any way; they are positive definite and always add as positive scalars. These terms, which appear on the diagonal of the sum if it is written as an N×N array, are what one would expect from a set of $N_1 + N_2 = N$ classical, incoherent sources.

Sum (III) has N×(N -1) terms and in many cases {e.g. in a laser} may be the most important contribution to $|\mathbf{E}|^2$ if, over the duration of the experiment, the apparatus dimensions are stable to within a small fraction of the wavelength, and the phases of the $\mathbf{E}_k$ are stable to within a small fraction of 2π. Indeed if conditions are favorable, the contributions from sum III will be on the order of N^2 times the net effect that one would expect from a single one of the sources. Sum (III) can be written in terms of real functions as:

$$\sum_{j\neq k}^{N_1} E_k\ E_j\ e^{i(\phi_k-\phi_j)} = \tfrac{1}{2}\sum_{j\neq k}^{N_1}\left[E_j\ E_k\ e^{i(\phi_k-\phi_j)}\right] + \left[E_k\ E_j\ e^{-i(\phi_k-\phi_j)}\right] \tag{4.12}$$
$$= \tfrac{1}{2}\sum_{j\neq k}^{N_1} E_k\ E_j\left[e^{i(\phi_k-\phi_j)} + e^{-i(\phi_k-\phi_j)}\right] = \sum_{j\neq k}^{N_1} E_k\ E_j\ \cos(\phi_k-\phi_j)$$

Sums IV, V, and VI are small since they involve the addition of vectors of random phase.

In Eq. (4.13) we show a 5×5 array that represents the $|\mathbf{E}|^2$ value expected from a situation with three coherent and two incoherent sources: $\mathbf{E}_{TOTAL} = \mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_3+\mathbf{F}_4+\mathbf{F}_5$. The first three contributions, $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ come from well-controlled sources and are in phase with one another. The remaining contributors, $\mathbf{F}_4$, and $\mathbf{F}_5$ have random phase as might arise from incoherent background or from general noise.

$$\vec{E}_{TOTAL} = \underbrace{\vec{E}_1 + \vec{E}_2 + \vec{E}_3}_{coherent} + \underbrace{\vec{F}_4 + \vec{F}_5}_{incoherent}$$

postive

coherent

partially coherent

$$\begin{aligned}\left|\vec{E}_1+\vec{E}_2+\vec{E}_3+\vec{F}_4+\vec{F}_5\right|^2 = {} & \vec{E}_1\ \vec{E}_1 + \vec{E}_1\ \vec{E}_2 + \vec{E}_1\ \vec{E}_3 + \vec{E}_1\ \vec{F}_4 + \vec{E}_1\ \vec{F}_5 + \\ & \vec{E}_2\ \vec{E}_1 + \vec{E}_2\ \vec{E}_2 + \vec{E}_2\ \vec{E}_3 + \vec{E}_2\ \vec{F}_4 + \vec{E}_2\ \vec{F}_5 + \\ & \vec{E}_3\ \vec{E}_1 + \vec{E}_3\ \vec{E}_2 + \vec{E}_3\ \vec{E}_3 + \vec{E}_3\ \vec{F}_4 + \vec{E}_3\ \vec{F}_5 + \\ & \vec{F}_4\ \vec{E}_1 + \vec{F}_4\ \vec{E}_2 + \vec{F}_4\ \vec{E}_3 + \vec{F}_4\ \vec{F}_4 + \vec{F}_4\ \vec{F}_5 + \\ & \vec{F}_5\ \vec{E}_1 + \vec{F}_5\ \vec{E}_2 + \vec{F}_5\ \vec{E}_3 + \vec{F}_5\ \vec{F}_4 + \vec{F}_5\ \vec{F}_5\end{aligned} \tag{4.13}$$

coherent

partially coherent

postive definite

4.6 WHEN ARE INTERFERENCE EFFECTS NOTICEABLE?

4.6.1 General Criteria for Observation of Interference

To see interference when radiation of frequency ω is incident on a structure with characteristic dimension L, we need to have a dimensional regularity in the structure (uniformity in space and stability over time) such that $\delta L/L < d\omega/\omega$ over the time of the measurement. Interference terms are seen only when the spatial and temporal stability of the system are adequate. Sometimes (as with the colors seen in thin soap films) this occurs without special interventions on the part of the experimenter, but the inadvertent occurrence of detectable interference phenomena in macroscopic experiments becomes less common as the wavelength decreases and as apparatus dimensions increase. The interference patterns that one might naively expect are often wiped out by random fluctuations in the phase of radiation and by inadvertent variations in the dimensions of the apparatus. This would be the case in Eq. (4.13) above, if the averaging time of the detector is long enough so that the contributors, $\mathbf{F}_4$, and $\mathbf{F}_5$ have plenty of time to randomize their fluctuations of phase so that the partially coherent contributions (and the totally incoherent terms $F_5 \bullet F_4$ and $F_4 \bullet F_5$) average to zero

Interference effects are easily seen with radiofrequency and microwave radiation because it is easy to keep apparatus geometry stable to within a small fraction of the wavelength, and because the radiation sources tend to have narrow linewidths ($d\omega/\omega \ll 10^{-3}$) with good long-term frequency stability. So in this range of the spectrum it is common to have interference phenomena, some of which is by design (e.g. directionality of TV antennas, Doppler radar speed monitors), and some of which is inadvertent (e.g. uneven heating in microwave oven, flutter of TV or FM signal when airplane flies overhead).

The observation of interference with visible light is also quite common, but unless one uses a laser for which $d\omega/\omega$ is small, the interference is usually seen only in structures that have dimensions comparable to the wavelength of the radiation. Optical interference is easy to obtain from fabricated structures (e.g. ruled gratings or self-supporting thin films).

To see X-ray interference, however, one requires periodic arrays with spacings that are too small for ordinary manufacture, so one turns to naturally- occurring periodic structures (e.g. crystals) or thin films that have been deposited on a carefully-prepared substrate. . .

Field Concept In Limit Of Weak Intensity

The above description is the one conventionally used to talk about the effects caused by electromagnetic radiation from one or more sources. If the experiments are done with very weak sources, then it becomes clear that the square of the net electric field, $|\mathbf{E}(x,t)|^2$, expresses the likelihood that a photon will arrive at location x at time t.

4.7 QUANTUM DESCRIPTION OF PARTICLES: E(x,t) → Ψ(x,t)

We find that much of the electric field representation described above can be taken over, nearly intact, for predicting the behavior of particles such as electrons, neutrons, protons, and even entire atoms. But the Ψ functions that express our predictions for the behavior of particles at ordinary energies are characterized by deBroglie wavelengths that are significantly shorter ($\lambda \ll 10^{-10}$ m) than the interatomic spacing of solids. Unless one takes unusual steps to assure dimensional stability or to achieve longer, well-defined deBroglie wavelengths, analogies to optical interference from constructed gratings and slits are hard to see in particle experiments because the randomness in phase variations almost always wipe out the interference terms in the prediction.

As a result, the observable part of the predictions for particle distributions in macroscopic experiments usually include only the classically expected result [i.e. analogs to terms (I) and (II) of the expression (4.11)above.

> *But it must not be supposed that we can dismiss particle interference as an effect seen only in special experiments. On the contrary, we shall see that the coherent addition of state functions with concomitant interference is absolutely fundamental to the quantum mechanical description of motion of any sort.*

With careful fabrication of a diffraction array from a large, single crystal it is possible to do interference experiments with slow neutrons.

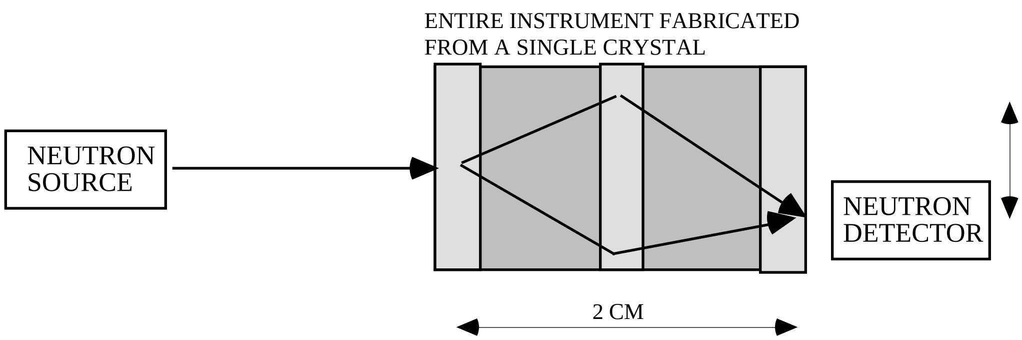

FIG:4.2 Single Crystal Neutron Interferometer