What is an ‘Argument’?

In ordinary speech, the word ‘argument’ can mean a fight, or a heated, vitriolic debate. In Philosophy, however, we have a more technical understanding of ‘argument’. An argument is just a bunch of reasons for believing some proposition. For instance, the following is an argument that you have probably seen advanced in the media:

We shouldn’t allow gays to marry, since marriage is a sacred institution which ought to be protected, and allowing gays to marry would undermine the sanctity of that institution.

What makes it an argument is that it gives reasons for believing that we shouldn’t allow gays to marry. Those reasons are the premises of the argument. That gays shouldn‘t be allowed to marry is is the conclusion of the argument—it’s the thing that the arguer is arguing for. Sometimes, when we want to make the structure of the argument more explicit, we can write it out in premise-conclusion form, as follows:

\[ P_1 \] We ought not undermine the sanctity of the institution of marriage.

\[ P_2 \] Allowing gays to marry would undermine the sanctity of the institution of marriage.

\[ C \] We ought not allow gays to marry.

The argument says that the propositions above the horizontal line give us some reason to think that the proposition below the horizontal line is true.
2 What Makes an Argument Good?

An argument attempts to provide premises which give us reason to think that the conclusion is true. One good kind of argument is one which provides premises which make the conclusion more likely. The following is an argument like that:

In the three and a half years that I’ve known Alex, he has never stayed out past midnight. So he won’t be staying out with us past midnight on Thursday.

or, in premise-conclusion form:

\[ P_1 \quad \text{In three and a half years, Alex has never stayed out past midnight.} \]

\[ C \quad \text{Alex will not stay out past midnight on Thursday.} \]

\( P_1 \) gives a good reason to believe that Alex won’t stay out past midnight on Thursday. However, \( P_1 \) doesn’t necessitate that Alex won’t stay out past midnight on Thursday. It could very well happen that on Thursday, Alex stays out past midnight for the first time. Nevertheless, \( P_1 \) does make it much more likely that Alex won’t be staying out late on Thursday.

In philosophy, we usually confine our attention to a different sort of argument, a stronger sort of argument, than this. In philosophy, we’re usually interested in arguments in which the premises necessitate the conclusion. In these arguments—known as deductively valid arguments—it is impossible for the premises to all be true and the conclusion to be simultaneously false. If the premises are all true, then the conclusion must be true as well. Here are a few examples of arguments like this:

\[ P_1 \quad \text{Obama is younger than 30.} \]

\[ C \quad \text{Obama is younger than 100.} \]

\[ P_1 \quad \text{Margaret Thatcher lives in New York City.} \]

\[ P_2 \quad \text{New York City is within New York State.} \]

\[ C \quad \text{Margaret Thatcher lives in New York State.} \]

\[ P_1 \quad \text{Ron Paul loves Freedom.} \]

\[ P_2 \quad \text{If Ron Paul loves Freedom, then he hates Canada.} \]

\[ C \quad \text{Ron Paul hates Canada.} \]
Notice that each of these arguments has a false premise. Nevertheless, these arguments are deductively valid. That's because, for an argument to be deductively valid, it doesn't have to have true premises, nor a true conclusion. There simply has to be the right connection between the premises and the conclusion. It just has to be the case that if the premises are all true, then the conclusion must be true as well. That is:

An argument is deductively valid if and only if the truth of the premises necessitates the truth of the conclusion.

Here's another (equivalent) definition of deductive validity:

An argument is deductively valid if and only if it is impossible for all the premises to be true and, simultaneously, for the conclusion to be false.

In each of the above arguments, this is the case. It is impossible for Obama to be younger than 30 and not be younger than 100. If Obama is younger than 30, then by necessity he will be younger than 100. Similarly, if Margaret Thatcher lives in New York City, and New York City is located within New York State, then by necessity Margaret Thatcher lives in New York State. Just because Obama isn't actually younger than 30, and just because Margaret Thatcher doesn't actually live in New York City, this doesn't destroy the connection between the premises and the conclusions of these arguments.

If an argument isn't deductively valid, then it is deductively invalid.

3 How to Resist the Conclusion of an Argument

There are two ways to resist the conclusion of a deductive argument (an argument that aspires to deductive validity). You can either 1) deny one of the premises, or 2) deny that the argument is deductively valid—that is, deny that the conclusion actually follows from the premises. For instance, I resist the conclusion of the argument about Margaret Thatcher by denying the first premise. Margaret Thatcher doesn't live in New York City.

Suppose, however, that I decide to deny that an argument is valid. Suppose, for instance, that I object to the following argument on the grounds that its conclusion doesn't follow from its premise:

| Pr | Life on earth was intelligently designed. |
| C  | God designed life on earth. |
If I deny that this argument is deductively valid, then I deny that it is impossible for the premise to be true and the conclusion to be simultaneously false. So, I must think that it is possible for the premise to be true and for the conclusion to be simultaneously false. So, if I deny that the argument is deductively valid, then I should be able to describe this possibility—the possibility in which the premise is true yet the conclusion is false.

Here’s how I can do that: I can point out that it’s possible that life on earth was intelligently designed by an alien civilization. If that were the case, then it would be true that life on earth was intelligently designed, but false that God designed life on earth. That is, if that were the case, then \( P1 \) would be true, yet \( C \) would be false. But if it’s possible for \( P1 \) to be true while \( C \) is false, then the argument is not deductively valid. If that’s possible, then the conclusion doesn’t follow from the premises.

An example like this—an example in which the premises are all true and yet the conclusion is still false—is known as a counterexample to the argument’s validity. If an argument is deductively valid, then there should be no counterexamples. So, if you can produce a counterexample, then you’ve shown that the argument isn’t deductively valid.

A counterexample to an argument is a possible scenario in which the premises are all true, yet the conclusion is simultaneously false.

**Deductive Validity (3).** An argument is deductively valid if and only if it has no counterexamples.

### 3.1 Examples of Counterexamples

Here are some more examples of deductively invalid arguments and the counterexamples which show them to be deductively invalid.

- **\( P1 \)**: Every Republican is an American.
- **\( P2 \)**: Every Republican is a Veteran.
- **\( C \)**: Every American is a Veteran.

**Counterexample:** Suppose that every Republican is an American, but that not every American is a Republican (there are some Democrats). Then, even if every Republican were a Veteran, it could still be the case that some Democrats are not Veterans. Since the Democrats are Americans, there would be some Americans
who are not Veterans. That is: P1 and P2 would be true, but C would be false. So, the argument is invalid. The conclusion doesn't follow from the premises.

\[ P1 \quad \text{John is taller than Sally.} \]
\[ C \quad \text{John is tall.} \]

**Counterexample:** If John were 4’8” and Sally were 4’6”, then even though it would be true that John is taller than Sally, it would still be false that John is tall. So the premise would be true and the conclusion would be false. So, from the fact that John is taller than Sally, it doesn't follow that John is tall.

\[ P1 \quad \text{If a fetus has rights, then abortion is immoral.} \]
\[ P2 \quad \text{Abortion is immoral.} \]
\[ C \quad \text{A fetus has rights.} \]

**Counterexample:** Grant that, if a fetus has rights, then abortion is immoral. However, deny that a fetus has rights. Still, abortion could be immoral for some reason other than the fact that the fetus has rights. For instance, if, in the past 30 years, only one woman has been able to conceive, and the fetus in her womb is humanity’s one hope for survival, then it could be immoral to abort that fetus even though that fetus doesn’t have any rights. In that possibility, abortion would be immoral. However, that wouldn't mean that the fetus has rights. So, that’s a possibility in which the premises are all true and the conclusion is false. So, the conclusion doesn't follow from the premises.

\[ P1 \quad \text{If it’s not Tuesday, then I’m not drinking.} \]
\[ P2 \quad \text{It’s Tuesday.} \]
\[ C \quad \text{I’m drinking.} \]

**Counterexample:** Suppose that I only drink on the first Tuesday of every month (for religious reasons, let’s say). Suppose also that it’s the second Tuesday of the month. Then, it would be true that, if it’s not Tuesday, then I’m not drinking (otherwise, I’d be breaking the rules of my religion). And it would also be true that it’s Tuesday, but it would be false that I’m drinking.

(Note that, in this scenario, it’s also true that ‘if it’s not the first Tuesday of the month, then I’m not drinking’. Perhaps this would be a more informative thing to say than P1 in the scenario I’ve described. However, P1 would still be true in this scenario—and that’s all that matters. For it to be a counterexample, the premises must be true and the conclusion must be false.)
4 Conditionals

Suppose that I have four cards, and I tell you that each of them has a letter printed on one side and a number printed on the other side. I lay them out on the table in front of you, like so:

```
   9  J  U  2
```

And I tell you that all four of these cards obey the rule

If there is a vowel printed on one side of the card, then there is an even number printed on the other.

Before reading on, take a moment to think about which of these cards you would have to turn over in order to check whether or not I am lying.

If you’re like most people, you said that you have to turn over the U and the 2 cards. That’s what about 95% of people say when they are asked this question. Unfortunately, it’s not right. In order to see whether I’m lying, you do have to turn over the U card, but you don’t have to touch the 2 card. And you do have to turn over the 9 card.

We humans have an incredibly hard time thinking about when sentences of the form

If $P$, then $Q$

(where $P$ and $Q$ are two arbitrary sentences) are true or false. Sentences like these are known as conditionals. They are called that because they don’t flat-out assert that $Q$; rather, they only assert $Q$, conditional on the supposition that $P$. And they don’t say anything about whether or not $P$.

Here’s a useful way to think about conditionals: the conditional claim ‘if $P$, then $Q$’ tells you that, given the circumstances, $P$’s being true is sufficient for $Q$’s being true—$P$’s being true is enough for $Q$ to be true.

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**Conditionals.** A conditional ‘if $P$, then $Q$’ is true if and only if, given the circumstances, the truth of $P$ is sufficient for the truth of $Q$. 

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6
However, ‘if $P$, then $Q$’ does not tell you whether $P$ is actually true, and it doesn’t tell you that $P$ is necessary for $Q$ to be true. In order to understand this, we have to talk for a bit about necessary and sufficient conditions.

### 4.1 Necessary and Sufficient Conditions

Being a Michigander is sufficient for being an American. Being an American is necessary for being a Michigander. There’s no way to be a Michigander without being an American.

However, being a Michigander is not necessary for being an American. You could be a New Yorker. Then, you’d be an American without being a Michigander. And being an American is not sufficient for being a Michigander. Georgians are Americans without being Michiganders. So just being an American isn’t enough to make you a Michigander.

---

**Necessary Condition.** $N$ is a necessary condition for $X$ if and only if every $X$ is an $N$—if and only if there is nothing which is $X$ which is not also $N$.

**Sufficient Condition.** $S$ is a sufficient condition for $X$ if and only if every $S$ is an $X$—if and only if there is nothing which is $S$ which is not also $X$.

We can visualize these definitions using the Venn diagram shown in figure 1.

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**Figure 1:** $N$ is necessary for $S$, and $S$ is sufficient for $N$.

In the diagram, $S$ is a sufficient condition for $N$, and $N$ is a necessary condition for $S$. Being inside $N$ is necessary for being inside $S$ and being inside $S$ is sufficient for being inside $N$.

Notice that if $N$ is a necessary condition for $S$, then $S$ is a sufficient condition for $N$, and *vice versa*.

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**Necessary Condition (2).** $N$ is a necessary condition for $X$ if and only if $X$ is a sufficient condition for $N$.

**Sufficient Condition (2).** $S$ is a sufficient condition for $X$ if and only if $X$ is a necessary condition for $S$. 

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7
Returning to conditionals,

**Conditionals.** *A conditional ‘if $P$, then $Q$’ is true if and only if, given the circumstances, the truth of $P$ is sufficient for the truth of $Q.*

In the Venn diagram in figure 1, ‘if $S$, then $N$’ is true. However, ‘if $N$, then $S$’ is not true. Think about the sentence

P: If Abraham Lincoln died in 1942, then he is dead today.

If $P$ is true, then every relevant possibility in which Lincoln died in 1942 is a possibility in which he is dead today. And every relevant possibility in which Lincoln died in 1942 is a possibility in which he is dead today. So we can conclude on the basis of the previous two sentences that $P$ is true. It is true, even though

the *if* part of $P$ is false. Even though Lincoln didn’t die in 1942, it’s still true that, *if he had*, he’d be dead today.

Go back to the four cards.

| 9 | J | U | 2 |

The rule that I claimed the cards obeyed was

*If there is a vowel printed on one side of the card, then there is an even number printed on the other.*

If this is true, then having a vowel is sufficient for having an even number. So, every card which has a vowel has an even number.

Notice that this *doesn’t* mean that every card which has an even number has a vowel. So there’s no need to flip over the 2 card. If it has a vowel on the other side, then it will conform to the rule, since it has an even number. On the other hand, if it has a consonant on the other side, then it will still conform to the rule. Look to the Venn diagram above. The conditional says that the cards with vowels is a subset of the cards with even numbers. But it’s consistent with that that there be a card with a consonant and an even number (this possibility lies in the logical
space marked with an ‘x’ in the diagram). So either way, the rule won't be violated. So there’s no need to check the 2 card.

On the other hand, you do have to check the 9 card. If the 9 card has a vowel on the other side, then the rule will be violated. So you must make sure that there’s no vowel on the other side of the 9 card. Also, you must make sure that there’s an even number on the other side of the U card—since the U card has a vowel, the rule says that it must have an even number on the other side. If it has an odd number on the other side, then the rule will be violated.

Now that I’ve told you a bit about deductive reasoning in philosophy, I’ll let you in on a secret: at one point during the previous eight pages, I made an egregious deductive error. I said that a conclusion followed when it actually didn’t. Look back through the previous pages and see if you can find the fallacy. The first three students to find the fallacious reasoning and report it to me via email will get an extra three points added to their final grade in the course.

**Worked Exercises**

1. Is the following argument deductively valid or deductively invalid?

   P1) If abortion is morally impermissible, then so is infanticide.

   P2) Infanticide is morally impermissible.

   C) Abortion is morally impermissible.

   A) Deductively Valid
   B) Deductively Invalid

   **Answer:** The argument is deductively invalid. Here is the counterexample: suppose that killing a developing life gets progressively worse the more developed the life is. If that were so, then it would be true that if abortion is morally impermissible, then infanticide would be morally impermissible also, since infanticide is morally worse than abortion. Suppose further that killing the developing life gets bad enough to count as morally impermissible at the moment of birth. In that case, it would be true that infanticide is morally impermissible. However, it would be false that abortion is morally impermissible, since abortion precedes birth, when killing the developing life gets so bad that it counts as impermissible.
So, in that possibility, the premises are both true, but the conclusion is false. So it is a counterexample. So the argument is deductively invalid.

2. Is the following argument deductively valid or deductively invalid?

\[ \text{P1) Obama will lose the 2012 U.S. Presidential election and marijuana is the world's most harmful drug.} \]

\[ \text{C) Obama will lose the 2012 U.S. Presidential election.} \]

A) Deductively Valid
B) Deductively Invalid

Answer: The argument is deductively valid. If it is true that Obama will lose the election and marijuana is the most harmful drug, then it must be true that Obama will lose the election. There's no way for the premises to be true and the conclusion to be simultaneously false. Notice that we don't have to consider whether the premises are true. We're not interested in the truth of the premises. We're just interested in the logical connection between the premises and the conclusion. We're just interested in whether the conclusion follows from the premises.

3. Is the following argument deductively valid or deductively invalid?

\[ \text{P1) Harold and Kumar Go to White Castle is the greatest film of all time} \]

\[ \text{C) Harold and Kumar Go to White Castle is the greatest film of all time} \]

A) Deductively Valid
B) Deductively Invalid

Answer: The argument is deductively valid. The premise and the conclusion are identical, so there's no way that the premise could be true and the conclusion could be simultaneously false. The argument is, by the way, really bad in certain respects—it wouldn't persuade anybody of the conclusion. But that doesn't mean that it's not deductively valid.

4. Is the following argument deductively valid or deductively invalid?

\[ \text{P1) Macbeth is the greatest work of literature ever written.} \]

\[ \text{P2) Ron Paul wrote Macbeth.} \]

\[ \text{C) Ron Paul wrote the greatest work of literature ever written.} \]

A) Deductively Valid
B) Deductively Invalid
**Answer:** The argument is deductively valid. If Ron Paul wrote Macbeth, and it is the greatest literary work, then Ron Paul wrote the greatest literary work. It doesn't matter at all that Ron Paul didn't write Macbeth. All that matters is that, if he did, and if it is the greatest literary work, then he wrote the greatest literary work.

5. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

\[ \text{P1) Bertrand Russell is the greatest philosopher of all time.} \]
\[ \text{P2) The greatest philosopher of all time died before any other philosopher was born.} \]
\[ \text{P3) Aristotle was a philosopher.} \]
\[ \text{C)} \text{ Bertrand Russell died before Aristotle was born.} \]

A) The argument is deductively valid. Therefore, there is no counterexample.

B) Bertrand Russell is the greatest philosopher of all time. However, he died after Aristotle was born.

C) Aristotle was the greatest philosopher of all time, and he died before Russell was born.

D) Plato is a better philosopher than either Aristotle or Russell, and he died before either of them were born.

**Answer:** A is the correct choice. The first two premises are actually false, but the conclusion follows necessarily from the truth of the premises. There's no possible way for the premises to be true and the conclusion to be false.

B is not a counterexample because, even though it makes one premise true and the conclusion false, it does not make all the premises true. C and D are not correct because both of them contradict the first premise. So, if C or D were true, then the first premise would be false. But a counterexample has to make all the premises true.

6. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

\[ \text{P1) All Indian food has chickpeas in it.} \]
\[ \text{P2) Chana Puri has chickpeas in it.} \]
\[ \text{C)} \text{ Chana Puri is a type of Indian food.} \]

A) The argument is deductively valid. Therefore, there is no counterexample

B) Chana Puri has chickpeas in it, and is a type of Indian food, but not all Indian food has chickpeas in it.
C) Chana Puri has chickpeas in it, but isn't a type of Indian food. All Indian food, however, has chickpeas in it.

D) Pani Puri is a type of Indian food which doesn't have chickpeas in it. Therefore, not all Indian food has chickpeas in it.

Answer: C is the correct choice. If C were true, then the first two premises would be true; however, the conclusion would be false. So it is a counterexample to the validity of the argument.

A is incorrect because the argument isn't deductively valid.

Neither B nor D provides a counterexample, since if either B or D were true, then the first premise would be false. But, in order to be a counterexample, all the premises must be true, and the conclusion simultaneously false.

7. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

\[
P_1) \text{All humans are mammals.} \\
P_2) \text{Some mammals have hair.} \\
C) \text{All humans have hair.}
\]

A) The argument is deductively valid. Therefore, there is no counterexample.

B) All humans are mammals and all whales are mammals. Whales have hair, but humans do not.

C) Humans are not mammals, and even though all mammals have hair, humans do not have hair.

D) Who's to say that all humans are mammals? Some humans believe that they are fish.

Answer: B is the correct choice. If B were true, then \( P_1 \) would be true, since all humans are mammals, and \( P_2 \) would be true, since some mammals (the whales) would have hair. However, the conclusion would be false, since no humans have hair. So it is a counterexample to the validity of the argument.

A is not correct because the argument is not valid.

C is not correct because, if \( C \) were true, then \( P_1 \) would be false. But, in order to be a counterexample, it has to be a scenario in which all the premises are true.

D is calling into question the truth of the first premise. However, the validity of the argument doesn't have anything to do with the truth of the premises. The premises could all be false and the argument could still be deductively valid. So D does not provide a counterexample.

8. Suppose that
If $P$, then $Q$

is true. Then, which of the following must be true?

A) Given the circumstances, the truth of $Q$ is sufficient for the truth of $P$.
B) Given the circumstances, the truth of $Q$ is necessary for the truth of $P$.
C) Given the circumstances, the truth of $P$ is necessary for the truth of $Q$.
D) Given the circumstances, the falsehood of $P$ is sufficient for the falsehood of $Q$.

Answer: The correct answer is B. We know that if $If P, then Q$ is true, then given the circumstances, the truth of $P$ is sufficient for the truth of $Q$. And we know that, if $S$ is sufficient for $N$, then $N$ is necessary for $S$. So if, given the circumstances, the truth of $P$ is sufficient for the truth of $Q$, then the truth of $Q$ must be necessary for the truth of $P$, given the circumstances.
Exercises

1. Is the following argument deductively valid or deductively invalid?

   P1) The American Constitution guarantees the right to bear arms.

   P2) If the American Constitution guarantees the right to bear arms, then we should be allowed to own guns without government interference.

   C) Gun control would not decrease the number of violent crimes.

2. Is the following argument deductively valid or deductively invalid?

   P1) George Washington was the first American President.

   P2) If George Washington was the first American President, then dinosaurs are not extinct.

   C) Dinosaurs are not extinct.

3. Is the following argument deductively valid or deductively invalid?

   P1) If Khrushchev had given missiles to Cuba, then there would have been a nuclear war.

   P2) There wasn't a nuclear war.

   C) Khrushchev didn't give missiles to Cuba.

4. Is the following argument deductively valid or deductively invalid?

   P1) If abortion is morally permissible, then so is infanticide.

   P2) Infanticide is not morally permissible.

   C) Abortion is not morally permissible.

5. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

   P1) Everything Saul Kripke has ever said (up to the present moment) has been true.

   P2) Tomorrow, Kripke will say "The earth is round."

   C) The earth is round.

   A) The argument is deductively valid. Therefore, there is no counterexample.

   B) Saul Kripke is a clever con artist who has been telling lies his whole life. However, he has tricked everybody into believing that he was telling the truth.

   C) Up until the present moment, Saul Kripke has never said anything that was false. Tomorrow, he’s going to say that the earth is round. However, he will be wrong, since the earth is flat.
D) Truth is subjective. Some people think that Saul Kripke is a liar. Others think he is telling the truth. There is no way to know for sure whether he is right or wrong.

6. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

P1) If Obama is President of the United States, then he is a natural-born U.S. citizen.

P2) Obama is a natural-born U.S. citizen.

C) Obama is President of the U.S.

A) The argument is deductively valid. Therefore, there is no counterexample.

B) Obama is both a natural born citizen and President of the U.S.

C) Obama is not a natural born citizen and is President of the U.S.

D) Obama is a natural-born U.S. citizen, but he was never elected President.

7. If the argument given below is deductively invalid, which of the answer choices is a counterexample to its validity?

P1) Obama was born in Kenya.

P2) You must be born in the U.S. to legitimately be President of the United States.

C) Obama is not legitimately President of the United States.

A) The argument is deductively valid. Therefore, there is no counterexample.

B) You don't need to be born in the U.S. in order to be the President.

C) McCain was born in Panama, so he's not eligible to be President of the U.S. either.

D) Obama wasn't born in Kenya. He is therefore legitimately President of the U.S.

8. If the argument given below is deductively invalid, provide a counterexample to its validity. If it is deductively valid, just write “deductively valid.”

P1) Colonel Mustard didn't do it in the Library.

P2) Mr. Plum didn't do it in the Lounge with the lead pipe.

P3) Ms. Peacock didn't do it with the rope.

C) Colonel Mustard didn't do it in the Library with the candle stick.
9. Suppose that I have four cards, and I tell you that each face of each card has either a number (1 through 9) or a suit (heart, diamond, club, spades). Just because there is a suit on one side, this doesn't mean that there's a number on the other; and just because there's a number on one side, this doesn't mean that there's a suit on the other. I lay the cards down on the table, and you see:

```
9 ♠ 4
```

I then tell you that all of the cards conform to the following rule: If there is a Heart on one side of a card, then there is the number 4 on the other side. Which of the cards would you have to flip over in order to tell whether I am lying?

A) You only have to flip over the Heart.
B) You have to flip over the Heart and the 4.
C) You have to flip over the 9, the Heart, and the 4.
D) You have to flip over the 9, the Heart, and the Club.

10. “P only if Q” is equivalent to (means the same thing as) “If P, then Q”. Given this, which of the following can you infer from the truth of (i)?

1) You'll have a window only if you're on the left side of the plane

A) Having a window is sufficient for being on the left side of the plane
B) Having a window is necessary for being on the left side of the plane
C) Being on the left side of the plane is sufficient for having a window
D) None of the Above

11. "P only if Q" is equivalent to (means the same thing as) "If P, then Q". Therefore, "P if and only if Q" is equivalent to (means the same thing as) "If P, then Q and if Q, then P."

Given this, which of the following can you infer from the truth of (2)?

2) You will have a window if and only if you are on the left side of the plane.

A) Everyone with a window is on the left side of the plane.
B) Everyone on the left side of the plane has a window
C) There's no one on the right side of the plane with a window
D) There's no one without a window on the left side of the plane
E) A and B
F) C and D
G) A, B, C, and D
5 Formal Deductive Logic

It turns out that many of the same words seem to show up again and again in deductively valid arguments. These include words like ‘it is not the case that...', ‘both... and...', ‘either...or...', ‘if... then...', and ‘...if and only if...’. These words are known as logical connectives, and they form the backbone of formal deductive logic.

What makes logic formal is that it doesn't pay any attention to the content of what is being said. It looks only to the logical form of the premises and the conclusion. To get a feel for this, notice that the following deductively valid arguments all share a common form:

P1) Either the Provost or the Dean is introducing the speaker.

P2) The Dean is not introducing the speaker.

C) The Provost is introducing the speaker.

P1) Rohan is either going to the Bahamas or Flint, Michigan.

P2) Rohan is not going to the Bahamas.

C) Rohan is going to Flint, Michigan.

P1) Obama is the president or I’m a monkey’s uncle.

P2) I’m not a monkey’s uncle.

C) Obama is the president.

The common form they share is this:

P1) Either \(X\) or \(Y\)

P2) It is not the case that \(Y\)

C) \(X\)

Here, \(X\) and \(Y\) are variables which stand in for sentences. In the first argument, \(X = \text{The Provost is introducing the speaker}\) and \(Y = \text{The Dean is introducing the speaker}\).
In the second argument, \( X = \text{Rohan is going to the Bahamas} \) and \( Y = \text{Rohan is going to Flint, Michigan} \). In the third argument, \( X = \text{Obama is the president} \) and \( Y = \text{I'm a monkey's uncle} \).

It's natural to think that it's not an accident that the arguments which share this form are all deductively valid. It's natural to think that there's something about the form of these sentences which guarantees that they will be deductively valid. It turns out that there is, and it comes from the meaning of 'either...or' and 'it is not the case that...'. Think about 'it is not the case that...' first. If I tell you that I have some sentence '\( p \)', and I tell you that '\( p \)' is true, can you tell me whether 'it is not the case that \( p \)' is true or false? Now suppose that I have some sentence '\( q \)', and I tell you that '\( q \)' is false. Can you tell me whether 'it is not the case that \( q \)' is true or false? It seems that you can. If you know that '\( p \)' is true, then you know that 'it is not the case that \( p \)' is false. If you know that '\( q \)' is false, then you know that 'it is not the case that \( q \)' is true. So, in general, in order to figure out the truth value\(^1\) of

\[
\text{It is not the case that } X
\]

You only need to know the truth value of \( X \). The truth value of 'it is not the case that \( X \)' is determined by the truth value of \( X \).

We can summarize this by writing up the following truth table, which tells us exactly how the truth value of 'it is not the case that \( X \)' is determined by the truth value of \( X \):

<table>
<thead>
<tr>
<th>( X )</th>
<th>It is not the case that ( X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

(Here, ‘\( T \)’ stands for ‘true’ and ‘\( F \)’ stands for ‘false.’)

We can do the same thing with 'either...or...'. If both \( X \) and \( Y \) are true, then 'either \( X \) or \( Y \)' will be true, too. If \( X \) is true and \( Y \) is not, then 'either \( X \) or \( Y \)' will be true. If \( X \) is false but \( Y \) is true, then 'either \( X \) or \( Y \)' will be true. And if both \( X \) and \( Y \) are false, then 'either \( X \) or \( Y \)' will be false. We can summarize this with the following truth table:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>either ( X ) or ( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

\(^1\) The truth value of a sentence is true if the sentence is true and false if the sentence is false.
This is what makes logical connectives interesting: they turn sentences into longer sentences, and the truth values of those longer sentences are entirely determined by the truth values of their constituent parts.

**Logical Connectives.** A logical connective forms a longer sentence out of smaller sentences, and the longer sentence that it forms has its truth value entirely determined by the truth values of the smaller sentences which make it up.

Notice that not every word which allows us to form longer sentences out of smaller sentences will be a logical connective. For instance, ‘...because...’ forms longer sentences out of smaller sentences. I can take the sentence ‘grass is green’ and the sentence ‘snow is white’ and stick them together with ‘...because...’ to get the sentence ‘grass is green because snow is white.’ And I can take the sentence ‘it’s cold in Ann Arbor’ and the sentence ‘Ann Arbor is in the north’ and stick them together with ‘...because...’ to get the sentence ‘it’s cold in Ann Arbor because Ann Arbor is in the north’. However, even though each of the sentences ‘grass is green’, ‘snow is white’, ‘it’s cold in Ann Arbor’, and ‘Ann Arbor is in the north’ are true, the sentence ‘grass is green because snow is white’ is false; whereas ‘it’s cold in Ann Arbor because Ann Arbor is in the north’ is true. But that means that there’s no truth table for because,

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>X because Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T/F</td>
</tr>
</tbody>
</table>

since sometimes, ‘X because Y’ is true when ‘X’ and ‘Y’ are both true, and sometimes ‘X because Y’ is false when ‘X’ and ‘Y’ are both true. So the truth values of ‘X’ and ‘Y’ don’t determine the truth value of ‘X because Y’. So ‘...because...’ is not a logical connective.

Once we recognize that phrases like it is not the case that... and either...or... are logical connectives, we can begin to see why every argument of the form

P1) Either X or Y
P2) It is not the case that Y
C) X

has to be deductively valid. The reason is that, given the truth tables for these logical connectives, it doesn't matter what truth values X and Y have—so long as
The Inclusive and the Exclusive ‘Or’

Note that sometimes, we use ‘either...or...’ to mean ‘either...or...but not both’. For instance, on restaurant menus, when it says that you may have either the soup or the salad, this does not mean that you may have both soup and salad. This is called the exclusive ‘or’, since it excludes the possibility of both the either and the or sentences being true. However, we often also use ‘either...or...’ to mean ‘either...or...and perhaps both’. For instance, when I say that you can come to either my tuesday or my thursday office hours, I don't mean to imply that you can't come to both. This is called the inclusive ‘or’, since it includes the possibility of both the either and the or sentences being true.

For the purposes of constructing a formal logic, we're going to make the arbitrary choice to focus on the inclusive ‘or’, but there's nothing more than convention motivating this choice. We could just as easily have chosen to focus on the exclusive ‘or’ instead. Given the inclusive or, we can define the exclusive or (xor) as either...or...and not both. Given the exclusive or (xor), we can define the inclusive or as either...xor...xor both. The important thing is just that we make sure not to confuse the inclusive and the exclusive or.
both ’either X or Y’ and ‘it is not the case that Y’ are true, ‘X’ must be true. Just look at the truth tables:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>either X or Y</th>
<th>it is not the case that Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The only way that both ’either X or Y’ and ‘it is not the case that Y’ could be true is if ‘X’ is true and ‘Y’ is false (this possibility is represented by the second row of the truth table. And, in that possibility, ‘X’ is true. That means that, so long as the premises of the argument are true, the conclusion will be true as well. And that means that every argument with this general form will be deductively valid. And we can know this without knowing anything about what the sentences ‘X’ and ‘Y’ are.

Here’s a list of all the logical connectives that get used in the most simple formal logical system—what’s known as sentential logic:

- *it is not the case that...*
- *both...and...*
- *either...or...*
- *if..., then...*
- *...if and only if...*

And here are their corresponding truth tables:

<table>
<thead>
<tr>
<th>X</th>
<th>It is not the case that X</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>both X and Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>either X or Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
The truth table for the conditional \( \text{if... then...} \) may look odd to you. If so, you're right. It is odd. (If it doesn't look odd to you, read the boxed text on the following page to see why the truth table is so odd.) In fact, this truth table probably doesn't do a very good job of capturing the meaning of the English \( \text{if... then...} \). That's because the English \( \text{if... then...} \) probably isn't a logical connective, in the sense we've defined it here. But logicians nevertheless want a logical connective which acts like the English \( \text{if... then...} \), and the truth table defined above does act an awful lot like the English \( \text{if... then...} \), so it's been introduced as a standard logical connective. To distinguish this from the English conditional \( \text{if... then...} \), we call the logical connective defined by the truth table above the material conditional.

Just to make things a bit easier to write down, logicians have introduced symbols to stand in for these logical connectives. Here is one set of commonly used symbols for the logical connectives:

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( \text{if } X, \text{ then } Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( X \text{ if and only if } Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

There are also other symbols that get used, with varying frequency. If you're interested, here's a table of the alternate symbols ('either... or...' only ever gets symbolized with \( \lor \)):

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( \text{if... or...} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

If you go on to study logic, you'll end up seeing these additional symbol being used (and you'll also see them in some of the readings over the course of the semester). However, I'm just going to stick to the first set: \( \neg, \land, \lor, \rightarrow, \) and \( \leftrightarrow \).
The Material Conditional

The logical connective defined by the truth table for ‘if X, then Y’ on the previous page has the following peculiarity: if ‘X’ is false, then ‘if X, then Y’ is true—no matter whether ‘Y’ is true or false, and no matter what kind of connection there is between ‘X’ and ‘Y’. This is odd because we ordinarily wouldn’t think that the sentence ‘If John Adams was America’s first president, then eating soap cures cancer’ is true, just in virtue of the fact that ‘John Adams was America’s first president’ is false.

I think that this is exactly right. The truth table does not do a very good job of capturing our English connective if..., then.... For this reason, logicians don’t call the logical connective defined by that truth table the conditional. They give it the special name ‘material conditional’. To clearly distinguish the material conditional from the English conditional, let’s use the symbol ‘→’ for the material conditional. Then, ‘X → Y’ is false just in case ‘X’ is true and ‘Y’ is false, and is true otherwise. That’s just what ‘→’ means.

Even though the material conditional is not completely like the English conditional, it is awfully close. For instance, the arguments

\[
\begin{array}{c}
\text{If } p, \text{ then } q \\
\hline
p \\
q
\end{array}
\]

are deductively valid, for any \( p \) and \( q \). And the corresponding arguments

\[
\begin{array}{c}
p \rightarrow q \\
p \\
q
\end{array}
\]

are deductively valid as well, for any \( p \) and \( q \). It is in virtue of this family resemblance between if..., then... and \( \rightarrow \) that we call the latter a ‘conditional.’
We can go ahead and just *define* these symbols as the logical connectives with the following truth tables:

\[
\begin{array}{c|c}
X & \neg X \\
\hline
T & F \\
F & T \\
\end{array}
\]

\[
\begin{array}{c|c|c}
X & Y & X \land Y \\
\hline
T & T & T \\
T & F & F \\
F & T & F \\
F & F & F \\
\end{array}
\quad
\begin{array}{c|c|c}
X & Y & X \rightarrow Y \\
\hline
T & T & T \\
T & F & F \\
F & T & T \\
F & F & T \\
\end{array}
\]

\[
\begin{array}{c|c|c}
X & Y & X \lor Y \\
\hline
T & T & T \\
T & F & T \\
F & T & T \\
F & F & F \\
\end{array}
\quad
\begin{array}{c|c|c}
X & Y & X \leftrightarrow Y \\
\hline
T & T & T \\
T & F & F \\
F & T & F \\
F & F & T \\
\end{array}
\]

Now that we have the truth tables for the logical connectives, we can see that the following argument forms are deductively valid by noting that, in every row of the truth table in which the premises are all true, the conclusion is true as well. (I’ve included the names that logicians give to these argument forms in italics.)

*modus ponens*  
\[
\begin{array}{c}
X \rightarrow Y \\
\hline
\neg Y \\
\end{array}
\]

*DeMorgan’s Law (I)*  
\[
\begin{array}{c}
\neg (X \lor Y) \\
\hline
\neg X \land \neg Y \\
\end{array}
\]

*DeMorgan’s Law (II)*  
\[
\begin{array}{c}
\neg (X \land Y) \\
\hline
\neg X \lor \neg Y \\
\end{array}
\]

*modus tollens*  
\[
\begin{array}{c}
\neg Y \\
\hline
\neg X \\
\end{array}
\]

*contraposition*  
\[
\begin{array}{c}
X \rightarrow Y \\
\hline
\neg Y \rightarrow \neg X \\
\end{array}
\]
**Worked Exercise**

Translate the following argument into formal symbolic logic (include a translation key saying which sentence letters stand for which sentences).

John either likes broccoli or ham.
John doesn't like ham.

\[ \text{John likes broccoli.} \]

**Solution**

Use the two sentence letters ‘p’ and ‘q’, where \( p = \text{John likes broccoli} \) and \( q = \text{John likes ham} \). Then, since the first premise is equivalent to *Either John likes broccoli or John likes ham*, it can be written

\[ p \lor q \]

And, since the second premise is equivalent to *it is not the case that John likes ham*, it can be written

\[ \neg q \]

And the conclusion is just \( p \), that John likes broccoli.
Exercises

1. Write out truth tables for each of the argument forms above (modus ponens, modus tollens, DeMorgan's Laws, and contraposition) and show that they are deductively valid.

2. Suppose that $p$ is the sentence Daniel goes to the store, $q$ is the sentence Daniel doesn’t remember that the store sells potato chips, and $r$ is the sentence Daniel buys potato chips. Then, provide an idiomatic English translation of the following formulae of sentential logic:

$$ p \rightarrow (\neg q \rightarrow r) $$
$$ q \rightarrow \neg p $$
$$ p \land \neg r $$

Suppose that Daniel goes to the store, remembers that they sell potato chips, and doesn’t buy potato chips. Then, using truth-tables, discover which of the above formulae are true and which are false (If the sentence includes three sentence letters, then you’ll have to include more than four rows in your truth table. You’ll have to include a row for every possible assignment of truth values to the three sentences. For the first formula, you should determine the truth-value of the expression inside parentheses first, and then determine the truth value of the entire expression).

3. Translate the following arguments into formal logic (provide a translation key) and show whether, so translated, they are deductively valid or invalid by drawing out a truth-table. If the argument, translated into sentential logic, is deductively invalid, then provide a counterexample (this will be a row of the truth-table). Say also whether the original argument, before being translated into formal logic, was deductively valid.

(a) P1. Everyone loves Mary.
P2. If everyone loves Mary, then Bob loves Mary.
C1. Bob loves Mary.

(b) P1. Samantha hates Bobby.
C1. Samantha hates somebody.

(c) P1. If Danny likes Mary, then Mary likes Danny.
P2. Marry doesn't like Danny.
C1. Danny doesn't like Mary.

(d) P1. Everyone loves Mary.
P2. Bob loves Mary.
C1. Bob loves Mary.
4. Determine whether the following arguments are deductively valid or deductively invalid. If it is deductively valid, explain why it is deductively valid using truth tables. If it is deductively invalid, then provide a counterexample (that is, an assignment of truth-values to the sentence letters which make the premises true but the conclusion false).

(a)  
\[
\begin{align*}
&h \rightarrow e \\
&\neg e \\
&\neg h
\end{align*}
\]

(b)  
\[
\begin{align*}
&h \rightarrow e \\
&\neg h \\
&\neg e
\end{align*}
\]

(c)  
\[
\begin{align*}
&h \rightarrow e \\
&e \\
&\neg h
\end{align*}
\]

(d)  
\[
\begin{align*}
&(h \land a) \rightarrow e \\
&\neg e \\
&\neg h
\end{align*}
\]

6 Properties

You may have had some difficulty with exercise 3 in the previous section. Though some of those arguments were deductively valid in English, their translations into formal logic didn’t allow us to conclude that they were deductively valid. So we need to soup-up our formal logic to allow it to capture some of the additional structure of the premises and conclusions of those arguments. That’s what we’ll do in this and the following sections.

First, we’ll have to think about properties. Many English sentences ascribe a property to an object or a person or an idea. For instance, the following sentences,
Pamela is tall
John hates broccoli
Calculus is cool
The bar is far away

Ascribe the property of being tall to Pamela, the property of hating broccoli to John, the property of being cool to Calculus, and the property of being far away to the bar.

Here's a way that we can formalize the ascription of a property to an object or person or idea. We'll use a capital letter like $P$ to stand for the property and a lowercase letter like $a$ to stand for the object or person or idea, and we'll write $Pa$ to mean that the object or person or idea $a$ has the property $P$.

If we use the lowercase letter '$p$' to stand for Pamela and the uppercase letter '$T$' to stand for the property of being tall, then we can write 'Pamela is tall' as

$$Tp$$

This just says that Pamela ($p$) has the property of being tall ($T$). Similarly, if we use the lowercase letter '$j$' to stand for John and we use the uppercase letter '$H$' to stand for the property of hating broccoli, then we can write 'John hate broccoli' as

$$Hj$$

This just says that John ($j$) has the property of hating broccoli ($H$). And if we use the lowercase letter '$c$' to stand for Calculus and the uppercase letter '$C$' to stand for the property of being cool, then we can write 'Calculus is cool' as

$$Cc$$

This says that Calculus ($c$) has the property of being cool ($C$).

These formal sentences work just like the sentence letters we've been using so far. So we can stick them together with the logical connectives, like so:

$$Hj \rightarrow (\neg Cc \vee Tp)$$

This formula says that, if John hates broccoli, then either Calculus isn't cool or Pamela is tall.

If we know whether the object possesses the property, then we can determine whether a sentence like $Pa$ is true or false. For any property $P$ and any object $a$, '$Pa$ is true if and only if the object which '$a$' refers to has the property which
‘P’ refers to. Then, we can use truth-tables to determine whether an arbitrary formula involving expressions like ‘Pa’ is true or false.

The language we get by including capital letters for properties and lowercase letters for things is called **predicate logic**. (That’s because another name for the property letters is **predicates**.)

**Worked Exercise**

Let ‘p’ refer to Pamela, ‘j’ refer to John, ‘T’ refer to the property of being tall, ‘H’ to the property of hating broccoli, and ‘C’ to the property of being cool, just as above. And suppose that John is tall and Pamela is not; Pamela hates broccoli and John does not; and that both Pamela and John are cool. Then, determine the truth value of the following formula.

\[ Hp \rightarrow \neg Hj \]

**Solution**

Since Pamela hates broccoli, ‘Hp’ is true. Since John does not hate broccoli, ‘Hj’ is false. Therefore, ‘\( \neg Hj \)’ is true, by the truth-table for ‘\( \neg \)’. So, the sentence to the left of ‘\( \rightarrow \)’ is true and the sentence to the right of ‘\( \rightarrow \)’ is true. By the truth-table for ‘\( \rightarrow \)’, the sentence ‘Hp \( \rightarrow \neg Hj \)’ is true.

**Exercises**

1. Let ‘p’ refer to Pamela, ‘j’ refer to John, ‘T’ refer to the property of being tall, ‘H’ to the property of hating broccoli, and ‘C’ to the property of being cool, just as above. And suppose that John is tall and Pamela is not; Pamela hates broccoli and John does not; and that both Pamela and John are cool. Then, determine the truth value of the following formulae (figure out the truth-values of the expressions inside parentheses first, and then figure out the truth-values of the entire formulae).
   \[
   \begin{align*}
   (a) \quad & Cp \rightarrow \neg Cj \\
   (b) \quad & (Cp \lor Hp) \rightarrow (Hj \land Cj) \\
   (c) \quad & (Tj \lor Tp) \lor (Cj \land Hj) \\
   (d) \quad & (Hj \rightarrow \neg Hj) \lor (Hj \lor \neg Hj)
   \end{align*}
   \]

2. Is the following argument deductively valid or deductively invalid? If it is deductively invalid, then provide a counterexample.

\[
\begin{align*}
Hp & \rightarrow (Cp \lor Cj) \\
\neg Cj & \\
\hline
\neg Hp
\end{align*}
\]
The Universal Quantifier

Sometimes, we don’t want to just talk about a particular object or person or idea possessing a certain property. Sometimes, we want to talk about every object or person or idea possessing a certain property. Suppose, then that we want to formalize the sentence “Everything is tall.” If there are only finitely many things—let’s say that there’s just John and Pamela—then we could formalize this by saying that both John and Pamela are tall, \( T_j \land T_p \). However, suppose that there are infinitely many people or things. Or suppose that we just don’t know how many or which people exist. Then, we won’t be able to utilize this strategy.

Here’s another way of formalizing “Everyone is tall.” We first use a variable \( x \). This variable is the kind of thing which can be replaced with a lowercase letter referring to a person or thing. We then get a formula like

\[ T_x \]

If we substitute in \( j \) for the variable \( x \), then we get the expression ‘\( T_j \)’—that is, John is tall. If we substitute in \( p \) for \( x \), then we get the expression ‘\( T_p \)’—Pamela is tall.

We can then introduce a formal object called a universal quantifier, written ‘\((\forall x)\)’. If we want to say that everything is tall, then we can write

\[ (\forall x)T_x \]

This formula is true if and only if every formula we get by removing the quantifier \((\forall x)\) and replacing \( x \) with a lowercase letter is true. So, if John and Pamela are the only things that there are, then \((\forall x)T_x\) is true if and only if both \( T_j \) and \( T_p \) are true.

Call the formula that we get by removing the universal quantifier \((\forall x)\) and then replacing every instance of the variable \( x \) with some lowercase letter denoting a person or thing a substitution instance of that formula. So, if we have the formula \((\forall x)(P_x \lor Q_x)\), and the lowercase letters \( a, b, c, \) and \( d \), all denote some person or thing, then the following are all substitution instances of this formula:

\[ Pa \lor Qa \]
\[ Pb \lor Qb \]
\[ Pc \lor Qc \]
\[ Pd \lor Qd \]
And \((\forall x)(P x \lor Q x)\) is true just in case all of its substitution instances are true.

Suppose now that we want to sayEverything that it tall is cool. Then, we can write that
\[(\forall x)(T x \rightarrow C x)\]
This is true if and only if every formula that we get by removing \((\forall x)\) and replacing every instance of \(x\) with a lowercase letter is true (if and only if, that is, every substitution instance of the formula is true). So, suppose that John is neither tall nor cool, and Pamela is both tall and cool, and John and Pamela are the only things that exist. Then,
\[T j \rightarrow C j\]
will be true, since ‘\(T j\)’ is false (by the truth-table for \(\rightarrow\)). And
\[T p \rightarrow C p\]
will be true, since both ‘\(T p\)’ and ‘\(C p\)’ are true (by the truth-table for \(\rightarrow\)). Since \(j\) and \(p\) are the only things that exist, and both the substitution instance of ‘\((\forall x)(T x \rightarrow C x)\)’ that swaps \(j\) for \(x\) and the substitution instance of ‘\((\forall x)(T x \rightarrow C x)\)’ that swaps \(p\) for \(x\) are true, this means that every substitution instance of ‘\((\forall x)(T x \rightarrow C x)\)’ is true. And that means that ‘\((\forall x)(T x \rightarrow C x)\)’ is true.

In general, a universally quantified formula is true if and only if every substitution instance of the formula is true.

**Worked Exercise**

Provide an interpretation of the properties \(P\) and \(Q\) and then find a counterexample to the following argument given that interpretation.

\[
(\forall x)(P x \rightarrow Q x)
\]
\[
(\forall x)\neg P x
\]
\[
(\forall x)\neg Q x
\]

**Solution**

Let \(P\) be the property of being taller than 5’11” and let \(Q\) be the property of being taller than 5’1’. Suppose that there is nobody who is taller than 5’11”. Suppose, however, that there is somebody, call her ‘Sarah’ who is 5’5”. Denote Sarah with the lowercase letter \(s\). The first premise is true, since everyone taller than 5’11” (all none of them) is also taller than 5’1’. That is to say, for every person \(p\), substituting \(p\) for \(x\) in the first premise will yield a formula \(P p \rightarrow Q p\). However, for every \(p\), \(P p\) will be false, since nobody is taller than 5’11”. So the entire formula \(P p \rightarrow Q p\) will be true (by the truth-table for \(\rightarrow\)). So
the formula \((\forall x)(Px \to Qx)\) is true. And the second premise is true, since everyone is not taller than 5'11". That is to say, for every person \(p\), substituting \(p\) for \(x\) in the second premise will yield a formula \(\neg Pp\). Since nobody is taller than 5’11”, this \(Pp\) will be false for every \(p\), so \(\neg Pp\) will be true for every \(p\) (by the truth table for \(\neg\)). So the formula \((\forall x)\neg Px\) is true.

However, the conclusion is false, since there is Sarah, \(s\), who is taller than 5’11”. So the substitution instance \(\neg Qs\) will be false. So \((\forall x)\neg Qx\) will be false.

Since there is a possibility in which the two premises are true but the conclusion is false, the argument is deductively invalid.

**Exercises**

1. Suppose that \(F\) is the property of being a fungus and \(E\) is the property of being located on the surface of the earth. Suppose that there are fungi on the surface of the earth, and that all fungi are located on the surface of the earth. However, all the other planets and stars still exist. Then, which of the following claims will be true, and which will be false?
   
   (a) \((\forall x)(Fx \land Ex)\)
   
   (b) \((\forall x)(Fx \lor Ex)\)
   
   (c) \((\forall x)(Fx \to Ex)\)
   
   (d) \((\forall x)Ex\)
   
   (e) \((\forall x)\neg Fx\)

2. Is this argument form deductively valid or deductively invalid? If it is deductively valid, explain why it is deductively valid. If it is deductively invalid, provide a counterexample and explain why it is a counterexample.

\[
\begin{align*}
Fa \land Ga \\
Fb \land Gb \\
Fc \land Fc \\
\hline
(\forall x)(Fx \to Gx)
\end{align*}
\]

3. Is the following argument form deductively valid or deductively invalid? If it is deductively valid, explain why it is deductively valid. If it is deductively invalid, provide a counterexample and explain why it is a counterexample.

\[
\begin{align*}
Ra \to Ba \\
\hline
\neg Ba \to \neg Ra
\end{align*}
\]

4. Is the following argument form deductively valid or deductively invalid? If it is deductively valid, explain why it is deductively valid. If it is deductively invalid,
provide a counterexample and explain why it is a counterexample.

\[
\begin{align*}
\forall x (Rx &\rightarrow Bx) \\
\forall x (\neg Bx &\rightarrow \neg Rx)
\end{align*}
\]