# Contents

1. **Basic Concepts of Logic**
   1.1 Discovering and Evaluating Arguments .............................................. 1
      1.1.1 Finding Argumentative Structure ........................................... 3
      1.1.2 Conditionals ................................................................. 5
      1.1.3 Necessary and Sufficient Conditions ......................................... 6
      1.1.4 Deductive Validity ............................................................ 7
      1.1.5 Inductive Strength ............................................................. 8
      1.1.6 Exercises .......................................................... .......... 9
   1.2 Proving Invalidity, take 1 ............................................................. 13
      1.2.1 Venn Diagrams ................................................................. 13
      1.2.2 Venn Diagrams, Counterexamples, and Validity .......................... 15
   1.3 Formal Deductive Validity .............................................................. 17
   1.4 Proving Invalidity, take 2 ............................................................. 20
      1.4.1 Exercises .......................................................... .......... 22

2. **Informal Fallacies**
   2.1 Fallacies .......................................................... .......... 25
   2.2 Fallacies of Irrelevance ............................................................... 27
      2.2.1 Argument Against the Person (Ad Hominem) .......................... 27
      2.2.2 Straw Man ................................................................. 28
      2.2.3 Appeal to Force (Ad Baculum) ........................................... 28
      2.2.4 Appeal to the People (Ad Populum) ..................................... 29
      2.2.5 Appeal to Ignorance (Ad Ignorantiam) ................................ 29
      2.2.6 Red Herring (Ignoratio Elenchi) ....................................... 30
   2.3 Fallacies Involving Ambiguity .......................................................... 32
      2.3.1 Equivocation ................................................................. 33
      2.3.2 Amphiboly ................................................................. 33
      2.3.3 Composition/Division ....................................................... 34
   2.4 Fallacies Involving Unwarranted Assumptions ................................... 35
      2.4.1 Begging the Question (Petitio Principii) ................................ 35
      2.4.2 False Dilemma ................................................................. 36
## 3 Propositional Logic

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Syntax for PL</td>
<td>42</td>
</tr>
<tr>
<td>3.1.1 Vocabulary</td>
<td>42</td>
</tr>
<tr>
<td>3.1.2 Grammar</td>
<td>42</td>
</tr>
<tr>
<td>3.1.3 Main Operators and Subformulae</td>
<td>43</td>
</tr>
<tr>
<td>3.2 Semantics for PL</td>
<td>45</td>
</tr>
<tr>
<td>3.2.1 The Meaning of the Statement Letters</td>
<td>45</td>
</tr>
<tr>
<td>3.2.2 The Meaning of ‘¬’</td>
<td>45</td>
</tr>
<tr>
<td>3.2.3 The Meaning of ‘•’</td>
<td>46</td>
</tr>
<tr>
<td>3.2.4 The Meaning of ‘∨’</td>
<td>46</td>
</tr>
<tr>
<td>3.2.5 The Meaning of ‘⊃’</td>
<td>47</td>
</tr>
<tr>
<td>3.2.6 The Meaning of ‘≡’</td>
<td>47</td>
</tr>
<tr>
<td>3.2.7 Determining the Truth-value of a wff of PL</td>
<td>48</td>
</tr>
<tr>
<td>3.3 Translation from PL to English</td>
<td>49</td>
</tr>
<tr>
<td>3.4 Translation from English to PL</td>
<td>52</td>
</tr>
<tr>
<td>3.4.1 Negation</td>
<td>52</td>
</tr>
<tr>
<td>3.4.2 Conjunction</td>
<td>53</td>
</tr>
<tr>
<td>3.4.3 Disjunction</td>
<td>54</td>
</tr>
<tr>
<td>3.4.4 The Material Conditional and Biconditional</td>
<td>54</td>
</tr>
<tr>
<td>3.5 Logical Properties of PL</td>
<td>56</td>
</tr>
<tr>
<td>3.5.1 How to Construct a Truth-table</td>
<td>56</td>
</tr>
<tr>
<td>3.5.2 PL-Validity</td>
<td>58</td>
</tr>
<tr>
<td>3.5.3 PL-Consistency and PL-Inconsistency</td>
<td>58</td>
</tr>
<tr>
<td>3.5.4 PL-Equivalence and PL-Contradiction</td>
<td>59</td>
</tr>
<tr>
<td>3.5.5 PL-tautologies, PL-self-contradictions, and PL-contingencies</td>
<td>59</td>
</tr>
<tr>
<td>3.5.6 The Relationship Between The Notions</td>
<td>59</td>
</tr>
</tbody>
</table>

## 4 Propositional Logic Derivations

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 The Basics</td>
<td>63</td>
</tr>
<tr>
<td>4.2 Rules of Implication</td>
<td>64</td>
</tr>
<tr>
<td>4.2.1 A Mistake to Avoid</td>
<td>69</td>
</tr>
<tr>
<td>4.3 Rules of Replacement</td>
<td>70</td>
</tr>
<tr>
<td>4.4 Four Final Rules of Inference</td>
<td>78</td>
</tr>
<tr>
<td>4.4.1 Subderivations</td>
<td>78</td>
</tr>
<tr>
<td>4.4.2 Conditional Proof</td>
<td>81</td>
</tr>
<tr>
<td>4.4.3 Indirect Proof</td>
<td>81</td>
</tr>
<tr>
<td>4.5 Proving PL-Tautologies</td>
<td>83</td>
</tr>
<tr>
<td>4.6 PL-Derivability and the Logical Notions of PL</td>
<td>85</td>
</tr>
<tr>
<td>4.7 Derivation Challenge</td>
<td>86</td>
</tr>
</tbody>
</table>

## 5 Quantification Logic

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Correctness and Completeness</td>
<td>95</td>
</tr>
<tr>
<td>5.2 Arguments that PL is Not Correct</td>
<td>95</td>
</tr>
<tr>
<td>5.2.1 A Counterexample to Modus Ponens ?</td>
<td>96</td>
</tr>
<tr>
<td>5.2.2 A Counterexample to Modus Tollens ?</td>
<td>97</td>
</tr>
<tr>
<td>5.2.3 A Counterexample to Disjunctive Syllogism and Modus Ponens ?</td>
<td>98</td>
</tr>
<tr>
<td>5.2.4 Why PL is Not Complete</td>
<td>99</td>
</tr>
<tr>
<td>5.3 The Language QL</td>
<td>102</td>
</tr>
</tbody>
</table>
5.3.1 The Syntax of QL .................................................. 103
5.3.2 Semantics for QL .................................................. 108
5.4 Notation ................................................................. 111
5.5 Translations from QL into English ............................... 112
  5.5.1 Translating Simple Quantified wffs of QL ................. 112
  5.5.2 Translating More Complicated Quantified wffs of QL .... 113
5.6 Translations from English into QL ............................... 116
5.7 QL-Validity ............................................................ 118
5.8 Proving QL-Invalidity ............................................... 118
5.9 QL-Derivations ......................................................... 119
  5.9.1 New Rules of Replacement .................................... 119
  5.9.2 New Rules of Implication .................................... 120

6  Inductive Strength ..................................................... 127
  6.1 Inductive Strength ................................................ 127
  6.2 The Theory of Probability ....................................... 128
  6.3 Rules for the Probability Calculus .............................. 128
    6.3.1 Restricted Conjunction Rule ............................. 128
    6.3.2 Restricted Disjunction Rule .............................. 129
    6.3.3 Negation Rule .............................................. 129
    6.3.4 Self-Contradiction Rule ................................. 130
    6.3.5 Equivalence Rule ....................................... 130
    6.3.6 General Conjunction Rule .............................. 131
    6.3.7 Total Probability Rules ................................. 132
    6.3.8 General Disjunction Rule .............................. 133
    6.3.9 Bayes’s Theorem ....................................... 134
  6.4 Probability Functions as Muddy Venn Diagrams ............... 135
  6.5 Deductive Validity and Inductive Strength ..................... 140
  6.6 Examples ........................................................... 141
    6.6.1 The Gambler’s Fallacy .................................. 141
Basic Concepts of Logic

1.1 Discovering and Evaluating Arguments

In many contexts, ‘argument’ can mean a fight, or a heated, vitriolic debate. In logic, we have a more technical understanding of what an argument is. In logic, we understand an argument to be something that provides reasons to believe some claim. The claim that the argument is arguing for is called the conclusion of the argument. The reasons that are adduced in the conclusion’s favor are known as the premises of the argument. An argument attempts to persuade its audience to accept its conclusion by providing premises that the audience is expected to accept, and showing that they support the conclusion.

Our lives are filled with arguments. Each day we make and listen to myriad arguments. These arguments are on matters both personal and political, both mundane and profound. Our ability to rationally decide what we think about these matters depends upon our ability to evaluate these arguments well. Consider the following example. On MSNBC in 2013, there was the following exchange between Tony Perkins, the president of the Family Research Council, and MSNBC host Luke Russert:

Perkins: You say that people ought to be able to marry whoever they love. If love becomes the definition of what the boundaries of marriage are, how do we define that going forward? What if someone wants to immigrate to this country that lives in a country that allows multiple spouses? They come here—right now they can’t immigrate with those spouses—but if the criteria or the parameters are simply love, how do we prohibit them from coming into the country? So, if it’s all about just love, as it’s being used, where do we set the lines?

Russert: So you equate homosexuality with polygamy?

Perkins: No, that’s not the argument.

Russert: But you just said that, sir.

Perkins: No, the argument being made by those wanting to redefine marriage is saying that it’s all based on love. You ought to be able to marry who you love. Isn’t that the argument that they’re using? If that’s the case, where do you draw the boundaries? That’s all that I’m asking.
§1.1. Discovering and Evaluating Arguments

In this passage, Perkins asks many rhetorical questions. It's not immediately obvious what the form of his argument is, what the conclusion might be, or even whether he is providing an argument at all. So, in order to evaluate what Perkins has to say, we must first decide whether he is making an argument, and, if so, what exactly that argument is. We might think, as Russert thought, that Perkins was making the following argument:

\[
\text{premises} \quad \begin{cases} \text{1. Gay marriage is morally tantamount to polygamy} \\ \text{2. Polygamy is wrong.} \end{cases} \\
\text{conclusion} \quad \text{3. So, gay marriage is wrong.}
\]

However, Perkins contends that this isn't the argument that he is making. What argument is he making? After the interview aired, some\(^1\) took Perkins to be making an argument like the following.

\[
\text{1. Legalizing gay marriage will lead to the legalization of polygamy.} \\
\text{2. We ought not legalize polygamy.} \\
\text{3. So, we ought not legalize gay marriage.}
\]

But perhaps not. Perhaps this passage is best understood in some other way. Perhaps Perkins isn't making a claim about what *would* happen if we legalized gay marriage. Perhaps he is making a claim about what *follows from* the claim that gay marriage ought to be legalized. Perhaps, that is, he is saying that, if we think gay marriage should be legal, then we are committed to thinking that polygamy should be legal as well. That is, perhaps we should understand his argument along the following lines:

\[
\text{1. If we ought to legalize gay marriage, then we ought to legalize polygamy.} \\
\text{2. We ought not legalize polygamy.} \\
\text{3. So, we ought not legalize gay marriage.}
\]

Then again, perhaps, rather than providing an argument *against* gay marriage, Perkins is simply providing an *objection* to somebody else's argument *for* gay marriage. Perhaps he is objecting to another's premise that all loving relationships deserve the rights of marriage. That is, perhaps his argument is best understood along these lines:

\[
\text{1. If all loving relationships deserve the rights of marriage, then loving polygamous relationships deserve the rights of marriage.} \\
\text{2. Loving polygamous relationships don't deserve the rights of marriage.} \\
\text{3. So, not all loving relationships deserve the rights of marriage.}
\]

As we'll see later on, good objections to one of these arguments are not necessarily going to be good objections to any of the others. So, what we ought to say about Perkins' statements here will depend upon how we ought understand them—whether we ought to understand

\(^1\) http://thinkprogress.org/lgbt/2013/03/27/1783301/top-conservative-says-marriage-equality-will-lead-to-influx-of-immigrant-polygamists/
them as implicitly making the first, second, third, or forth argument above (or whether we ought to understand them in some other way).

**Logic** is the study of arguments. The goal of logic is to give a theory of which arguments are good and which are bad, and to explain what it is that makes arguments good or bad. Since this is our goal, we ought not understand ‘argument’ in such a way that an argument has to be any good. So, in this class, we’ll understand an argument to be any collection of statements, one of which is presented as the conclusion, and the others of which are presented as the premises.

A statement is a sentence which is capable of being true or false. Questions, commands, suggestions, and exclamations are not statements, since they are not capable of being true or false. It doesn't make sense to say ‘It’s true that Damn it!’ or “It’s false that when did you arrive?’, so ‘Damn it!’ and ‘When did you arrive?’ are not statements. It does make sense to say, e.g., ‘It’s true that the store closes at eleven’, so ‘the store closes at eleven’ is a statement.

### A Test

Given some sentence, P, if “It is true that P” makes sense, then P is a statement. If “It is true that P” does not make sense, then P is not a statement.

#### 1.1.1 Finding Argumentative Structure

As we saw with Tony Perkins above, given a passage, it is not always obvious whether the passage constitutes an argument or not. Given that it is an argument, it is not always obvious which sentences are premises, which are conclusions, and which sentences are extraneous (asides which are not a part of the argument).

Some clues are provided by indicator words. For instance, if any of the following words precede a statement which occurs in an argument, then that statement is almost certainly the argument’s conclusion:

- therefore, ...
- hence, ...
- so, ...
- thus, ...
- this entails that...
- as a result, ...
- for this reason, ...
- we may conclude ...
- consequently, ...
- accordingly, ...
- this implies that...
- this entails that...

Similarly, if any of the following words precede a statement in an argument, then that statement is almost certainly one of the argument’s premises:

- since...
- for...
- as...
- because...
- given that...
- may be inferred from...
- in that...
- for the reason that...
- seeing that...
- seeing as...
- as is shown by...
- owing to...

However, often, indicator words are missing, and one must infer from the context and other clues both 1) whether the passage is an argument; and 2) which statements are premises and which are conclusions. For (1), it is important to consider the author’s goal in writing the passage. If their goal is to persuade the reader, then the passage is an argument. If their goal is anything else, then it is not providing an argument. In particular, if the passage is
§1.1. Discovering and Evaluating Arguments

providing an explanation, or providing information, then it is not an argument. Stories may very well contain indicator words like ‘because’ and ‘consequently’, but this does not mean that they are arguments. For instance, if I tell you

Sabeen is visiting New York because her company was hired to do a workshop there.

my goal is not to persuade you that Sabeen is visiting New York. Rather, I’m simply telling you something about why she is there. This is not an argument, even though it contains the indicator word ‘because’.

For (2), you should work with a principle of charity—figure out which potential argument the author might be making is the best argument.

PRINCIPLE OF CHARITY: When searching for argumentative structure within a passage, attempt to find the argument which is most persuasive.

For instance, the following passage lacks indicator words:

We must give up some privacy in the name of security. If the homeland is not secure, terrorist attacks orders of magnitude larger than 9/11 will find their way to our shores. No amount of privacy is worth enduring an attack like this.

So, there are a few arguments we could see the author making. They might be making this argument:

1. We must give up some privacy in the name of security.
2. If the homeland is not secure, terrorist attacks orders of magnitude larger than 9/11 will find their way to our shores.
3. So, no amount of privacy is worth enduring an attack like this.

Alternatively, they might be making this argument:

1. We must give up some privacy in the name of security.
2. No amount of privacy is worth enduring an attack orders of magnitude larger than 9/11.
3. So, if the homeland is not secure, terrorist attacks like this will find their way to our shores.

Finally, they might be making this argument:

1. If the homeland is not secure, terrorist attacks orders of magnitude larger than 9/11 will find their way to our shores.
2. No amount of privacy is worth enduring an attack like this.
3. So, we must give up some privacy in the name of security.
Which of these is correct? Well, the first two arguments are just really bad arguments. With respect to the first one, ask yourself: “suppose that there would be a large attack, and suppose, moreover, that we must give up privacy in the name of security. Does this tell me anything about the relative worth of privacy and avoiding such an attack?” Perhaps the first premise (we must give up some privacy in the name of security) does tell us something about the relative worth of privacy and attacks like this, but then the second premise would be entirely unneeded. So there wouldn’t have been any good reason for the arguer to include it. So this argument looks pretty poor.

The second argument is even worse. Ask yourself “suppose that we must give up privacy in the name of security, and suppose, moreover, that no amount of privacy is worth enduring an attack worse than 9/11. Does this tell me anything about whether terrorists will be able to find their way to our shores if we don’t secure the homeland?” Again, perhaps the first premise does tell us that there must be some reason that we must give up some privacy in the name of security, and perhaps this reason is that if the homeland is not secure, then terrorist attacks will find their way to our shores. However, again, that would make the second premise entirely unnecessary. Moreover, it looks like the only reason one would have for accepting the first premise is that one accepts the conclusion, so the argument is entirely unpersuasive.

The third argument is much stronger. In that argument, both premises are required, and they actually lend support to the conclusion. The principle of charity tells us to attribute this argument to the author.

### 1.1.2 Conditionals

Suppose that I have four cards, and I tell you that each of them has a letter printed on one side and a number printed on the other side. I lay them out on the table in front of you, like so:

```
9  J  U  2
```

And I tell you that all four of these cards obey the rule

> If there is a vowel printed on one side of the card, then there is an even number printed on the other.

Which of these cards would you have to flip over in order to figure out whether or not I am lying?

The correct answer is that you would have to flip over the cards whose exposed faces read ‘9’ and ‘U’. Most people get this question wrong. They are inclined to think that we have to flip over the cards whose exposed faces read ‘U’ and ‘2’. To see why this is wrong, consider the following case, which you will see is structurally identical to the first. Suppose that I have four cards, and I tell you that each of them has an age printed on one side and a
beverage printed on the other. I lay them out on the table in front of you, like so:

```
  Beer   22   16   Coke
```

I then tell you that each of the cards obeys the following rule.

If there is an age under 21 on one side of the card, then there is a non-alcoholic beverage printed on the other.

Which cards must you flip over to figure out whether I am lying? Which cards must you flip over in order to discover whether the cards comply with the rule. People find this case much easier to think about, and most find it intuitively obvious that you must flip over the cards with 'Beer' and '16' on their exposed face, whereas you needn't flip over the cards with '22' or 'Coke' on their exposed face. Note, however, that this precisely the same problem that I presented earlier, except with 'under 21' swapped out for 'vowel' and 'non-alcoholic' swapped out for 'even number'. You must check the card which has an age under 21, as well as the card with an alcoholic drink; similarly, you must check the card which has a vowel, as well as the card which has a non-even number. You didn't have to check the card with an age over 21, nor the card with a non-alcoholic drink; similarly, you needn't check the card with a consonant, nor the card with an even number.

We seem to have a very hard time reasoning about claims like these—claims of the form 'If P, then Q.' Claims of this form as known as 'conditionals.' That's because they don't flat out assert that Q, but rather, they only assert that Q, conditional on its being the case that P. Here's a good way to think about these kinds of claims: 'if P, then Q' says that the truth of P is sufficient for the truth of Q.

### 1.1.3 Necessary and Sufficient Conditions

One condition, X, is necessary for another condition, Y, if and only if everything which is Y is also X. That is, X is necessary for Y if and only if there's no way to be Y without being X.

**NECESSARY CONDITION:** Being X is necessary for being Y iff there's no way to be Y without also being X.

**NECESSARY CONDITION:** The truth of X is necessary for the truth of Y iff there's no way for Y to be true without X also being true.

For instance, being an American citizen is necessary for being the American president. There's no way to be president without also being an American citizen. For another: being a triangle is necessary for being an equilateral triangle. There's no way to be an equilateral
triangle without also being a triangle. The truth of ‘the car is coloured’ is necessary for the truth of ‘the car is red.’ There’s no way for the car to be red without the car being coloured.

One condition, $X$, is sufficient for another condition, $Y$, if and only if everything which is $X$ is also $Y$. That is, $X$ is sufficient for $Y$ if and only if there’s no way to be $X$ without also being $Y$.

**Sufficient condition:** Being $X$ is sufficient for being $Y$ iff there’s no way to be $X$ without also being $Y$.

**Sufficient condition:** The truth of $X$ is sufficient for the truth of $Y$ iff there’s no way for $X$ to be true without $Y$ also being true.

For instance, being French is sufficient for being European. There’s no way to be French without also being European. For another: being square is sufficient for being rectangular. There’s no way to be square without also being rectangular. And the truth of ‘Sabeen is older than 27’ is sufficient for the truth of ‘Sabeen is older than 20.’

We can visualize this with the Venn Diagram shown in figure 1.1. In that diagram, being inside the circle $S$ is sufficient for being inside the circle $N$—everything inside $S$ is also inside $N$. And being inside the circle $N$ is necessary for being inside the circle $S$—everything inside $S$ is also inside $N$. This diagram also makes it clear that $S$ is a sufficient condition for $N$ if and only if $N$ is a necessary condition for $S$.

### 1.1.4 Deductive Validity

Our goal in Logic is to separate out the good arguments from the bad. Here’s one very good property that an argument can have: it can be **deductively valid**. An argument is deductively valid if and only if the truth of its premises is sufficient for the truth of its conclusion.

An argument is **deductively valid** if and only if the truth of its premises is sufficient for the truth of its conclusion.
Equivalently, an argument is deductively valid if and only if there is no way for its premises to all be true while its conclusion is simultaneously false.

**An argument is deductively valid if and only if it is impossible for its premises to all be true while its conclusion is simultaneously false.**

For instance, each of the following arguments are deductively valid:

1. If Obama is president, then he is the commander in chief.
2. Obama is president.
3. So, Obama is the commander in chief.

1. Gerald is either in Barcelona or in New York.
2. Gerald is not in New York.
3. So, Gerald is in Barcelona.

1. Obama is younger than 30.
2. So, Obama is younger than 40.

(I will often just say that the argument is ‘valid’, rather than ‘deductively valid’.)

Just because an argument is deductively valid, it doesn’t follow that the conclusion of the argument is true. The third argument above is deductively valid, but its conclusion is false. Obama is not younger than 40. If, however, a deductively valid argument has all true premises, then its conclusion must be true as well. If a deductively valid argument has all true premises, then we say that the argument is **deductively sound**.

**An argument is deductively sound if and only if it is deductively valid and all of its premises are true.**

If an argument is deductively sound, then its conclusion will be true. Of all the good making features of arguments that we will discuss today, none is finer than deductive soundness. Of all the honorifics of arguments that we’ll discuss today, there is no finer compliment to an argument than to say that it’s deductively sound.

### 1.1.5 Inductive Strength

Not every good argument is deductively valid. For instance, the following argument is not deductively valid:

1. Every human born before 1880 has died.
2. So, I will die.
However, it is still an excellent argument. Its premise gives us spectacular reason to believe its conclusion. Arguments like these are inductively strong, even though they are not deductively valid. An argument is inductively strong if and only if its conclusion is sufficiently probable given its premises.

An argument is **inductively strong** to the extent that its conclusion is probable, given the truth of its premises.

This means that inductive strength, unlike deductive validity, is the kind of thing that comes in degrees. Some arguments can be inductively stronger than others. We could, if we like, set some arbitrary threshold and say that an argument is inductively strong—full stop—if and only if its premises probabilify its conclusion above that threshold. For instance, we could say that

An argument is **inductively strong** if and only if

\[ \Pr(\text{conclusion} \mid \text{premises}) > 0.5 \]

If an argument is inductively strong with all true premises, then it is **inductively cogent**.

An argument is **inductively cogent** if and only if it is inductively strong and all of its premises are true.

---

**Important Concepts:**
- STATEMENT
- ARGUMENT
- PREMISE
- CONCLUSION
- NECESSARY CONDITION
- SUFFICIENT CONDITION
- CONDITIONAL
- DEDUCTIVE VALIDITY
- DEDUCTIVE SOUNDNESS
- INDUCTIVE STRENGTH
- INDUCTIVE COGENCY

### 1.1.6 Exercises

A. Which of the following are statements? Put a checkmark in the blank next to each statement, and leave the blanks next to non-statements blank.

1. George Washington is a famous movie star.

2. When does the movie start?
3. The University of Michigan is located in China.
4. That’s disgusting.
5. Let us rejoice and be glad.
6. Murder is wrong.
7. Remember to take out the trash.
8. If you don’t remember to take out the trash, then you won’t be allowed to go to the dance on Sunday.
9. If you don’t remember to take out the trash, then screw you!

B. For each of the passages below, figure out whether the passage is presenting an argument or not. If the passage is presenting an argument, then underline its conclusion and circle its premises (not every sentence other than the conclusion is necessarily a premise of the argument). If it is not presenting an argument, leave the passage alone.

1. Tax cuts without any decrease in government expenditure would be disastrous for the U.S. economy. For, if we cut taxes without lowering government expenditure, then the government will not be able to pay its debts. And if the government isn’t able to pay its debts, then the value of the dollar will deflate. If the value of the dollar deflates, then America will not be able to afford its imports, and both GDP and employment will drop. I, for one, will be voting Republican this election!

2. Seeing that he was out of milk, John went to the grocery store before heading off to work. For this reason, he ran into his friend Bob, who was at the grocery store because he had run out of M&M’s. Since he spent so much time catching up with Bob, John ended up being late for work. Consequently, John’s boss made him come in over the weekend.

3. If a moral theory is studied empirically, then examples of conduct will be considered. But if examples of conduct are considered, principles for selecting examples will be used. But if principles for selecting examples are are used, then moral theory is not being studied empirically. Therefore, moral theory is not being studied empirically.²

4. Since life begins at conception, abortion is akin to murder.³

5. Almost all handguns are banned from civilian possession, ownership, purchase, or sale in the United Kingdom as a result of the Second Firearms Act of 1997. This was in response to the Dunblane Massacre, in which 43-year-old Thomas Hamilton walked into an elementary school and shot dead 16 children, aged six or younger, and one teacher before killing himself. He used four handguns.⁴

6. Gun control laws wouldn’t work since criminals won’t follow them.⁵

² from Immanuel Kant (cited by Greg Restall)
³ from http://womensissues.about.com/od/reproductiverights/a/AbortionArgument.htm
⁴ from http://listverse.com/2013/12/12/10-arguments-against-gun-control/
⁵ from http://nyulocal.com/national/2013/09/19/shooting-holes-in-four-common-objections-to-gun-control/
7. Wages have declined in America relative to inflation since 2000. American working people are hurting; many of the jobs created today are part-time so it makes no sense at all to see a dramatic increase in the legal flow of immigration while we’re not even reducing the illegal flow.6

8. Every sentence in my book is well written. Accordingly, my book is well written.7

9. The forest fire started because campers did not properly extinguish their campfire.

10. If good intentions make good sermons, then Reverend McGuire is a good preacher. Unfortunately, they don’t; so he’s not.8

C. Decide whether the following claims are true or false.

1. Being American is sufficient for being a Michigander.

2. The truth of ‘Irrfan is a single twenty-something’ is necessary for the truth of ‘Irrfan is a twenty-something’.

3. Being a premise in a valid argument is sufficient for a statement being valid.

4. Being greater than 5 is sufficient for being greater than 7.

5. If being X is sufficient for being Y, then everything that is X is Y.

6. If being X is necessary for being Y, then everything that is Y is X.

7. Having true premises and a true conclusion is sufficient for an argument to be deductively valid.

8. Having true premises and a true conclusion is necessary for an argument to be deductively valid.

9. Having true premises and a true conclusion is necessary for an argument to be deductively sound.

10. If an argument is deductively valid, then its conclusion is true.

11. If an argument is deductively invalid, then its conclusion is false.

12. If an argument has a false conclusion, then it has a false premise.

13. If an argument has a false conclusion, then it is not deductively valid.

14. If an argument has a true conclusion, then it is deductively valid.

15. If an argument is deductively sound, then it has a true conclusion.

7 from Howard-Snyder et al. (2013, p. 174)
8 from Howard-Snyder et al. (2013, p. 147)
§1.1. Discovering and Evaluating Arguments

D. Suppose that I show you the following six cards.

\[
\begin{array}{c}
12 \\
A \\
U \\
13 \\
Z \\
7
\end{array}
\]

I don't tell you anything about what kinds of things the cards have written on them. They may have a number on both sides, or a letter on both sides, or nothing at all on either side, for all I've told you. Suppose that I then tell you that all of the cards obey the following rule:

If the card has a prime number on one side, then it has a vowel on the other.

Which of the cards must you flip over in order to figure out whether I am lying? (For those who don't math: 13 and 7 are both prime; 12 is not)

E. For each argument, decide whether the argument is deductively valid or deductively invalid. If the argument is deductively valid, write 'valid' on the line. If the argument is deductively invalid, write 'invalid' on the line.

1. P1. Everyone who owns a baseball glove loves baseball.
   P2. John owns a baseball glove.
   C. So, John loves baseball.

2. P1. Everyone who owns a baseball glove loves baseball.
   P2. John doesn't own a baseball glove.
   C. So, John doesn't love baseball.

3. P1. Being American is sufficient for being a Michigander.
   P2. John is a Michigander.
   C. So, John is an American.

   P2. Kesha once spelled her name with a dollar sign.
   P3. Dmitri does not love anyone who ever spelled their name with a dollar sign.
   C. So, Obama was born in Kenya.

5. P1. Being American is sufficient for being a Michigander.
   C. So, being a Michigander is necessary for being American.

6. P1. Argument #5 above is deductively invalid.
   C. The conclusion of argument #5 is false.

7. P1. Argument #6 above is not deductively sound.
   P2. So, argument #6 above is not deductively valid.
8. P1. The truth of argument #1’s conclusion isn’t necessary for the truth of all of its premises.
   C. Argument #1 is not deductively valid.

9. P1. Al Gore is President of the U.S.
   P2. Obama is Vice President of the U.S.
   C. So, either Obama was born in Kenya or Obama was not born in Kenya.

### 1.2 Proving Invalidity, Take I

Suppose that you want to show that $X$ is not sufficient for $Y$. How would you show that? For instance, suppose that you want to show that being human is not sufficient for being a woman. How would you show that? One thing you could do is point to a human man. This is an example of something that is human but not a woman. So, if there is something like that, then it can’t be that being human is sufficient for being a woman.

We can do the very same thing with arguments. For instance, suppose that you wanted to show that the truth of the argument’s premises is not sufficient for the argument’s conclusion. One thing you could do is point to a possibility in which the premises are true, yet the conclusion is false. Call a possibility like that a counterexample to the validity of an argument.

---

A COUNTEREXAMPLE to the validity of an argument from premises $p_1, p_2, \ldots, p_N$ to the conclusion $c$ is a specification of a possibility in which $p_1, p_2, \ldots, p_N$ are all true, yet $c$ is false.

---

### 1.2.1 Venn Diagrams

Let’s talk a bit about Venn diagrams. A Venn diagram has 2 components: a box and some number of circles inside of the box. One example is shown in figure 1.2. In order to interpret this diagram, we must say two things: first, what the domain, $D$, of the diagram...
§1.2. Proving Invalidity, Take 1

is. That is, we must say what the box contains. Secondly, we must say what each of the circles, $F$ and $G$, represent.

An interpretation of a Venn diagram says
1) what the domain $\mathcal{D}$ is; and
2) what each circle represents

In general, a circle will represent a set of things inside the box. An object is represented as belonging to the set if and only if it is inside of the circle. For instance, I could interpret the Venn diagram in figure 1.2 by saying that the domain $\mathcal{D}$ is all animals. That is, every animal is located somewhere inside of the box. I could then say that $F$ is the set of all frogs and that $G$ is the set of all green animals. Alternatively, I could interpret this diagram by saying that the domain is the set of all people, $F$ is the set of all fathers, and $G$ is the set of all grandfathers. Thus, either of the following would be an interpretation of the Venn diagram in figure 1.2:

$\mathcal{D}$ = the set of all animals  $\mathcal{D}$ = the set of all people
$F$ = the set of all frogs  $F$ = the set of all fathers
$G$ = the set of all green animals  $G$ = the set of all grandfathers

Let’s start with the first interpretation. There are some animals who are neither frogs nor green (zebras). They lie outside of both the circle $F$ and the circle $G$. There are some animals who are frogs but not green (brown frogs). They lie within the circle $F$ yet outside of the circle $G$. There are some animals who are both frogs and green (green frogs). They lie inside both the circles $F$ and $G$. Finally, there are green animals which are not frogs (crocodiles). They lie inside the circle $G$, but not inside the circle $F$.

Think now about the second interpretation. There are people who are neither fathers nor grandfathers. There are also people who are fathers but not grandfathers. And there are people who are both fathers and grandfathers. However, there are no people who are grandfathers but not fathers. So there is nobody who is outside of the circle $F$ but still inside of the circle $G$. Suppose that we want to express the idea that this area is unoccupied. We may do so by crossing out that area of the graph, as shown in figure 1.3. The lines in figure 1.3 make the claim that all $G$s are $F$s. Equivalently: they make the claim that there are no $G$s which are not $F$. Equivalently: they make the claim that being $G$ is not necessary for being $F$. Equivalently: they make the claim that being $F$ is not sufficient for being $G$. (Make sure that you understand why all of these claims are equivalent.)

Suppose that we wish to say that some area of the Venn diagram is occupied. Perhaps, that is, we wish to make the claim that there are some fathers who are not grandfathers. That is, we wish to claim that there are some $F$s that are not $G$s. We may indicate this by putting a single ‘$\times$’ in the diagram which is inside the circle $F$ yet outside of the circle $G$, as in figure 1.4. In figure 1.4, the ‘$\times$’ makes the claim that some $F$s are not $G$s. Equivalently: it makes the claim that not all $G$s are $F$s. Equivalently: it makes the claim that being $F$ is not sufficient for being $G$. Equivalently: it makes the claim that being $G$ is not necessary for being $F$. (Make sure that you understand why all of these claims are equivalent.)
Suppose that we’ve got an argument from the premises $p_1$ and $p_2$ to the conclusion $c$. This argument is deductively valid if and only if it is impossible for $p_1$ and $p_2$ to both be true and yet for $c$ to be simultaneously false. Let’s think about this claim using Venn diagrams. Consider the Venn diagram in figure 1.5. Let us give this diagram the following interpretation. The domain $\mathcal{D}$ is the set of all possibilities. If any state of affairs is possible, then that state of affairs is included in $\mathcal{D}$. $P_1$ is the set of possibilities in which $p_1$ is true. $P_2$ is the set of possibilities in which $p_2$ is true. And $C$ is the set of possibilities in which $c$ is true.

\[ \mathcal{D} = \text{the set of all possibilities} \]
\[ P_1 = \text{the set of possibilities in which } p_1 \text{ is true} \]
\[ P_2 = \text{the set of possibilities in which } p_2 \text{ is true} \]
\[ C = \text{the set of possibilities in which } c \text{ is true} \]

The diagram in figure 1.5 makes the claim that there are no possibilities in which both $p_1$ and $p_2$ are true, yet $c$ is false. But to say this is just to make the claim that the truth of $p_1$ and $p_2$ is sufficient for the truth of $c$. But to say this is just to make the claim that the argument from $p_1$ and $p_2$ to $c$ is deductively valid. (Make sure that you understand why these three claims are equivalent.)
On the other hand, suppose that there is some possibility in which both $p_1$ and $p_2$ are true, yet $c$ is false. This claim is illustrated with the Venn diagram in figure 1.6. (There, we are using the same interpretation that we used for the Venn diagram in figure 1.5.) If the claim made in figure 1.6 is correct—if there is some possibility in which $p_1$ and $p_2$ are both true, yet $c$ is false—then the claim made in figure 1.5—that there is no possibility in which $p_1$ and $p_2$ are both true yet $c$ is false—cannot be true. So, if the claim made in figure 1.6 is correct, then the argument from $p_1$ to $p_2$ to $c$ cannot be deductively valid. But the claim made in figure 1.6 is just the claim that there is some counterexample to the validity of the argument from $p_1$ and $p_2$ to $c$. So, if there is a counterexample to the validity of an argument, then the argument cannot be deductively valid.

This affords us a new definition of deductive validity which is equivalent to the earlier two.

An argument is **deductively valid** if and only if it has no counterexample.

(Make sure that you understand why this new definition is equivalent to the earlier two.) So, one way to establish that an argument is deductively invalid is to provide a counterexample.
Consider the following arguments:

1. The earth moves around the sun.
2. So, the sun does not move.

1. Raising the minimum wage reduces employment.
2. Obama wants to raise the minimum wage.
3. So, Obama wants to reduce employment.

1. We have not discovered life on other planets.
2. So, there is no life on other planets.

Each of these arguments are deductively invalid. And we may demonstrate that they are deductively invalid by providing the following counterexamples. For the first argument, consider the following state of affairs: the earth moves around the sun, and the sun itself moves. In this state of affairs, the premise of the first argument is true, yet the conclusion is false. So, since this state of affairs is possible (it is actual), the argument is invalid. For the second argument, consider the following state of affairs: raising the minimum wage does reduce employment; however, Obama does not know this. Obama wants to raise the minimum wage, but does not want to reduce employment. Since this state of affairs is possible (though perhaps not actual), the argument is invalid. For the third argument, consider the following state of affairs: life on other planets is hidden somewhere we would be unlikely to have yet found it. Though we have not yet found it, it is still out there. In this state of affairs, the premise of the argument is true, yet its conclusion is false. Since this state of affairs is possible (though perhaps not actual), the argument is invalid.

1.3 Formal Deductive Validity

Up until this point, both Hurley and I have been defining deductive validity as necessary truth-preservation—that is, a valid argument is one such that, necessarily, if its premises are all true, then its conclusion will be true as well. In §1.5 of Hurley, however, a new idea shows up: that “the validity of a deductive argument is determined by the argument form.”9 Understanding this definition requires understanding what an argument form is, as well as what it is for a given argument to have a certain form.

Let’s start with the idea of a variable. A variable is just a kind of place-holder for which you can substitute a certain kind of thing—perhaps a number, perhaps a statement, perhaps a name, perhaps something else entirely. Those entities that can take the place of the variable are the variable’s possible values. For instance, we could use ‘x’ as a variable whose possible values are names. We could similarly use ‘p’ as a variable whose possible values are whole statements. Specifying a variable means specifying what its possible values are—those are known as the values over which the variable ranges.

Next, consider the idea of a statement form. A statement form is a string of words

---

9 Hurley, §1.5.
§1.3. Formal Deductive Validity

containing variables such that, if the variables are substituted for the appropriate values, then you get a statement. For instance, if ‘p’ and ‘q’ are variables ranging over statements, then

if p, then q

is a statement form. If we plug in statements for p and q, then we get a substitution instance of this statement form. For instance, the following is a substitution instance of ‘if p, then q’:

If Zoë is hungry, then Barcelona is in France.

Here, we have set p = ‘Zoë is hungry’ and q = ‘Barcelona is in France’. Since both of these are statements, they are both appropriate values for p and q. On the other hand, this is not a substitution instance of ‘if p, then q’:

If Bob, then Mary.

Since ‘Bob’ and ‘Mary’ are not statements, the variables p and q do not range over them, and they may not be substituted in for p and q. Similarly, if ‘x’ and ‘y’ are variables ranging over names, then

x loves y

is a statement form. A substitution instance of this statement form is

Bob loves Mary.

Since, if we set x = ‘Bob’ and y = ‘Mary’, in the statement form ‘x loves y’, we get the statement ‘Bob loves Mary’.

Finally, a argument form is a collection of statements and/or statement forms, one of which is presented as the conclusion, the others of which are presented as the premises. The following are all argument forms (where ‘p’ and ‘q’ are variables ranging over statements, and ‘x’ and ‘y’ are variables ranging over names).

1. p and q
2. So, q

1. If p, then q
2. p
3. So, q

1. x loves y
2. So, y loves x

If we look at the first and second argument form, we might notice that it looks as though we can figure out that, no matter which statements we substitute in for p and q, the resulting argument will be valid. Additionally, we might notice, when we look at the third argument, that it looks as though we can figure out that, no matter which names we substitute in for x and y, the resulting argument will be invalid. This observation suggests the following incredibly bold and daring and provocative thesis about deductive validity: what it is for an argument to be deductively valid is for it to be a substitution instance of a form which necessarily preserves truth.
A bit more carefully: let’s start by defining the notion of a *deductively valid argument form*. An argument form is deductively valid if and only if every substitution instance of the argument form has the following property: if the premises are all true, then the conclusion is true as well.

An argument form is **deductively valid** if and only if every substitution instance of the argument form with all true premises has a true conclusion as well.

An argument form is **deductively invalid** if and only if there is some substitution instance with true premises and a false conclusion.

Then, we may define a corresponding notion of *formal deductive validity*. An argument is formally deductively valid if and only if it is a substitution instance of a deductively valid argument form.

An argument is **formally deductively valid** if and only if it is a substitution instance of a deductively valid argument form.

Here’s the bold and daring and provocative thesis: deductive validity *just is* formal deductive validity.

**BOLD AND DARING AND PROVOCATIVE THESIS:** An argument is deductively valid if and only if it is formally deductively valid.

To see some *prima facie* motivation for this thesis, consider the examples of deductively valid arguments that we encountered last time.

1. If Obama is president, then he is the commander in chief.
2. Obama is president.
3. So, Obama is the commander in chief.

1. Gerald is either in Barcelona or in New York.
2. Gerald is not in New York.
3. So, Gerald is in Barcelona.

1. Obama is younger than 30.
2. So, Obama is younger than 40.

Each of these arguments has a deductively valid argument form, namely,
§1.4. Proving Invalidity, Take 2

Consider the following arguments:

1. If Russia invades the Ukraine, there will be war.
2. Russia won’t invade the Ukraine.
3. So, there won’t be war.

1. If it’s raining, then it’s raining and Romney is president.
2. It’s not raining.
3. So, it’s not the case that it is raining and Romney is president.

Both of these arguments are of the same general form, namely

1. If $p$, then $q$
2. It is not the case that $p$
3. So, it is not the case that $q$

(In the first argument, $p$ = ‘Russia invades the Ukraine’ and $q$ = ‘there will be war’. In the second argument, $p$ = ‘it’s raining’ and $q$ = ‘it’s raining and Romney is president’.)

However, this general form is invalid. We can show that the general form is invalid by pointing out that it has a substitution instance with true premises and a false conclusion, namely,

1. If Romney is president, then a man is president.
2. It is not the case that Romney is president.
3. So, it is not the case that a man is president.

Despite this strong *prima facie* motivation, the bold and daring and provocative thesis is still controversial; many philosophers dispute it. Nevertheless, I will assume it in what follows. As it turns out, very little of what we will do in this class will depend upon the thesis. (The following section provides a noteworthy exception to this general rule.)
(where \( p = \text{‘Romney is president’} \) and \( q = \text{‘a man is president’}. \)) In this substitution instance, the premises are true, yet the conclusion is false. Therefore, the argument form ‘if \( p \), then \( q \); it is not the case that \( p \); therefore, it is not the case that \( q \)’ is invalid.

Earlier, I said that

An argument is **FORMALLY DEDUCTIVELY VALID** if and only if it is a substitution instance of a deductively valid argument form.

What I **didn’t** say, because it was **false**, was

---

**THIS IS FALSE!!!**

An argument is **FORMALLY DEDUCTIVELY INVALID** if and only if it is a substitution instance of a deductively invalid argument form.

**THIS IS FALSE!!!**

To see why this is false, note that the argument considered above, namely,

1. If it’s raining, then it’s raining and Romney is president.
2. It’s not raining.
3. So, it’s not the case that it is raining and Romney is president.

**is** a substitution instance of the deductively invalid form

\[
\begin{align*}
1. & \text{ If } p, \text{ then } q \\
2. & \text{ It is not the case that } p \\
3. & \text{ So, it is not the case that } q
\end{align*}
\]

**However, it is also** a substitution instance of the deductively **valid** form

\[
\begin{align*}
1. & \text{ If } p, \text{ then } (p \text{ and } q) \\
2. & \text{ It is not the case that } p \\
3. & \text{ So, it is not the case that } (p \text{ and } q).
\end{align*}
\]

This argument form is deductively valid because the conclusion follows straightaway from the second premise. If it’s not the case that \( p \), then it can’t be the case that \( p \) and \( q \). The first premise is unnecessary, but the argument form is still formally deductively valid.

For another example, consider the deductively valid argument

\[
\begin{align*}
1. & \text{ If Romney is president, then a man is president.} \\
2. & \text{ Romney is president.} \\
3. & \text{ So, a man is president.}
\end{align*}
\]

This is a deductively valid argument, since it is of the valid form (known as **modus ponens**).
1. If \( p \), then \( q \).
2. \( p \).
3. So, \( q \).

(with \( p = 'Romney is president' \) and \( q = 'a man is president'. \) However, it is also of the invalid form

1. \( p \)
2. \( q \)
3. So, \( r \).

(with \( p = 'If Romney is president, then a man is president', q = 'Romney is president', and \( r = 'a man is president'. \) So, formally deductively valid arguments can have invalid forms. In fact, every argument whatsoever will have an invalid form. What it takes to show that an argument is deductively invalid is that you’ve uncovered the right form. How much of the form of the argument must we represent in order to be sure that we’ve uncovered the right form? That difficult question will be the one we face when we learn about propositional and predicate logic.

\[ \text{Most Important To Understand:} \]
- \text{COUNTEREXAMPLES and their relationship to DEDUCTIVE VALIDITY}
- \text{FORMAL DEDUCTIVE VALIDITY}
- \text{FORMAL COUNTEREXAMPLES and their relationship to FORMAL DEDUCTIVE VALIDITY}

\[ \text{Very Important to Understand:} \]
- just because an argument is of an invalid form, this does not mean that the argument is formally invalid.

\[ \text{Important to Understand:} \]
- \text{VENN DIAGRAMS}

1.4.1 Exercises

A. Which of the following interpretations of the Venn diagram in figure 1.7 makes the claims of that diagram correct? (That is, which interpretation makes it the case that, if the Venn diagram says that an area is unoccupied, then it is unoccupied, and, if the Venn diagram says that an area is occupied, then it is occupied?)

\[ \mathcal{P} = \text{the set of all positive integers (1, 2, 3, 4, ...)} \]
\[ F = \text{the set of all even positive integers} \]
\[ G = \text{the set of all odd positive integers} \]
\[ H = \text{the set of all prime integers (2, 3, 5, 7, 11,...)} \]
Figure 1.7

\[ D = \text{the set of all celebrities} \]
\[ F = \text{the set of all married celebrities} \]
\[ G = \text{the set of all single (i.e., not married) celebrities} \]
\[ H = \text{the set of all Academy Award-winning celebrities} \]

\[ D = \text{the set of all animals} \]
\[ F = \text{the set of all mammals} \]
\[ G = \text{the set of all reptiles} \]
\[ H = \text{the set of all aquatic animals} \]

\[ D = \text{the set of all people} \]
\[ F = \text{the set of all fathers} \]
\[ G = \text{the set of all grandfathers} \]
\[ H = \text{the set of all daughters} \]

B. For each of the following arguments, if they are deductively invalid, circle every answer choice that provides a counterexample to their validity.

1. Sammy loves everyone who owns a horse.
   2. Bobby doesn't own a horse.
   2. So, Sammy doesn't love Bobby.

   a. The argument is deductively valid, so there is no counterexample.
   b. Cameron is the only person who owns a horse, and Sammy loves Cameron.
       Bobby doesn't own a horse, and Sammy doesn't love him.
   c. Cameron is the only person who owns a horse, but Sammy doesn't love Cameron.
       Bobby doesn't own a horse, and Sammy doesn't love him.
   d. Cameron is the only person who owns a horse, but Sammy doesn't love Cameron.
       Bobby doesn't own a horse, and Sammy loves him.
   e. Cameron is the only person who owns a horse, and Sammy loves Cameron.
       Bobby doesn't own a horse, but Sammy still loves him.
§1.4. Proving Invalidity, Take 2

1. Margaret Thatcher doesn't live in New York City
2. New York City is located in New York State.

a. The argument is deductively valid, so there is no counterexample.
b. Margaret Thatcher is dead, so she doesn't live anywhere any longer.
c. Margaret Thatcher lives in London, which is not in New York State.
d. Margaret Thatcher lives in Buffalo, which is in New York State, but not in New York City.
e. Margaret Thatcher lives in Manhattan, which is in New York City and New York State.

1. All humans are mammals.
2. Some mammals have hair.
3. All humans have hair.

a. The argument is deductively valid. Therefore, there is no counterexample.
b. All humans are mammals and all whales are mammals. Whales have hair, but humans do not.
c. Humans are not mammals, and even though all mammals have hair, humans do not have hair.
d. Who’s to say that all humans are mammals? Some humans believe that they are fish.
e. Humans are mammals, but they are hairless; so too is every other mammal—no mammals have any hair.

C. Which of the arguments provided below are of the following form.

1. If $p$, then both $q$ and $r$
2. It is not the case that both $p$ and $q$
3. So, it is not the case that $r$.

1. If today is Sunday, then both tomorrow is Monday and yesterday is Saturday.
2. It is not the case that both today is Sunday and yesterday is Saturday.
3. So, it is not the case that yesterday is Saturday.

1. If Rand Paul is a Senator, then both Paul Ryan is a Senator and Marsha Brady is a Senator.
2. It is not the case that both Paul Ryan is a Senator and Marsha Brady is a Senator.
3. So, it is not the case that Rand Paul is a Senator.

1. If I will sleep in, then both I will miss my appointment and I will not have time to study.
2. It is not the case that both I will sleep in and I will miss my appointment.
3. So, it is not the case that I will not have time to study.

1. If I live in Manhattan, then both I live in New York City and I live in New York State.
2. It is not the case that both I live in Manhattan and I live in New York City.
3. So, it is not the case that I live in New York State.
Informal Fallacies

2.1 Fallacies

A fallacy is an error in reasoning. Simply because an argument contains false premises, this is not enough to make the argument fallacious. It must make a mistake in inferring the conclusion from the premises. When an argument commits a fallacy, something has gone wrong with the inference from the premises to the conclusion.

A formal fallacy is a fallacy that we may diagnose as bad simply by looking at the argument’s form. For instance, the following is a formal fallacy:

1. If Russia invades Ukraine, then Russia wants war.
2. Russia wants war.
3. So, Russia will invade Ukraine.

We can diagnose this argument as fallacious by noting that it is of a deductively invalid form, namely,

\[
\begin{align*}
1. & \quad \text{If } p, \text{ then } q. \\
2. & \quad q. \\
3. & \quad \text{So, } p.
\end{align*}
\]

(\text{where } p = \text{ ‘Russia invades Ukraine’, and } q = \text{ ‘Russia wants war’.)} We may show that this form is invalid by pointing to a substitution instance on which the premises are uncontroversially true, yet the conclusion is uncontroversially false. The following example will do:

1. If Sylvester Stallone was governor of California, then a former action star was governor of California.
2. A former action star was governor of California.
3. So, Sylvester Stallone was governor of California.
This argument has all true premises, and a false conclusion. And it is a substitution instance of the argument form ‘if \( p \), then \( q \); \( q \); so, \( p \)’ (with \( p = ‘\text{Sylvester Stallone was governor of California}’ \) and \( q = ‘\text{a former action star was governor of California}’ \)). So the argument form is invalid.

However, there are other common fallacies which we may not detect merely by inspecting the arguments form; we must additionally look to the content of the argument.

An informal fallacy is a fallacy which we cannot diagnose by simply inspecting the argument’s form; in order to diagnose the fallacy, we must look additionally to the argument’s content.

For instance, here is an informal fallacy:

1. Zoë has more energy than Daniel.
2. Energy is proportional to mass.
3. Zoë has more mass than Daniel.

This argument is fallacious; however, if we try to extract its logical form, we might only get the following argument form, which appears to be deductively valid.

1. \( x \) has more \( F \) than \( z \).
2. \( F \) is proportional to \( G \).
3. \( x \) has more \( G \) than \( z \).

This is an example of the informal fallacy of equivocation. The word ‘energy’ has two different meanings in the original argument. In premise 1, it means something like ‘the personality trait of being excitable’ (‘personality energy’, for short); whereas, in premise 2, it means ‘the theoretical physical quantity of energy’ (‘physical energy’, for short). The argument will be valid so long as we mean the same thing by ‘energy’ throughout. However, while both of the following arguments are valid, neither are at all persuasive.

1. Zoë has more physical energy than Daniel.
2. Physical energy is proportional to mass.
3. Zoë has more mass than Daniel.

1. Zoë has more personality energy than Daniel.
2. Personality energy is proportional to mass.
3. Zoë has more mass than Daniel.

Both of these arguments are valid; however, there is no reason whatsoever to accept their premises. In the first argument, premise 1 is obviously false. Just because Zoë is more excitable than Daniel, that doesn’t mean that she has more physical energy than he does. In the second argument, premise 2 is obviously false. Just because \( E = mc^2 \), this doesn’t mean that the personality trait of being excitable is proportional to mass.
There are three broad classes of informal fallacies that we will study here. They are fallacies of irrelevance, fallacies involving ambiguity, and fallacies involving unwarranted assumptions. For each informal fallacy we study, we should be on our guard and not be too hasty to call some piece of reasoning fallacious simply because it fits the general mold. For most of these fallacies, though there are a great many arguments that fit the basic mold and which are incredibly poor arguments, there are also some arguments that fit the basic mold but which are perfectly good arguments. For each fallacy, we’ll have to think about why an argument of that general character is bad, when it is bad, and why it might be good, when it is good.

### 2.2 Fallacies of Irrelevance

#### 2.2.1 Argument Against the Person (Ad Hominem)

This is a fallacy in which one fails to properly engage with another person’s reasoning. An *ad hominem* is a way of responding to an argument that attacks the person rather than the argument. It comes in three flavors: firstly, an abusive ad hominem attempts to discredit an argument by discrediting the person making that argument.

**Example:** After Sandra Fluke argued before Congress that healthcare should include birth control, since it is used to combat ovarian cysts, Rush Limbaugh responded with: "What does it say about the college co-ed Sandra Fluke, who goes before a congressional committee and essentially says that she must be paid to have sex, what does that make her? It makes her a slut, right? It makes her a prostitute."

Secondly, a circumstantial ad hominem attempts to discredit an argument by calling attention to some circumstantial features of the person making the argument (even though those features might not in and of themselves be bad-making features).

**Example:** Robert Kennedy argues that we shouldn’t have a wind farm in the Nantucket Sound because the wind turbines would kill thousands of migrating songbirds and sea ducks each year. However, Robert Kennedy is only opposed to the wind farm because he and his family have property in Hyannis Port whose value would be hurt by the building of the wind farms. So, songbirds and sea ducks are just a distraction; we should build the wind farm.

Thirdly, a tu quoque attempts to discredit an argument by pointing out that the person making the argument themselves hypocritically rejects the conclusion in other contexts. For example,

**Example:** Newt Gingrich called for Bill Clinton to be impeached for lying about his affair with Monica Lewinsky. However, at the same time, Gingrich was lying about his own affair. So, Clinton ought not to have been impeached.
§2.2. Fallacies of Irrelevance

Why this is fallacious: the argument swings free of the person who happens to be making it. Even if the person who happens to be advancing the argument has some personal flaw, or stands in principle, somebody else without those flaws could make the very same argument.

A closely-related but non-fallacious argument: If the issue under discussion is whether the arguer is a good person, then personal attacks may not be fallacious; they might be entirely relevant to the question at hand. If the arguer is appealing to their own authority, then questioning the arguer’s authority could be a perfectly reasonable way of rejecting one of the argument’s premises.

2.2.2 Straw Man

A straw man fallacy occurs when one misrepresents somebody else’s position or argument (usually making it more simplistic or naive than their actual position or argument), and then argues against the misrepresented position or argument, rather than the person’s actual position or argument.

Example: Mr. Goldberg has argued against prayer in the public schools. Obviously Mr. Goldberg advocates atheism. But atheism is what they used to have in Russia. Atheism leads to the suppression of all religions and the replacement of God by an omnipotent state. Is that what we want for this country? I hardly think so. Clearly Mr. Goldberg’s argument is nonsense.

Why this is fallacious: simply because a misrepresentation of somebody’s view is false, this doesn’t give us any reason to think that their correctly represented view is false.

2.2.3 Appeal to Force (Ad Baculum)

An ad baculum fallacy occurs when a conclusion is defended, or an argument attacked, by making a threat to the well-being of those who make it (or implying that bad things will happen to those who accept the conclusion or argument).

Example: Anusar argues that workers are entitled to more of the firm’s profits than management because they contribute more to the product. But no firm wants to hire an employee with radical views like that. That’s why Anusar’s been unemployed for so long. So it doesn’t matter how much workers contribute; workers are entitled to what they get. If you think otherwise, you’ll end up out of work like Anusar.

Why this is fallacious: simply because you can avoid harm by rejecting a certain statement or argument, that doesn’t give you any reason to suppose that the statement is false or that the argument is bad. So the premises of an ad baculum argument don’t give you any reason to believe that the conclusion is true; even though they might make it a good idea to pretend that the conclusion is true.

A closely-related but non-fallacious argument: if the harm being threatened is relevant to the truth of the conclusion, then an ad baculum might be perfectly reasonable. E.g.,
Not An Example: You shouldn’t smoke, or else you’ll likely get lung cancer.

2.2.4 Appeal to the People (Ad Populum)

Ad populum is a fallacy which attempts to argue for a conclusion by in some way appealing to people’s innate desire to be accepted or desired by others. It can occur when an arguer appeal to nationalism, as in

Example: We Americans have always valued freedom. We understand that this freedom comes with a price, but it is a price we are willing to pay. True Americans resist the more extreme measures of the war on terror, like the Patriot Act. So, we need to repeal the Patriot Act.

Or, it could occur when an arguer appeals to the audience’s desire to have mainstream opinions, as in

Example: I can’t believe that you think we should curtail the freedom of speech in order to protect minority rights. Only fascists and kooks think that! So you should really reconsider your opinion.

Why this is fallacious: That holding a certain opinion will make you stand out from the group does not, on its own, provide any reason to think that that opinion is false. Even though most people generally want to be included in the group and hold the majority opinion, this doesn’t give you any reason to think that the majority opinion is true.

A closely-related but non-fallacious argument: If the arguer is pointing to the consensus of people who are in a better position to evaluate the evidence, then they could be making an appeal to authority, which needn’t be fallacious.

Not An Example: The biological community has reached a near-unanimous consensus that the hypothesis of evolution by natural selection is correct. Since they are experts on the subject, we should trust them that there is excellent reason to believe in the hypothesis of evolution by natural selection.

2.2.5 Appeal to Ignorance (Ad Ignorantiam)

An appeal to ignorance occurs when somebody argues in favor of a conclusion that we don’t antecedently have any reason to accept (or which we antecedently have reason to reject) on the grounds that there’s no evidence either way. Alternatively, it occurs when somebody argues against a conclusion that we don’t antecedently have any reason to reject (or which we antecedently have reason to accept) on the grounds that there’s no evidence either way.

Example: The studies purporting to show that barefoot running is good for you have been discredited. However, there aren’t any studies showing that it’s not good for you—the jury’s still out. So, you should keep running barefoot.
Example: There’s no evidence showing that there’s life on other planets. So we should stop looking—it’s not there.

Why it’s fallacious: If we don’t antecedently have any reason to accept or reject a claim, then, in the absence of evidence, we should suspend judgment. Just because no reason has been offered to think that the conclusion is false, that doesn’t mean that we should think that it is true. Similarly, just because no reason has been offered to think that the conclusion is true, that doesn’t mean that we should think that it is false.

Two closely-related but non-fallacious arguments:

a) If you do have antecedent reason to accept or reject a conclusion, then the absence of any defeating evidence can provide good reason to continue believing the conclusion.

Not an Example: The studies showing that circumcision reduces HIV transmission were badly methodologically flawed, so circumcision probably doesn’t reduce HIV transmission.

b) If certain evidence was to be expected if a certain statement were true (false), and we don’t find that evidence, that can count as good reason to think that the statement is false (true).

Not an Example: If he had been poisoned, the toxicology report would have revealed poison in his blood; it didn’t; so, he probably wasn’t.

2.2.6 Red Herring (Ignoratio Elenchi)

The red herring fallacy occurs when somebody presents premises which might be psychologically compelling, but which are irrelevant to the conclusion. As such, every other fallacy in this section constitutes an instance of the red herring fallacy. It is the most general fallacy of irrelevance. (Nevertheless, we should use ‘red herring’ to refer only to fallacies of irrelevance which do not fall into the other categories of this section. If a fallacy falls into one of the other categories, identifying it as a red herring, on, e.g., a test, will not be correct.)

Example: Jamal says that we shouldn’t have a central bank because central banking is responsible for the economic fluctuations of the business cycle. But people have been banking for centuries. Bankers aren’t bad people, and they provide the valuable service of providing credit to people who don’t have their own capital.

A. Underline the conclusion of each passage below, and say which, if any, informal fallacies are being committed. (A passage may contain more than one fallacy. If the passage doesn’t commit any informal fallacies, write ‘none’.)

1. Katie Roiphe says that Colin McGinn’s behavior wasn’t unethical. But any true feminist knows that Colin McGinn abused his authority. We feminists have had experiences with people like McGinn our whole lives. We’ve learned that apologists
like Roiphe will bend over backwards to justify their bad behavior. Roiphe claims to be a feminist, but if she were a true feminist, she would join us in condemning McGinn. So, Roiphe is wrong; McGinn's behavior was absolutely unethical.

Answer: 

2. "[Congresswoman Wasserman-Schultz] expressed her contempt for Governor Palin as follows: 'She knows nothing...Quite honestly, the interview I saw and that Americans saw...was similar to when I didn't read a book in high school and had to read the Cliff's Notes and phone in my report.' Good for you, Representative Wasserman-Schultz; you admitted, publicly, to cheating in high school. What other examples of cutting corners, fudging data, embellishing claims of competence, and the like, drive you to attack others with such intensity? [So, the Congresswoman is wrong about Governor Palin.]"

Answer: 

3. [Context: Gottlob Frege is a German mathematician, widely regarded as the father of modern logic; his diary reveals that he was an anti-semit]

"Hitler...guided by sentiments not unlike the ones expressed in Frege's diary, worked out the master-logic of National Socialism...The applications of logic to action that Frege had promised came readily to hand. If Jews are a mongrel race, they must be exterminated. 'A thought like a hammer' [Frege's phrase] demanded instant obedience to the dictates of logic...[Therefore,] Logic in its final perfection is insane."

Answer: 

4. Neil Degrasse Tyson says that the scientific community has unanimously concluded that humans evolved from other species; and that for this reason, we ought to accept that humans evolved from other species. However, the scientific community is full of atheists who hate God and want to mislead people into discarding their faith. We shouldn't listen to people with such an irrational hatred of God—they aren't trustworthy. So their consensus doesn't give us any reason to think that humans evolved from other species.

Answer: 

5. "Examples of the *tu quoque* fallacy occur all the time. For instance...BBC Sport reported:

Manchester United have hit their fans with a 12.3% average rise in season ticket prices for the next campaign. A top-price ticket will cost £38 and the cheapest £23...But United have defended the price rises, saying they compare favourably with the rest of the Premiership. 'We do not know what most of our rivals will charge next year, buy even a price freeze across the rest of the Premiership would mean that next year only seven

---


2. from Andrea Nye's *Words of Power: A Feminist Reading of the History of Logic* (Routledge, 1990)
clubs will have a cheaper ticket than £23 and nine clubs will have a top price over £39—in some cases almost double,' said Humby [Manchester United finance director].

The representative of Manchester United’s argument was essentially this: ‘Other Premiership clubs charge more, therefore our ticket prices are justified.’ This commits the *tu quoque* fallacy because it’s quite possible that all clubs, including Manchester United, overcharge for their tickets.”

Answer: 

6. You abortion advocates say that it’s o.k. to murder someone so long as their existence causes you some burden. You should be careful with that position, ‘cause if you keep claiming that murder is permissible, you’re going to start to cause me some burden. So, abortion is not permissible—got it?

Answer: 

7. You can’t give me any good reason to think that Elvis isn’t still alive or that UFOs don’t exist. There’s just not any good evidence either way. So Elvis lives on a UFO with aliens.

Answer: 

8. “Evolution teaches that energy, such as lightning or heat, plus matter, can occasionally create new life. Yet our entire food industry rests on the fact that this can never happen. If we examine a jar of peanut butter, it contains matter and is exposed to light and heat, yet we never find new life inside. [So, evolution is probably false.]”

Answer: 

9. My opponent has criticized me for accepting donations from big corporations. He says that this will make me care more about the welfare of large corporations than the interests of my constituency. But my opponent has *himself* accepted donations from large corporations. We’re politicians; this is what we do. So I can take the money without losing sight of the needs of my constituents.

Answer: 

10. If you keep making obnoxious jokes like that, I’m going to break up with you. So you should stop making obnoxious jokes like that.

Answer: 

2.3 **Fallacies Involving Ambiguity**

These are all fallacies that arise because of some ambiguity in the language appearing in the statements in the argument.

2.3.1 Equivocation

The fallacy of equivocation occurs when a single word is used in two different ways at two different stages of the argument, where validity would require that the word be used in the same way at both stages.

**Example:** In order to be a theist, as opposed to an agnostic, you must claim to know that God exists. But, even if you believe that God exists, you don't know it. Thus, you shouldn't be a theist. It follows that you should either be an agnostic or an atheist. However, once you've ruled out theism, what is there to be agnostic about? Once theism has been ruled out, atheism is the only remaining position. Therefore, you shouldn't be agnostic. Hence, you should be an atheist.

‘Agnostic’ can mean either 1) not claiming knowledge that God exists, or 2) not having belief either way about whether God exists. The first stage of the argument relies upon the first meaning; while the second stage of the argument relies upon the second meaning.

*Why it’s fallacious:* the argument gives the appearance of validity if we don’t realize that the word is being used in two different senses throughout the argument. However, once we are clear about what the words mean, the argument either becomes invalid, or else has an obviously false premise.

*A closely related but non-fallacious argument:* If an argument uses a word that has multiple meanings, but the premises are all true on a single disambiguation, then the argument does not equivocate.

**Not an Example:** [Suppose that I am a fisherman who works at the riverside]

I work at the bank, and there are fish at the bank. So there are fish where I work.

2.3.2 Amphiboly

The fallacy of amphiboly occurs when multiple meanings of a sentence are used in a context where a) validity would require a single meaning, and b) the multiple meanings are due to sentence structure.

**Example:** You say that you don’t keep your promises because it’s in your interest to do so. People who don’t keep their promises are immoral. So, you are immoral.

‘You don’t keep your promises because it’s in your interest to do so’ has two different readings: either that you keep your promises, but not because it’s in your interest—that is, your
reason isn’t that it’s in your interest. Or that you don’t keep your promises, and that’s be-
cause it’s in your interest—that is, that the fact that it’s in your interest is your reason for
not keeping your promises. In most contexts, the former would be the reading intended.
So the first sentence is only true if the sentence is interpreted in the first way. However,
the conclusion only follows if it is interpreted in the second way.

**Example:** Nothing is better than Game of Thrones, and Duck Dynasty is
better than nothing. We can infer that Duck Dynasty is better than Game of
Thrones.

‘Nothing is better than Game of Thrones’ could either mean that there isn’t anything which
is better than Game of Thrones, or it could mean that not watching anything at all is better
than Game of Thrones. The argument is only valid if we interpret the sentence in the
second way. However, the sentence is only true if we interpret it in the first way.

**Why it’s fallacious:** the argument gives the appearance of validity if we don’t realize that the
sentence is being understood in two different ways in the argument. However, once we are
clear about what the sentence means, the argument either becomes invalid, or else has an
obviously false premise.

**A closely related but non-fallacious argument:** If an argument uses a sentence that has multiple
meanings, but the premises are true and the argument valid on a single disambiguation,
then the argument is not amphibolous.

**Not an Example:** Flying planes can be dangerous. You should avoid danger-
ous things. So, you should avoid flying planes.

### 2.3.3 Composition/Division

The fallacy of **composition** occurs when 1) a property of the parts of an
object is improperly transferred to the object itself, or 2) a property of the in-
dividuals belonging to a group is improperly transferred to the group.

**Example:** Atoms are invisible, and I am made of atoms. So I am invisible.

**Example (?)**: Every part of the world is caused. So, the world is caused.

The fallacy of **division** occurs when 1) a property of an object is improperly
transferred to the parts of the itself, or 2) a property of a group is improperly
transferred to the individuals belonging to the group.

**Example:** About 70 million people watch sitcoms. So *How I Met Your Mother*
has about 70 million viewers.

**Example:** China and India consume more natural resources than America.
So, Chinese and India citizens consume more resources than American citi-
zens.
**Why it’s fallacious:** wholes and parts can have different properties from one another, as can individuals and groups. Simply because parts have a property, that doesn’t necessarily mean that the whole does; and simply because individuals have a property, that doesn’t necessarily mean that the group does. Similarly, simply because the whole has a property, that doesn’t necessarily mean that the parts do; and simply because the group has a property, that doesn’t necessarily mean that the individuals in the group do.

**A closely related but non-fallacious argument:** There are some properties which can be properly transferred from parts to wholes (or wholes to parts), or from individuals to groups (or groups to individuals). These arguments are valid. We must, therefore, look to the properties in question in order to decide whether the argument is valid or invalid. (That’s what makes this an informal fallacy.)

**Not an Example:** Every part of the train is made of metal; so the train is made of metal.

### 2.4 Fallacies Involving Unwarranted Assumptions

These fallacies all occur when an arguer assumes something in their argument which they are unwarranted in assuming.

#### 2.4.1 Begging the Question (Petitio Principi)

An argument commits the fallacy of BEGGING THE QUESTION when it assumes the very conclusion that it is trying to establish.

**Example:** Surely Anthony loves me. For he told me he loves me, and he wouldn’t lie to someone he loves.

**Example:** My scale is working perfectly. I weighed this textbook, and it said that it was 12 ounces. And, as I just learned by looking at the scale, it is 12 ounces. So the scale got it exactly right!

Note: question-begging arguments are deductively valid. They’re just not especially persuasive.

A word of caution: it is incredibly difficult in some cases to distinguish good, valid arguments from question-begging arguments. For instance, the argument

**Example:** There are numbers greater than 4. Therefore, there are numbers.

Might be thought to be question-begging, because we’ve simply assumed that there are numbers. However, many people end up finding this argument persuasive. While everyone
accepts that some arguments are question-begging, and therefore defective, there is no consensus on the question of when arguments are question-begging and when they are not.

### 2.4.2 False Dilemma

Two statements are *contraries* when they cannot both be true at once, but they *can* both be false at once. Two statements are *contradictories* when they cannot both be true at once, *nor* can they both be false at once (at least, and at most, one of them must be true).

The fallacy of **false dilemma** occurs when an argument makes use of a premise that presents contraries as though they were contradictories.

**Example:** Either you are with us or you are with the terrorists. If you’re leaking classified information about our government, then you’re not with us. So, you are with the terrorists.

**Example:** It would be terrible if the government regulated every aspect of a person’s life—their clothes, their love life, their personal beliefs. So we shouldn’t have government regulation; let the free market decide.

**Example:** It would be terrible if there were no government regulation of any behavior. There would be total anarchy, and those with the most money and influence would exert their arbitrary authority over everyone else. So we need the government to regulate the marketplace.

*An closely-related but non-fallacious argument:* if we have good reason to set certain cases aside, then, so long as the argument is explicit that it is setting those cases aside, the argument will not be posing a false dilemma. What makes the argument a false dilemma is that it pretends as though two contraries are contradictories—not that it asserts, with good reason, that one of two contraries are true.

**Not an Example:** Given that it’s around noon, Dmitri is either in his office or at lunch. But he’s not in his office, so he’s probably at lunch.

### 2.4.3 False Cause Fallacy

The fallacy of **false cause** occurs when a merely possible cause is assumed to be a cause without evidence.

**Example:** Good philosophers write books; so if you want to be a good philosopher, you should write a book.
Example: The weather channel usually knows what the weather will be. The conclusion is inescapable: the weather channel is causing the weather.

This fallacy has some special sub-cases. First, there is the post-hoc, ergo proper hoc fallacy. In Latin, this means ‘afterwards, therefore because of’. In this fallacy, it is assumed that, because $E$ follows $C$, $C$ must be a cause of $E$.

The post hoc, ergo propter hoc fallacy occurs when one assumes that $C$ caused $E$ solely on the basis that $E$ followed $C$.

Example: Since Obama’s policies were enacted, unemployment has stopped growing. We can conclude that Obama’s policies worked.

Example: After Obama’s speech, the stock market took a nose dive. Good work, Obama.

A closely-related but non-fallacious argument: If we have good antecedent reason to think that two factors may be causally related, then the fact that they regularly appear together could be good reason to think that they might be causally related.

Another flavor is the slippery slope fallacy, in which it is assumed, without evidence, that one action will set off a causal chain leading to several other bad actions, and then argued that we shouldn’t perform the first action.

The slippery slope fallacy occurs when one assumes in an argument against some action that performing the action will set off a chain reaction of several bad consequences, when there is insufficient evidence to support the claim that performing the action will have these consequences.

Example: If we legalize gay marriage, then we’ll soon be legalizing polygamous marriages, bestiality, dendrophilia, and other perversions. So we shouldn’t legalize gay marriage.

A closely related but non-fallacious argument: If there is good reason to suppose that performing the first action will set off a chain reaction of consequences which are bad, then one is not reasoning fallaciously to suggest that we shouldn’t undertake the first action.

Not an Example: If start going down that slippery slope, you’ll just start slipping further and further down, until you fall off the edge of the cliff. So you shouldn’t start going down that slippery slope.
§2.4. Fallacies Involving Unwarranted Assumptions

Exercise

A. Underline the conclusion of each passage below, and say which, if any, informal fallacies are being committed. (A passage may contain more than one fallacy. If the passage doesn’t commit any informal fallacies, write ‘none’.)

1. Nuclear weapons are more destructive than conventional weapons. Therefore, over the course of human history, more destruction has resulted from nuclear weapon than from conventional weapons.

   Answer: 

2. America is still a free country, right? You bet it is. That being so, how can you doubt that we are free to choose between good and evil? Every real American is free and knows it. I’m starting to wonder what country you’re from.

   Answer: 

3. All philosophers are not losers. Therefore, all losers are non-philosophers.

   Answer: 

4. According to the Declaration of Independence, all men are created equal. Well, I disagree. It is obvious that human beings differ in important respects from birth, for example, in intelligence, athletic ability, and physical attractiveness. Therefore, contrary to the Declaration of Independence, it is not the case that human beings are all created equal.

   Answer: 

5. Either God created everything in six days or human life evolved gradually out of lower life forms over a very long period of time apart from any divine activity. But you are not a religious fanatic, so you know about fossils. And hence, you know that human life evolved gradually out of lower life forms over a very long period of time. Thus, God did not create everything in six days. I mean, I hate to break the news, but you are just about the last person on earth who believes that humans were created by God.

   Answer: 

6. Time is composed of moments. Moments have no duration. Therefore, time has no duration.

   Answer: 

7. We cannot intervene militarily in all countries with wicked dictators: Syria, Yemen, Zimbabwe, etc. To be consistent, we have to intervene everywhere or nowhere. So, we should intervene nowhere.

   Answer: 

38
8. The New York Times correctly reported the outcome of last night’s game. For the
New York Times reported that the Braves won last night, and the Braves did win last
night (as I just learned from the New York Times).

Answer: ________________________________

9. Night causes day, since day always follows night.

Answer: ________________________________

10. Sleeping pills work because they cause people to go to sleep.

Answer: ________________________________

11. Before television came along, we didn’t have much of a problem with illegal drugs.
But people learn about drugs on TV, and then they want the drugs. So, TV is ruining
our country.

Answer: ________________________________
Propositional Logic

In this chapter, we’re going to construct an artificial language, call it ‘PL’ (for ‘propositional logic’) within which we can be incredibly precise about which arguments are deductively valid and which are deductively invalid. This, together with a method for translating from English into PL (and out of PL into English) will allow us to theorize about which English-language arguments are deductively valid and which are deductively invalid. One advantage to theorizing about deductive validity in this way is that we won’t have to worry about the kinds of ambiguities that we encountered in our discussion of informal fallacies (e.g., equivocation and amphiboly), because the sentences of our artificial language won’t admit of any ambiguity. Their meaning will always be perfectly precise.

In general, we can specify a language by doing three things: 1) giving the vocabulary for the language, 2) giving the grammar of the language—that is, specifying which ways of sticking together the expressions from the vocabulary are grammatical, and 3) saying what the meaning of every grammatical expression is. For instance, in English, the vocabulary consists of all of the words of English. The grammar for English consists of rules saying when various strings of English words count as grammatical English sentences. ‘Bubbie makes pickles’ and ‘Colorless green ideas sleep furiously’ will count as grammatical sentences, whereas ‘Up bouncy ball door John variously catapult’ does not count as a grammatical sentence. Finally, the meaning of every English sentence is given by providing a dictionary entry for every word of English and providing rules for understanding the meaning of sentences in terms of the meanings of the words appearing in the sentence. The first two tasks are the tasks of specifying the syntax of the language. The final task is the fast of specifying the semantics of the language.

\[\text{syntax} \rightarrow \begin{cases} \text{1. Vocabulary} \\ \text{2. Grammar} \end{cases} \text{semantics} \rightarrow \text{3. Meaning}\]

That’s exactly what we’re going to do for our artificial language PL. However, our task will be much simpler than the task of specifying English, as we will have a far simpler vocabulary, a far simpler grammar, and a far simpler semantics.
3.1 Syntax for PL

3.1.1 Vocabulary

The vocabulary of PL includes the following symbols:

1. An infinite number of statement letters:
   \[ A, B, C, ..., Y, Z, A_1, B_1, C_1, ..., Y_1, Z_1, A_2, B_2, C_2, ... \]

2. Logical operators:
   \[ \sim, \cdot, \lor, \supset, \equiv \]

3. Parentheses
   \[ (, ) \]

Nothing else is included in the vocabulary of PL.

3.1.2 Grammar

Any sequence of the symbols in the vocabulary of PL is a formula of PL. For instance, all of the following are formulae of PL:

\[
\begin{align*}
((())A_23 & \cdot \supset \lor Z \\
P & \supset (Q) \supset (B) \\
(P & \supset (Q \supset (R \supset (S \supset T)))) \\
A & \cdot B \cdot (C \sim D))
\end{align*}
\]

However, only one—the third—is a well-formed formula (or 'wff') of PL. We specify what it is for a string of symbols from the vocabulary of PL to be a wff of PL with the following rules.

SL) Any statement letter, by itself, is a wff.

\( \sim \) If ‘\( p \)’ is a wff, then ‘\( \sim p \)’ is a wff.

\( \cdot \) If ‘\( p \)’ and ‘\( q \)’ are wffs, then ‘(\( p \cdot q \))’ is a wff.

\( \lor \) If ‘\( p \)’ and ‘\( q \)’ are wffs, then ‘(\( p \lor q \))’ is a wff.

\( \supset \) ‘\( p \)’ and ‘\( q \)’ are wffs, then ‘(\( p \supset q \))’ is a wff.

\( \equiv \) ‘\( p \)’ and ‘\( q \)’ are wffs, then ‘(\( p \equiv q \))’ is a wff.

– Nothing else is a wff.
Note: ‘p’ and ’q’ do not appear in the vocabulary of PL. They are not themselves wffs of PL. Rather, they are being used here as formulae variables—they are variables whose potential values are formulae of PL.

All and only the strings of symbols that can be constructed by repeated application of rules 1–6 above are well-formed formulae. To show that ‘(¬(P ∨ Q) ⊃ R)’ is a wff, we could walk through the following steps to build the formula up:

a) ‘P’ is a wff [from (SL)]

b) ‘Q’ is a wff [from (SL)]

c) So, ‘(P ∨ Q)’ is a wff [from (a) and (b) and (∨)]

d) So, ‘¬(P ∨ Q)’ is a wff [from (c) and (¬)]

e) ‘R’ is a wff [from (SL)]

f) So, ‘(¬(P ∨ Q) ⊃ R)’ is a wff [from (d), (e), and (⊃)]

The rule (⊃) requires that we include the outermost parentheses around the expression ‘(¬(P ∨ Q) ⊃ R)’. However, I will adopt the standard convention of omitting the outermost parentheses, writing, e.g., ‘¬(P ∨ Q) ⊃ R’. This convention is harmless, but you should bear in mind that, strictly speaking, formula like ‘¬(P ∨ Q) ⊃ R’ are not wffs of PL.

3.1.3 Main Operators and Subformulae

Given the rules for wffs provided above, we can give a simple definition of what a wff’s main operator is. The wff’s main operator is just the operator associated with the last rule which would have to be applied if we were building the formula up by applying the rules for wffs above. For instance, if we want to know what the main operator is for the wff ‘¬P • Q’, we would just imagine running through the following proof that ‘¬P • Q’ is a wff of PL, by applying to the rules for well formed formulae, i.e.,

a) ‘P’ is a wff [from (SL)]

b) So, ‘¬P’ is a wff [from (a) and (¬)]

c) ‘Q’ is a wff [from (SL)]

d) So, ‘(¬P • Q)’ is a wff [from (b), (c), and (•)]

Here, the fact that we had to appeal to the rule (•) in the final step of building up ‘¬P • Q’ tells us that • is the main operator. Imagine that we had tried to build up the formula in some other way. For instance, suppose we had attempted to first apply the rule (•) and then the rule (¬). Then, our derivation would have gone line this.

a) ‘P’ is a wff [from (SL)]
§3.1. Syntax for PL

b) ‘Q’ is a wff [from (SL)]

c) So, ‘(P • Q)’ is a wff [from (a), (b), and (•)]

d) So, ‘∼(P • Q)’ is a wff [from (c) and (∼)]

This is an entirely different wff. ‘∼(P • Q)’ is not the same as ‘(∼ P • Q)’. While the main operator of ‘∼ P • Q’ is •, the main operator of ‘∼(P • Q)’ is ∼.

We can also use the rules for wffs to give a definition of what a wff’s subformulae are. p is a subformula of q if and only if, in the course of building up q by applying the rules for wffs, p appears on a line before q. So, for instance ‘∼ P’ is a subformula of ‘∼ P • Q’ (because it shows up on line (b) of that wff’s derivation), whereas ‘∼ P’ is not a subformula of ‘∼(P • Q)’ (since it does not show up at any point in that wff’s derivation).

A formula’s immediate subformulae are those wffs whose lines were appealed to in the final step of building to formula up. For instance, the immediate subformulae of ‘∼ P • Q’ are ‘∼ P’ and ‘Q’, whereas the immediate subformula of ‘∼(P • Q)’ is ‘P • Q’. A wff’s immediate subformulae are just those formulae on which the wff’s main operator operates.

Another way of notating the proofs that certain formulae are wffs of PL is with syntax trees. For instance, we could represent our proof that ‘(∼ (P ∨ Q) ⊃ R)’ is a wff of PL with the following syntax tree.

```
(∼ (P ∨ Q) ⊃ R)  (⊃)
  ~ (P ∨ Q)    R
    (∼)   (SL)
      (∨)
    P
     (SL)  Q  (SL)
```

This tree tells us, firstly, that ‘P’ and ‘Q’ are wffs of PL (by rule (SL)). Then, by rule (∨), ‘(P ∨ Q)’ is a wff. Then, by rule (∼), ‘∼ (P ∨ Q)’ is a wff. And, since ‘R’ is a wff, by (SL), ‘(∼ (P ∨ Q) ⊃ R)’ is a wff (by rule (⊃)).

If we want to leave out the rules, we can represent this syntax tree more simply as follows.
We can similarly write out the syntax trees for ‘(P ∨ Q)’ and ‘(P • Q)’ like so.

These trees give us the syntactic structure of a wff of PL. They highlight what the parentheses were already telling us about what the main operator of the sentence is, what its subformulae are, and how the various subformulae are interrelated (how the sentence is built up out of its subformulae). For instance, the tree on the left tells us that the immediate subformulae of ‘(P ∨ Q)’ are ‘P’ and ‘Q’. And the tree on the right tells us that the immediate subformula of ‘(P • Q)’ is ‘(P ∨ Q)’.

3.2 Semantics for PL

We now need to say something about the meaning of the wffs appearing in PL. Throughout, our assumption will be that what it is to understand the meaning of an expression is just to understand the circumstances in which it is true.

There are three components to the vocabulary of PL: the statement letters, the logical operators, and the parentheses. The parentheses do not add anything to the meaning of the sentences of PL. They merely serve as notational tools that help us avoid ambiguity. Put them aside. We must then say what the meanings of the statements letters are and what the meanings of the logical operators are.

3.2.1 The Meaning of the Statement Letters

Each statement letter represents a statement in English. The statement letter is true if and only if the statement in English is true. That is: statement letters inherit their meaning from their English translations.

3.2.2 The Meaning of ‘∼’

The operator ‘∼’ is known as the tilde. A wff whose main operator is the tilde is called a negation. Its immediate subformula is called the negand. If a wff ‘p’, is true, then ‘∼ p’ is false. If a wff p is false, then ‘∼ p’ is true. To write this a bit more perspicuously, we
can use the letters ‘T’ and ‘F’ to stand for the truth-values true and false. Then, for any wff ‘p’, if ‘p’ is T, then ‘¬ p’ is F. If ‘p’ is F, then ‘¬ p’ is T. We can summarize this with the following truth table.

<table>
<thead>
<tr>
<th>p</th>
<th>¬ p</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This table tells us how the truth-value of a wff of the form ‘¬ p’ is determined by the truth-value of ‘p’. If we understand the circumstances under which ‘p’ is true, then the above definition gives us all that we need to understand the circumstances under which ‘¬ p’ is true. So we’ve said enough to say what the meaning of ‘¬’ is.

Note that ‘p’ is not a wff of PL—statement letters must be capitalized. Rather, we are using the lowercase ‘p’ and ‘q’ as variables ranging over the wffs of PL.

3.2.3 The Meaning of ‘•’

The operator ‘•’ is known as the dot. A wff whose main operator is the dot is known as a conjunction. Its immediate subformulae are called conjuncts. A conjunction is true if and only if both of its conjuncts are true. Using a truth-table, this means that:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p • q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

This table tells us how the truth-value of a wff of the form ‘p • q’ is determined by the truth-values of ‘p’ and ‘q’. If we understand the circumstances under which ‘p’ and ‘q’ are true, then this definition gives us enough to understand the circumstances under which ‘p • q’ is true. So we’ve said enough to say what the meaning of ‘•’ is.

3.2.4 The Meaning of ‘∨’

The operator ‘∨’ is known as the wedge. A wff whose main operator is the wedge is known as a disjunction. Its immediate subformulae are called disjuncts. A disjunction is true if and only if at least one of its disjuncts is true.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p ∨ q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
This table tells us how the truth-value of a wff of the form \( p \lor q \) is determined by the truth-value of \( p \) and \( q \). If we understand the circumstances under which \( p \) and \( q \) are true, then this definition gives us enough to understand the circumstances under which \( p \lor q \) is true. So we’ve said enough to say what the meaning of \( \lor \) is.

3.2.5 \textbf{The Meaning of ‘\( \supset \)’}

The operator ‘\( \supset \)’ is known as the \textbf{horseshoe}. A wff whose main operator is the horseshoe is known as a \textbf{material conditional}. The immediate subformulae which precedes the horseshoe is known as the \textbf{antecedent}. The immediate subformulae which follows the horseshoe is known as the \textbf{consequent}. A material conditional is true if and only if either its antecedent is false or its consequent is true.

\[
\begin{array}{c|c|c}
 p & q & p \supset q \\
\hline
 T & T & T \\
 T & F & F \\
 F & T & T \\
 F & F & T \\
\end{array}
\]

As before, the above table gives us enough to understand the circumstances under which a wff of the form \( p \supset q \) is true, assuming that we understand the circumstances under which \( p \) and \( q \) are true. So this table defines the meaning of the operator ‘\( \supset \)’.

Note that this is the only binary operator which is not symmetric. That is, while \( p \bullet q \) has the same meaning as \( q \bullet p \), \( p \lor q \) as the same meaning as \( q \lor p \), and \( p \equiv q \) has the same meaning as \( q \equiv p \), ‘\( p \supset q \) does not have the same meaning as ‘\( q \supset p \)’.

3.2.6 \textbf{The Meaning of ‘\( \equiv \)’}

The operator ‘\( \equiv \)’ is known as the \textbf{triple bar}. A wff whose main operator is the triple bar is known as a \textbf{material biconditional}. The immediate subformula which appears before the triple bar is known as the biconditional’s \textbf{left hand side}, and the immediate subformula which appears after the triple bar is known as the biconditional’s \textbf{right hand side}. A material biconditional is true if and only if its right hand side and its left hand side have the same truth-value.

\[
\begin{array}{c|c|c}
 p & q & p \equiv q \\
\hline
 T & T & T \\
 T & F & F \\
 F & T & F \\
 F & F & T \\
\end{array}
\]

Again, this table gives us enough to understand the circumstances under which a wff of the form \( p \equiv q \) is true, assuming that we understand the circumstances under which \( p \) and \( q \) are true. So this table defines the meaning of the operator ‘\( \equiv \)’.
§3.2. Semantics for PL

3.2.7 Determining the Truth-value of a wff of PL

If we know the truth-value of all the statement letters appearing in a wff of PL, then we can use our knowledge of the syntactic structure of the wff to determine its truth value. For instance, suppose that we know that ‘P’ is true and that ‘Q’ is false. Then, we know that ‘~P • Q’ is false, and that ‘~(P • Q)’ is true.

\[
\begin{array}{c|c}
\sim P & Q \\
\hline
T & F \\
F & T \\
\end{array}
\]

\[
\begin{array}{c|c}
\sim (P • Q) & T \\
\hline
F & T \\
F & T \\
\end{array}
\]

We can do the very same thing with truth-tables. For instance, to construct the truth-table for the wff ‘~P • Q’, begin by writing out all the possible truth-values for P and Q.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ P • Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Then, copy the column of truth-values for P, placing it beneath every appearance of the statement letter P, and do the same for Q.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ P • Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Then, begin working your way up the syntactic structure of the sentence by calculating the truth-values of the subformulae appearing in the wff. We know how to calculate the truth-value of ‘~P’, given the truth-value of ‘P’ (from the truth-table for ‘~’ which tells us the meaning of ‘~’), so do that first, placing the appropriate truth-values beneath the main connective of the subformulae ‘~P’.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ P • Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Now, we have to calculate the column of truth-values of ‘~P • Q’, writing them out beneath the main connective of that wff—the ‘•’. The truth-value of ‘~P • Q’ is a function of the truth-values of ‘~P’ and ‘Q’, and not the truth values of ‘P’ and ‘Q’, so
we must look at the bolded columns of truth-values below.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ P • Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Now, we can simply look to the truth-table for ‘•’ to figure out what column of truth-values ought to go beneath the ‘•’ in ‘~ P • Q’. Since ‘•’ is the main operator of the wff, this tells us the column of truth-values associated with the wff ‘~ P • Q’. To indicate that this column of truth-values is the column associated with the main operator of the wff ‘~ P • Q’, we put a box around this column.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ P • Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This truth-table tells us how the truth-value of ‘~ P • Q’ is determined by the truth-values of ‘P’ and ‘Q’. If ‘P’ is false and ‘Q’ is true, then ‘~ P • Q’ is true. Otherwise, ‘~ P • Q’ is false.

If we do the same thing with the wff ‘~ (P • Q)’, we will arrive at the following truth-table.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>~ (P • Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

This shows us how important it is to pay attention to the syntactic structure of the different wffs of PL—they end up making a difference to the meaning of those sentences. If we’re not careful with our parentheses, we’ll lose a big advantage of moving to a formal language—namely, that the sentences in PL are not ambiguous between different meanings.

### 3.3 Translation from PL to English

The meanings of ‘~, •, ∨, ⊃, and ≡ are given by the truth-tables in the previous section. However, when we look at those meanings, it is difficult to not see some commonalities between these operators and some common English words. In particular, it appears that there’s a very close connection between the meaning of ‘~’ and the meaning of ‘it is not the case that’; a very close connection between ‘∨’ and ‘or’; a very close connection between ‘•’ and ‘and’.
§3.3. Translation from PL to English

Submitted for your approval: the following provides a translation guide from PL to English.

\[ \sim p \quad \rightarrow \quad \text{It is not the case that } p \]
\[ p \land q \quad \rightarrow \quad \text{Both } p \text{ and } q \]
\[ p \lor q \quad \rightarrow \quad \text{Either } p \text{ or } q \]
\[ p \supset q \quad \rightarrow \quad \text{If } p, \text{ then } q \]
\[ p \equiv q \quad \rightarrow \quad p \text{ if and only if } q \]

This translation guide requires some provisos. In the first place: there appears to be an important difference between the meaning of ‘\(p \supset q\)’ and ‘if \(p\), then \(q\)’. The difference is this: if ‘\(p\)’ is false, then ‘\(p \supset q\)’ is automatically true, no matter what statement \(q\) represents, and no matter what kind of connection there is between \(p\) and \(q\). However, we wouldn’t ordinarily think that the sentence ‘if John Adams was America’s first president, then eating soap cures cancer’ is true, just in virtue of the fact that ‘John Adams was America’s first president’ is false. So it must be that ‘if \(p\), then \(q\)’ differs in meaning from ‘\(p \supset q\)’. I think that this is exactly right. However, there is still some close connection between the meanings of these two claims. To bring that connection out, suppose that I make the following claim:

If it’s a weekday, then I’m on campus.

And suppose that Steve makes the claim,

If I’m on campus, then it’s a weekday.

Think about the circumstances under which you could justly say that Steve or I had lied. If it’s a weekday, but I’m not on campus, then I have lied. If, however, it’s a weekday but Steve is not on campus, then he hasn’t lied. After all, he never said that he would be on campus every weekday. He just said that, if he’s on campus, then it’s a weekday. But he did not commit himself to ever coming to campus at all. On the other hand, suppose that I’m on campus during the weekend. Then, you wouldn’t be able to say that I had lied. For I never said that I would stay home during the weekend. I just said that, if it’s a weekday, then I’m on campus. However, if Steve is on campus during the weekend, then Steve has lied. After all, he said that he’d only be on campus on weekdays. Using ‘\(D\)’ to represent the statement ‘Dmitri is on campus’, ‘\(S\)’ to represent ‘Steve is on campus’ and ‘\(W\)’ to represent ‘it is a weekday’, then it looks like the possibilities in which you can say that I have lied are just the possibilities in which the material conditional ‘\(W \supset D\)’ is false.

<table>
<thead>
<tr>
<th>(D)</th>
<th>(W)</th>
<th>if (W), then (D)</th>
<th>(W \supset D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>lied</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
</tbody>
</table>

And it looks like the possibilities in which you can say that Steve has lied are just the possibilities in which you can say that the material conditional ‘\(S \supset W\)’ is false.
So, even though the translation isn’t perfect, it’s still pretty good. Moreover, even if a PL wff of the form ‘p ⊃ q’ might be better translated into English with ‘Either it is not the case that p or q’, it appears as though ‘p ⊃ q’ is the best possible PL-translation of the English ‘if p, then q’. So that’s how we’ll be translating it here. But if you think the translation is less than perfect, you’re absolutely correct. There are more advanced logics which attempt to give a better translation of the English conditional, but they are beyond the purview of this course.

In the second place: ‘or’ is used in English in two different senses. In one sense, called the ‘inclusive or’, a statement of the form ‘p or q’ is true if and only at least one of ‘p’ and ‘q’ are true—that is, it is true if and only if either ‘p’ is true, or ‘q’ is true, or both are true. For instance, if I say to you ‘either the elevator or the escalator is working’, then I haven’t lied to you if they are both working. To see this more clearly, think about the sentence ‘if either the elevator or the escalator is working, then you will be in compliance with the Americans with Disabilities Act’. If both are working and you are not in compliance with the ADA, then I have lied to you. However, if ‘either the elevator or the escalator is working’ were false when they are both working, then I couldn’t have lied to you.

Inclusive ‘or’: In the inclusive sense ‘p or q’ means ‘Either p or q or both.’

In another sense, called the ‘exclusive or’, a statement of the form ‘p or q’ is true if and only if at least and at most one of ‘p’ and ‘q’ are true. That is, in the exclusive sense, ‘p or q’ means ‘p or q, but not both’. For instance, if your parent tells you, ‘Either you clean your room, or you’re grounded’, you clean your room, and your parent grounds you, then you can fairly complain that they lied.

Exclusive ‘or’: In the exclusive sense ‘p or q’ means ‘Either p or q, but not both.’

When I say that ‘p ∨ q’ may be translated as ‘p or q’, I am using ‘or’ in its inclusive sense—that is, I am using it to mean ‘p or q or both’.

‘∨’ translates to the inclusive ‘or’

Let’s call the phrases on the right-hand-side of the translation guide above the canonical logical expressions of English. If the logical structure of an English statement is written in this form, then that statement is in canonical logical form. For instance, the following claim is in canonical logical form:

<table>
<thead>
<tr>
<th>S</th>
<th>W</th>
<th>if S, then W</th>
<th>S ⊃ W</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>lied</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>didn’t lie</td>
<td>T</td>
</tr>
</tbody>
</table>
§3.4. Translation from English to PL

If both John loves Andrew and it is not the case that Andrew loves John, then it is not the case that John and Andrew will be friends.

Because the sentence is in canonical logical form, it is simple to translate it into PL. We simply introduce the statement letters 'J', 'A', and 'F', where J = 'John loves Andrew', A = 'Andrew loves John', and F = 'John and Andrew will be friends'. Then, the translation into PL is

\[(J \land \sim A) \supset \sim F\]

On the other hand, this English sentence, which has the same meaning as the first, is not written in canonical logical form.

John and Andrew won't be friends if John loves Andrew but Andrew doesn't love him back.

So we'll have to say a bit more about how to translate sentences like this into PL.

3.4 Translation from English to PL

3.4.1 Negation

In English, the word 'not' can show up in many places in a sentence. In order for an English sentence to be translated into a wff of PL with a '\(\sim\)', it need not contain the words 'it is not the case that'. For instance, if we let 'H' stand in for the English sentence 'Harry likes chestnuts', then we may translate the English sentence

Harry doesn't like chestnuts

as '\(\sim H\)'. The reason is that '\(\sim H\)' is true if and only if 'H' is false, and 'Harry doesn't like chestnuts' is true if and only if 'Harry likes chestnuts' is false. So our translation has the same meaning as the sentence we wanted to translate. Here's a more general strategy for translating English sentences into PL: re-write the sentences in the canonical logical form given by the translation schema from the previous section, and check to see whether the re-written sentence has the same meaning as the sentence that you started out with. If it does, then you may substitute the canonical logical forms for the logical operators of PL according to the translation schema of the previous section. If not, then you may not.

For instance, we could re-write 'Harry doesn't like chestnuts' as

It is not the case that Harry likes chestnuts.

Since this contains the canonical logical form 'it is not the case that', we may swap this phrase of English out for PL's '\(\sim\)' to get

\(\sim\)Harry likes chestnuts.

52
We may then use the statement letter ‘$H$’ to represent ‘Harry likes chestnuts’, and we will get the PL wff

$$\sim H$$

A word of warning: just because an English statement contains the word ‘not’, that does not mean that it should be translated into a wff of PL with a ‘$\sim$’. In order to see whether it can, we have to see whether re-writing the statement in canonical logical form preserves meaning. For instance, the following sentence contains the word ‘not’:

I hate not getting what I want and I hate getting what I want.

We might attempt to translate this into canonical logical form like so,

It is not the case that I hate getting what I want, and I hate getting what I want.

substitute ‘$\sim$’ for ‘it is not the case that’ and ‘$\land$’ for ‘and’, and get

$$\sim I$$ hate getting what I want $\land$ $I$ hate getting what I want.

If we then used ‘$H$’ to represent the English ‘I hate getting what I want’, we would get the PL wff

$$\sim H \land H$$

However, this wff of PL is necessarily false, as the following truth-table shows

<table>
<thead>
<tr>
<th></th>
<th>$\sim H$</th>
<th>$\land H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

But the sentence we started with $\text{wasn't}$ necessarily false. For it is possible that I both hate not getting what I want and getting what I want. If this were possible, then I’d hate everything, but surely it’s not a logical truth that I don’t hate everything. So something went wrong. What went wrong was that ‘I hate not getting what I want’ doesn’t have the same meaning as ‘It is not the case that I hate getting what I want’. So we must make sure that translation into canonical logical form preserves meaning in English before we translate that canonical logical form into PL.

### 3.4.2 Conjunction

Many expressions in English have subtle shades of meaning which must be lost when we translate into PL. In particular, the following two English expressions will both have the same PL translation:
Hannes loves peaches and he loves apples.
Hannes loves peaches but he loves apples.

The second sentence implies some kind of contrast between ‘Hannes loves peaches’ and ‘Hannes loves apples’; whereas the first sentence does not. This subtle difference in meaning will be lost when we translate into PL, since both of these claims are true under exactly the same conditions: namely, the condition in which Hannes loves peaches and apples. So, using ‘P’ to represent ‘Hannes loves peaches’ and ‘A’ to represent ‘Hannes loves apples’, they will both be translated into PL as ‘P • A’.

All of the following expressions of English will also be translated into PL with the ‘ • ’.

\[
\begin{align*}
    p & \text{ and } q \\
    p, \text{ but } q \\
    p; \text{ however, } q \\
    p, \text{ though } q \\
    p \text{ as well as } q \\
\end{align*}
\]

\[\rightarrow p \cdot q\]

3.4.3 Disjunction

Both ‘p or q’ and ‘p unless q’ are translated into PL as ‘p \lor q’. If you’re unhappy about this translation, you should recognize that ‘p unless q’ could be translated as ‘\sim q \supset p’, and that this is expression has the very same meaning, in PL, as ‘p \lor q’ (they have the very same truth-table).

\[
\begin{align*}
    p \text{ or } q \\
    p \text{ unless } q \\
\end{align*}
\]

\[\rightarrow p \lor q\]

3.4.4 The Material Conditional and Biconditional

Any of the following English expressions are appropriately translated in PL as ‘p \supset q’.

\[
\begin{align*}
    \text{If } p, \text{ then } q \\
    p \text{ only if } q \\
    q \text{ if } p \\
    p \text{ is sufficient for } q \\
    q \text{ is necessary for } p \\
\end{align*}
\]

\[\rightarrow p \supset q\]

And any of the following are appropriately translated in PL as ‘p \equiv q’.

\[
\begin{align*}
    p \text{ if and only if } q \\
    p \text{ is necessary and sufficient for } q \\
\end{align*}
\]

\[\rightarrow p \equiv q\]
Chapter 3. Propositional Logic

Exercises

A. well formed formulae. Which of the following are well-formed formulae of PL?

1. \((P \land Q) \land R\)
2. \(P \equiv (S \equiv R)\)
3. \((p \supset (q \supset r))\)
4. \((\sim A) \supset A\)
5. \(\sim \sim \sim \sim (\sim B \supset \sim A) \supset (A \supset B)\)

B. main operators. What is the main operator of the following wffs of PL? What is (are) the subformula(e) on which the main operator is operating?

1. \(\sim A \equiv (B \supset C)\)
2. \((C \equiv A) \lor B\)
3. \(A \supset (B \supset C)\)
4. \((A \cdot B) \lor C\)
5. \(\sim B \equiv B\)

C. truth conditions. Suppose that \(A\) is true, \(B\) is false, and \(C\) is true. Then, are the following wffs of PL true or false?

1. \(\sim A \equiv (B \supset C)\)
2. \((C \equiv A) \lor B\)
3. \(A \supset (B \supset C)\)
4. \((A \cdot B) \lor C\)
5. \(\sim B \equiv B\)

D. translations Let \(B = \text{‘The Braves will win the pennant’}, \ L = \text{‘I will wear my lucky shirt’}, \) and \(C = \text{‘Clinton will win in 2016’}. \) Then, translate the following wffs of PL into English.

1. \(B \cdot \sim C\)
2. \(\sim L \supset (\sim B \lor \sim C)\)
3. \(\sim L \supset (B \cdot C)\)
3.5 Logical Properties of PL

3.5.1 How to Construct a Truth-table

If there are \(n\) distinct statement letters appearing in your wff/argument/set of wffs of PL, then create \(2^n\) rows in your truth table. Arrange the statement letters alphabetically (lower subscripts first), and then put \(2^n/2\) ‘T’s, followed by \(2^n/2\) ‘F’s, under the first statement letter. For the next statement letter (if there is on), put \(2^n/4\) ‘T’s, followed by \(2^n/4\) ‘F’s, followed by \(2^n/4\) ‘T’s, followed by \(2^n/4\) ‘F’s. In general, for the \(i\)th statement letter, put \(2^n/2^i\) ‘T’s, followed by \(2^n/2^i\) ‘F’s, and so on, until all the rows are filled. Complete this until you’ve written out a row for every statement letter.

Why we do it this way: because this way, we’ll end up representing every possible assignment of truth-values to the statement letters appearing in the wff/argument/set of wffs. So we’ll be sure to consider every possible case. If we didn’t do it in this systematic way, we might end up leaving some possibility out, and incorrectly concluding that something was a tautology when it’s not, or that an argument is valid when it’s not, or what-have-you.

Example: if \(n = 1\), then we need \(2^1 = 2\) rows in our truth-table. Under the first (i.e., only) statement letter, we put \(2^1/2 = 1\) ‘T’ followed by \(2^1/2 = 1\) ‘F’, and we’re done.

\[
\begin{array}{c|c}
A & \hline
T & \\
F & \\
\end{array}
\]

If \(n = 2\), then we need \(2^2 = 4\) rows in our truth-table. Under the first statement letter, we put \(2^2/2 = 2\) ‘T’s, followed by \(2^2/2 = 2\) ‘F’s. Under the second statement letter, we put \(2^2/2^2 = 1\) ‘T’, followed by \(2^2/2^2 = 1\) ‘F’, followed by \(2^2/2^2 = 1\) ‘T’, followed by \(2^2/2^2 = 1\) ‘F’, and we’re done.

\[
\begin{array}{c|c}
A & B \\
T & T \\
T & F \\
F & T \\
F & F \\
\end{array}
\]

If \(n = 3\), then we need \(2^3 = 8\) rows in our truth-table. Under the first statement letter, we put \(2^3/2 = 4\) ‘T’s, followed by \(2^3/2 = 4\) ‘F’s. Under the second statement letter, we put \(2^3/2^2 = 2\) ‘T’s, followed by \(2^3/2^2 = 2\) ‘F’s, followed by \(2^3/2^2 = 2\) ‘T’s, and so on, until we fill the column. Under the third statement letter, we put \(2^3/2^3 = 1\) ‘T’ followed
by $2^3/2^3 = 1 \ 'F'$, followed by $2^3/2^3 = 1 \ 'T'$, and so on, until we fill the column.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

If $n = 4$, then we need $2^4 = 16$ rows in our truth-table. Under the first statement letter, we write $2^4/2^2 = 8 \ 'T'$'s, followed by $2^4/2 = 8 \ 'F'$'s. Under the second statement letter, we write $2^4/2^2 = 4 \ 'T'$'s, followed by $2^4/2 = 4 \ 'F'$'s, followed by $2^4/2 = 4 \ 'T'$'s, and so on, until we fill the column. Under the third statement letter, we write $2^4/2^3 = 2 \ 'T'$'s, followed by $2^4/2^3 = 2 \ 'F'$'s, followed by $2^4/2^3 = 2 \ 'T'$'s, and so on, until we fill the column. Finally, under the final statement letter, we write $2^4/2^4 = 1 \ 'T'$, followed by $2^4/2^4 = 1 \ 'F'$, followed by $2^4/2^4 = 1 \ 'T'$, and so on, until we fill the column.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
§3.5. Logical Properties of PL

3.5.2 PL-Validity

A bit of new notation: If we have an argument from the premises \( p, q, r \) to the conclusion \( s \), then, rather than writing this as we have been, like so

\[
p \\
quill \\
r \\
s
\]

we will denote the argument by putting single forward slashes between premises, and putting a double forward slash between the premises and the conclusion, like so:

\[
p / q / r // s
\]

We can now define a whole host of interesting logical notions in the language \( PL \) with the aid of truth-tables.

An argument is **PL-valid** if and only if every row of the truth-table in which all of the premises are true is a row in which the conclusion is true, also.

An argument is **PL-invalid** if and only if there is some row of the truth-table in which all of the premises are true and in which the conclusion is false.

A **PL-counterexample** to the PL-validity of an argument is a row of the truth table in which all of the premises are true and the conclusion is false.

An argument is **PL-valid** if and only if it has no PL-counterexample.

3.5.3 PL-Consistency and PL-Inconsistency

A set of wffs of PL is **PL-consistent** if and only if there is some row of the truth table in which all of the wffs are true.

A set of wffs of PL is **PL-inconsistent** if and only if there is no row of the truth table in which all of the wffs are true (i.e., if and only if, in every row of the truth table, at least one of the wffs in the set is false).
3.5.4 PL-Equivalence and PL-Contradiction

Two wffs are PL-equivalent if and only if there is no row of the truth table in which they have different truth values (i.e., if and only if their truth values match in every row of the truth table).

Two wffs are PL-contradictory if and only if there is no row of the truth table in which they have the same truth value (i.e., if and only if they have different truth values in every row of the truth table).

3.5.5 PL-tautologies, PL-self-contradictions, and PL-contingencies

A wff is a PL-tautology if and only if it is true in every row of the truth-table.

A wff is a PL-self-contradiction if and only if it is false in every row of the truth-table.

A wff is PL-contingent if and only if it is true in some rows of the truth table and false in some other rows.

3.5.6 The Relationship Between The Notions

A set of wffs of PL, \( \{q_1, q_2, ..., q_N\} \) is PL-consistent iff the argument

\[
q_1 / q_2 / ... / q_{N-1} // \sim q_N
\]

is PL-invalid

A set of wffs of PL, \( \{q_1, q_2, ..., q_N\} \) is PL-inconsistent iff the argument

\[
q_1 / q_2 / ... / q_{N-1} // \sim q_N
\]

is PL-valid

Two wffs of PL, \( p \) and \( q \), are PL-equivalent iff

\[
A \lor \sim A // p \equiv q
\]

is PL-valid
§3.5. **Logical Properties of PL**

Two wffs of PL, \( p \) and \( q \), are PL-contradictory iff

\[
p \equiv q \Leftrightarrow A \cdot \sim A
\]

is PL-valid

A wff of PL, \( p \), is a PL-tautology iff

\[
A \lor \sim A \Leftrightarrow p
\]

is PL-valid

A wff of PL, \( p \), is a PL-self-contradiction iff

\[
p \Leftrightarrow A \cdot \sim A
\]

is PL-valid

A wff of PL, \( p \), is PL-contingent iff both

\[
A \lor \sim A \Leftrightarrow p
\]

and

\[
p \Leftrightarrow A \cdot \sim A
\]

are PL-invalid

---

**Exercises**

A. **Translations.** Translate the following English sentences into PL, using the following translation guide: \( A \) = ‘Abelard loved Heloise’, \( B \) = ‘Abelard loved philosophy’ and \( C \) = ‘Heloise loved philosophy’.

1. Abelard either loved Heloise or philosophy.
2. If Abelard didn’t love philosophy, then he didn’t love Heloise, either.
3. If Heloise didn’t love philosophy, then Abelard didn’t love her—unless Abelard didn’t love philosophy either.
4. Abelard loved Heloise only if she loved either philosophy or him.

B. **Validity.** Use truth-tables to determine whether the following arguments are deductively valid or deductively invalid.

1. \( P \supset R \Leftrightarrow \sim (P \supset P) \supset R \)
Chapter 3. Propositional Logic

2. \( P \lor Q / P \supset \sim Q / \sim Q \)
3. \( (A \equiv B) \supset A / \sim A / B \)

C. consistency/inconsistency. Use truth-tables to determine whether the following pairs of wffs of PL are consistent or inconsistent.

1. (a) \( (K \lor M) \supset \sim K \)
   (b) \( \sim M \)
2. (a) \( (L \cdot W) \equiv (W \lor L) \)
   (b) \( \sim W \cdot \sim L \)

D. equivalent/contradictory. Use truth-tables to determine whether the following pairs of wffs of PL are equivalent or contradictory.

1. (a) \( \sim (A \lor B) \)
   (b) \( \sim A \cdot \sim B \)
2. (a) \( P \supset Q \)
   (b) \( \sim (\sim P \lor Q) \)

D. tautologous/self-contradictory/contingent. Use truth-tables to determine whether the following wffs of PL are tautologous, self-contradictory, or contingent.

1. \( P \supset (Q \supset P) \)
2. \( P \supset (Q \supset \sim P) \)
3. \( \sim ((P \cdot (Q \lor R)) \supset ((P \lor Q) \cdot (P \lor R))) \)
Propositional Logic Derivations

The truth-table method of checking for PL-validity and PL-invalidity can be prohibitively difficult when the number of statement letters appearing in the argument are large. For instance, consider the following argument:

\[(P \equiv Q) \supset R / R \equiv S / S \equiv T / T \equiv U / U \equiv V / \sim V \equiv (P \bullet \sim Q) \vee (\sim P \bullet Q)\]

This argument is PL-valid. However, checking the validity of this argument with a truth table would require a table with \(2^7 = 128\) rows.

In this section of the course, we’re going to learn how to establish the validity arguments involving many statement letters much more simply. We will, at the same time, acquire the ability to think through which wffs PL-follow from which other wffs.

4.1 The Basics

To begin with: a PL-derivation consists of a certain number of lines, each one numbered. On each line of the derivation, we have a wff of PL along with a justification explaining why we get to write that wff down on that line—unless that wff is one of the premises of the argument we are attempting to show to be valid.

If the derivation is to be legal, then the formulae appearing on each line must be wffs of PL. Additionally, each line with a justification must follow from the lines cited in the justification, along with the rule cited in the justification. Moreover, the lines cited must precede the line on which the justification is written. You may not justify a line by citing a line beneath it in the derivation. Only lines preceding a given line are accessible from that line; and only accessible lines may be legally cited in a justification.

Fact: If there is a legal PL-derivation which has the wffs \(p_1, p_2, \ldots, p_N\) as assumptions and has \(q\) appearing on its final line, then \(p_1 / p_2 / \ldots / p_N / / q\) is a PL-valid argument.
4.2 Rules of Implication

The first set of rules are rules of implication. What makes these rules of implication are that they are one way. While the lines cited in the justification do entail the wff which is so justified (i.e., the argument from the lines cited in the justification to the justified wff is PL-valid), the justified wff does not entail the lines cited in the justification (i.e., the argument from the justified wff to the lines cited in the justification is not PL-valid). You could check the PL-validity with truth-tables, if you wanted.

The first rule is known as modus ponens.

\[
\text{Modus Ponens (MP)}
\]

\[
\begin{array}{c}
p \supset q \\
p \\
\therefore q
\end{array}
\]

Here’s how to read this rule. It says: if you have a wff of the form ‘p’ written down on an accessible line, and you have a wff ‘p \supset q’ written down on an accessible line, then you can write down ‘q’. When you justify your use of this rule, you should cite the line numbers that ‘p’ and ‘p \supset q’ were written on, and write ‘MP’.

The next rule is known as modus tollens.

\[
\text{Modus Tollens (MT)}
\]

\[
\begin{array}{c}
p \supset q \\
\sim q \\
\therefore \sim p
\end{array}
\]

This rule says: if you have a wff of the form ‘p \supset q’ written down on an accessible line, and you have a wff of the form ‘\sim q’ written down on an accessible line, then you may write down ‘\sim p’. When you justify your use of this rule, you should cite the line numbers on which ‘p \supset q’ and ‘\sim q’ appeared and write ‘MT’.

A Sample Derivation

\[
\begin{array}{ll}
1 & \sim C \supset (A \supset C) \\
2 & \sim C \quad \frac{! \sim A}{! \sim A} \\
3 & A \supset C \quad 1, 2, MP \\
4 & \sim A \quad 2, 3, MT
\end{array}
\]
In the derivation, lines 1 and 2 don’t have any justifications written next to them. That's because they are the premises of the argument, and don’t require justification. The ‘!/\ A’ written on line 2 indicates that ‘!/\ A’ is the conclusion to be derived from the wffs appearing on lines 1 and 2.

\[
\text{Hypothetical Syllogism (HS)}
\]

\[
\begin{align*}
p & \supset q \\
q & \supset r \\
\therefore p & \supset r
\end{align*}
\]

This rule says: if you have a wff of the form ‘p \supset q’ on an accessible line, and you have a wff of the form ‘q \supset r’ on an accessible line, then you may write down ‘p \supset q’ on an accessible line. When you justify your use of this rule, you must cite the line numbers on which ‘p \supset q’ and ‘q \supset r’ appeared and write ‘HS’.

\[
\text{Disjunctive Syllogism (DS)}
\]

\[
\begin{align*}
p & \lor q \\
\neg p & \\
\therefore q
\end{align*}
\]

This rule says: if you have a wff of the form ‘p \lor q’ on an accessible line, and you have a wff of the form ‘\neg p’ on an accessible line, then you may write down ‘q’. In your justification, you should write the lines on which ‘p \lor q’ and ‘\neg p’ appear, and ‘DS’.

**NOTE:** In DS, the order of the disjuncts in a disjunction matters. The following is not a legal derivation:

\[
\begin{align*}
1 & A \lor B \\
2 & \neg B \\
3 & A & 1, 2, DS & \leftarrow \text{MISTAKE!!!}
\end{align*}
\]

For lines 1 and 2 are of the form ‘p \lor q’ and ‘\neg q’. However, DS only tells us what we can do with lines of the form ‘p \lor q’ and ‘\neg p’. So DS does not tell us that we may infer ‘A’ from ‘A \lor B’ and ‘\neg B’.

This, however, is a legal derivation:

\[
\begin{align*}
1 & A \lor B \\
2 & \neg A \\
3 & B & 1, 2, DS
\end{align*}
\]
§4.2. Rules of Implication

NOTE: It is not enough to have a line which is PL-equivalent to \( \sim p \). The line must actually be of the form \( \sim p \). For instance, the following derivation is not legal:

\[
\begin{align*}
1 & \quad \sim A \lor B \\
2 & \quad A \\
3 & \quad B & 1, 2, DS & \text{MISTAKE!!!}
\end{align*}
\]

This derivation, on the other hand, is legal:

\[
\begin{align*}
1 & \quad \sim A \lor B \\
2 & \quad \sim \sim A \\
3 & \quad B & 1, 2, DS
\end{align*}
\]

A Sample Derivation

\[
\begin{align*}
1 & \quad B \supset S \\
2 & \quad S \supset (T \lor U) \\
3 & \quad B \\
4 & \quad \sim T & /U \\
5 & \quad B \supset (T \lor U) & 1, 3, HS \\
6 & \quad T \lor U & 3, 5, MP \\
7 & \quad U & 4, 6, DS
\end{align*}
\]

This rule says: if you have a formula of the form \( p \cdot q \) written on an accessible line, then you may write down \( p \). Your justification should cite the line number on which \( p \cdot q \) appears and say ‘Simp’.

NOTE: Here, too, the order of the conjuncts in \( p \cdot q \) matters. The following is not a legal derivation:

\[
\begin{align*}
1 & \quad (A \equiv B) \cdot \sim (C \supset D) \\
2 & \quad \sim (C \supset D) & 1, \text{Simp} & \text{MISTAKE!!!}
\end{align*}
\]
However, the following is a legal derivation:

1. \((A \equiv B) \cdot \sim (C \supset D)\)
2. \(A \equiv B\)  1. *Simp*

### Conjunction (Conj)
- \(p\)
- \(q\)
- \(\triangleright p \cdot q\)

This rule says that, if you have a wff of the form ‘\(p\)’ written on an accessible line, and you have a wff of the form ‘\(q\)’ written on an accessible line, then you may write ‘\(p \cdot q\)’. Your justification should cite the line number of the line on which ‘\(p\)’ appears, the line number of the line on which ‘\(q\)’ appears, and say ‘*Conj*’.

### Addition (Add)
- \(p\)
- \(\triangleright p \lor q\)

This rule says that, if you have a wff of the form ‘\(p\)’ written on an accessible line, then you may write any wff of the form ‘\(p \lor q\)’. Your justification should cite the line number of the line on which ‘\(p\)’ appears and say ‘*Add*’.

**NOTE:** the order of the disjuncts in ‘\(p \lor q\)’ matters. For instance, the following is not a legal derivation:

1. \(C \supset (D \supset E)\)
2. \((Z \equiv W) \lor (C \supset (D \supset E))\)  1. *Add*  ← **MISTAKE!!!**

However, this is a legal derivation.

1. \(C \supset (D \supset E)\)
2. \((C \supset (D \supset E)) \lor (Z \equiv W)\)  1. *Add*
§4.2. Rules of Implication

**Constructive Dilemma (CD)**

\[
(p \supset q) \land (r \supset s) \\
p \lor r \\
\therefore q \lor s
\]

This rule says the following: if you have a wff of the form \((p \supset q) \land (r \supset s)\) written on an accessible line and a wff of the form \(p \lor r\), then you may write down \(q \lor s\). In your justification, you should cite the line numbers of the lines on which \((p \supset q) \land (r \supset s)\) and \(p \lor r\) appear, and write 'CD'.

**NOTE:** one of the lines appealed to must be of the form \((p \supset q) \land (r \supset s)\). You may not appeal to two lines, one of the form \(p \supset q\) and one of the form \(r \supset s\). For instance, the following derivation is not legal:

1. \(A \supset (Q \lor R)\)
2. \(B \supset (T \equiv V)\)
3. \(A \lor B\)
4. \((Q \lor R) \lor (T \equiv V)\) \(1, 2, 3, CD \leftarrow \text{MISTAKE!!!}\)

However, the following derivation is legal:

1. \(A \supset (Q \lor R)\)
2. \(B \supset (T \equiv V)\)
3. \(A \lor B\)
4. \((A \supset (Q \lor R)) \land (B \supset (T \equiv V))\) \(1, 2, \text{Conj}\)
5. \((Q \lor R) \lor (T \equiv V)\) \(3, 4, CD\)

**NOTE:** Here, too, the order of both the conjuncts in \((p \supset q) \land (r \supset s)\) and the disjuncts in \(p \lor r\) and \(q \lor s\) matters. For instance, the following derivations are not legal:

1. \((A \supset B) \land (C \supset D)\)
2. \(C \lor A\)
3. \(B \lor D\) \(1, 2, CD \leftarrow \text{MISTAKE!!!}\)

1. \((A \supset B) \land (C \supset D)\)
2. \(A \lor C\)
3. \(D \lor B\) \(1, 2, CD \leftarrow \text{MISTAKE!!!}\)
This, however, is a legal derivation:

\[
\begin{align*}
1 & \quad (A \supset B) \cdot (C \supset D) \\
2 & \quad A \lor C \\
3 & \quad B \lor D & 1, 2, CD
\end{align*}
\]

A Sample Derivation

\[
\begin{array}{ll}
1 & A \cdot B \\
2 & (A \lor C) \supset ((D \supset E) \cdot F) \\
3 & G \supset H \\
4 & D \lor G & 1, \text{Simp} \\
5 & A & 5, \text{Add} \\
6 & A \lor C \\
7 & (D \supset E) \cdot F & 2, 6, \text{MP} \\
8 & D \supset E & 7, \text{Simp} \\
9 & (D \supset E) \cdot (G \supset H) & 3, 8, \text{Conj} \\
10 & E \lor H & 4, 9, \text{CD}
\end{array}
\]

4.2.1 A Mistake to Avoid

**Rules of implication may not be applied to subformulae.** For instance, the following derivation is not legal.

\[
\begin{align*}
1 & \quad P \supset (Q \supset R) \\
2 & \quad Q \\
3 & \quad P \supset R & 1, 2, \text{MP} \quad \leftarrow \text{MISTAKE!!!}
\end{align*}
\]

Modus Ponens allows you to write down ‘R’ if you have ‘Q \supset R’ written down on an accessible line and ‘Q’ written down on an accessible line. However, it does not allow you to swap out ‘R’ for ‘Q \supset R’ if you have ‘Q’ written on an accessible line, and ‘Q \supset R’ is merely a subformulae of a wff on an accessible line.
4.3 Rules of Replacement

The rules in this section are known as rules of replacement. What makes them rules of replacement is that they are two way. They allow you to substitute one wff of PL for another, where the substituted wff of PL is PL-equivalent to the one it replaces. You could check this with truth-tables, if you wanted.

Rules of Replacement, unlike Rules of Implication, may be applied to subformulae.

\[
\begin{align*}
\text{DeMorgan's (DM)} \\
\sim (p \cdot q) & \Leftrightarrow \sim p \lor \sim q \\
\sim (p \lor q) & \Leftrightarrow \sim p \cdot \sim q
\end{align*}
\]

This rule actually allows four distinct replacements (one corresponding to each ‘\(\Leftrightarrow\)’). It says:

1. if you have a subformula of the form ‘\(\sim (p \cdot q)\)’ within a wff on an accessible line, you may replace that subformula with ‘\(\sim p \lor \sim q\)’. When you do so, cite the line on which the wff containing ‘\(\sim (p \cdot q)\)’ appears and write ‘DM’.

2. Similarly, if you have a subformula of the form ‘\(\sim p \lor \sim q\)’ within a wff on an accessible line, you may replace that subformula with ‘\(\sim (p \cdot q)\)’. When you do so, cite the line on which the wff containing ‘\(\sim p \lor \sim q\)’ appears and write ‘DM’.

3. Additionally, if you have a wff of the form ‘\(\sim (p \lor q)\)’ within a wff on an accessible line, you may replace that subformula with ‘\(\sim p \cdot \sim q\)’. When you do so, cite the line on which the wff containing ‘\(\sim (p \lor q)\)’ appears and write ‘DM’.

4. Similarly, if you have a subformula of the form ‘\(\sim p \cdot \sim q\)’ within a wff on an accessible line, you may replace that subformula with ‘\(\sim (p \lor q)\)’. When you do so, cite the line on which the wff containing ‘\(\sim p \cdot \sim q\)’ appears and write ‘DM’.

**NOTE**: In order for DeMorgan’s rule to apply, ‘\(\sim (p \lor q)\)’ (for example) must actually be a subformula of a wff on an accessible line. For instance, the following derivation is not legal:

\[
\begin{align*}
1 & \quad A \supset (\sim B \lor C) \\
2 & \quad A \supset (\sim B \cdot \sim C) \quad 1, \, DM \quad \leftarrow \text{MISTAKE!!!}
\end{align*}
\]

Here, ‘\(\sim B \lor C\)’ is not of the form ‘\(\sim (p \lor q)\)’ (because it is missing the parentheses). However, the following derivation is legal:

\[
\begin{align*}
1 & \quad A \supset \sim (B \lor C) \\
2 & \quad A \supset (\sim B \cdot \sim C) \quad 1, \, DM
\end{align*}
\]
Chapter 4. Propositional Logic Derivations

\[
\text{Commutativity (Com)}
\]
\[
\begin{align*}
p \lor q & \Rightarrow q \lor p \\
p \land q & \Rightarrow q \land p
\end{align*}
\]

This rule says:

1. If you have a subformula of the form \( p \lor q \) within a wff appearing on an accessible line, then you may replace that subformula with \( q \lor p \). When you do so, cite the line number on which the wff containing \( p \lor q \) appears and write \( \text{Com} \).

2. If you have a subformula of the form \( p \land q \) within a wff appearing on an accessible line, then you may replace that subformula with \( q \land p \). When you do so, cite the line number on which the wff containing \( p \land q \) appears and write \( \text{Com} \).

\[
\text{Associativity (Assoc)}
\]
\[
\begin{align*}
(p \lor q) \lor r & \Leftarrow p \lor (q \lor r) \\
(p \land q) \land r & \Leftarrow p \land (q \land r)
\end{align*}
\]

This rule says:

1. If you have a subformula of the form \( (p \lor q) \lor r \) within a wff appearing on an accessible line, then you may replace that subformula with \( p \lor (q \lor r) \). When you do so, cite the line number on which the wff containing \( (p \lor q) \lor r \) appears and write \( \text{Assoc} \).

2. Similarly, if you have a subformula of the form \( p \lor (q \lor r) \) within a wff appearing on an accessible line, then you may replace that subformula with \( (p \lor q) \lor r \). When you do so, cite the line number on which the wff containing \( p \lor (q \lor r) \) appears and write \( \text{Assoc} \).

3. Additionally, if you have a subformula of the form \( (p \land q) \land r \) within a wff appearing on an accessible line, then you may replace that subformula with \( p \land (q \land r) \). When you do so, cite the line number on which the wff containing \( (p \land q) \land r \) appears and write \( \text{Assoc} \).

4. Similarly, if you have a subformula of the form \( p \land (q \land r) \) within a wff appearing on an accessible line, then you may replace that subformula with \( (p \land q) \land r \). When you do so, cite the line number on which the wff containing \( p \land (q \land r) \) appears and write \( \text{Assoc} \).

\textbf{NOTE:} It is important that the subformula contains only \( \lor \)'s or only \( \land \)'s. For instance, the following derivation is not legal:

\[
\begin{align*}
1 & \quad A \equiv (B \lor (C \land D)) \\
2 & \quad A \equiv ((B \lor C) \land D) \quad \text{1, Assoc} \quad \text{MISTAKE!!!}
\end{align*}
\]
§4.3. Rules of Replacement

This derivation, on the other hand, is legal:

1. \[ A \equiv (B \lor (C \lor D)) \]
2. \[ A \equiv ((B \lor C) \lor D) \quad 1, \text{Assoc} \]

\begin{align*}
\text{Distribution (Dist)} \\
p \cdot (q \lor r) &\leftrightarrow (p \cdot q) \lor (p \cdot r) \\
p \lor (q \cdot r) &\leftrightarrow (p \lor q) \cdot (p \lor r)
\end{align*}

This rule says:

1. If you have a subformula of the form ‘\( p \cdot (q \lor r) \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( (p \cdot q) \lor (p \cdot r) \)’.
   When you do so, cite the line number on which the wff containing ‘\( p \cdot (q \lor r) \)’ appears and write ‘Dist’.

2. Similarly, if you have a subformula of the form ‘\( (p \cdot q) \lor (p \cdot r) \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( p \cdot (q \lor r) \)’.
   When you do so, cite the line number on which the wff containing ‘\( (p \cdot q) \lor (p \cdot r) \)’ appears and write ‘Dist’.

3. Additionally, if you have a subformula of the form ‘\( (p \cdot q) \cdot r \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( p \cdot (q \cdot r) \)’.
   When you do so, cite the line number on which the wff containing ‘\( (p \cdot q) \cdot r \)’ appears and write ‘Dist’.

4. Similarly, if you have a subformula of the form ‘\( p \cdot (q \cdot r) \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( (p \cdot q) \cdot r \)’.
   When you do so, cite the line number on which the wff containing ‘\( p \cdot (q \cdot r) \)’ appears and write ‘Dist’.

**NOTE:** When you apply Distribution to a subformula, the main operator of that subformula should change either from a ‘\( \lor \)’ to a ‘\( \cdot \)’ or from a ‘\( \cdot \)’ to a ‘\( \lor \)’. For instance, the following is not a legal derivation:

1. \[ P \equiv (A \cdot (B \lor C)) \]
2. \[ P \equiv ((A \lor B) \cdot (A \lor C)) \quad 1, \text{Dist} \quad \text{MISTAKE!!!} \]

This derivation, on the other hand, is legal:

1. \[ P \equiv (A \cdot (B \lor C)) \]
2. \[ P \equiv ((A \cdot B) \lor (A \cdot C)) \quad 1, \text{Dist} \]
NOTE: As with Disjunctive Syllogism and Simplification, the order of the disjuncts and conjuncts matters. The following derivation is not legal:

1  \( P \equiv ((B \lor C) \cdot A) \)
2  \( P \equiv ((B \cdot A) \lor (C \cdot A)) \)  \( 1, \text{Dist} \leftarrow \text{MISTAKE!!!} \)

This derivation, however, is legal:

1  \( P \equiv ((B \lor C) \cdot A) \)
2  \( P \equiv (A \cdot (B \lor C)) \)  \( 1, \text{Com} \)
3  \( P \equiv ((A \cdot B) \lor (A \cdot C)) \)  \( 2, \text{Dist} \)

\[ \text{Double Negation (DN)} \]
\[ p \iff \neg\neg p \]

This rule says:

1. If you have a subformula of the form ‘\( p \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( \neg\neg p \)’. When you do so, cite the line number on which the wff containing ‘\( p \)’ appears and write ‘DN’.

2. If you have a subformula of the form ‘\( \neg\neg p \)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\( p \)’. When you do so, cite the line number on which the wff containing ‘\( p \)’ appears and write ‘DN’.
§4.3. Rules of Replacement

A Sample Derivation

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(A \lor (B \cdot C))</td>
</tr>
<tr>
<td>2</td>
<td>(\sim B)</td>
</tr>
<tr>
<td>3</td>
<td>(\sim \sim A \supset (D \lor E))</td>
</tr>
<tr>
<td>4</td>
<td>(\sim D \supset F)</td>
</tr>
<tr>
<td>5</td>
<td>((A \lor B) \cdot (A \lor C))</td>
</tr>
<tr>
<td>6</td>
<td>(A \lor B)</td>
</tr>
<tr>
<td>7</td>
<td>(B \lor A)</td>
</tr>
<tr>
<td>8</td>
<td>(A)</td>
</tr>
<tr>
<td>9</td>
<td>(\sim \sim A)</td>
</tr>
<tr>
<td>10</td>
<td>(\sim \sim A \supset (\sim D \cdot \sim E))</td>
</tr>
<tr>
<td>11</td>
<td>(\sim D \cdot \sim E)</td>
</tr>
<tr>
<td>12</td>
<td>(\sim D)</td>
</tr>
<tr>
<td>13</td>
<td>(F)</td>
</tr>
</tbody>
</table>

\[
\text{Transposition (Trans)}
\]
\[
\frac{p \supset q \iff q \supset \sim p}{\text{Trans}}
\]

This rule says:

1. If you have a subformula of the form ‘\(p \supset q\)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\(\sim q \supset \sim p\)’. When you do so, cite the line number on which the wff containing ‘\(p \supset q\)’ appears and write ‘Trans’.

2. If you have a subformula of the form ‘\(\sim q \supset \sim p\)’ within a wff appearing on an accessible line, then you may replace that subformula with ‘\(p \supset q\)’. When you do so, cite the line number on which the wff containing ‘\(\sim q \supset \sim p\)’ appears and write ‘Trans’.

**NOTE**: The subformula which you replace and the one with which you replace it **must actually be of the forms** ‘\(p \supset q\)’ and ‘\(\sim q \supset \sim p\)’. It is not enough that they are PL-equivalent to wffs of those forms. For instance, the following derivation is **not** legal:

1. \(A \equiv (B \supset \sim C)\)
2. \(A \equiv (C \supset \sim B)\) | 1, Trans ← **MISTAKE!!!**
This derivation, however, is legal:

1. \[ A \equiv (B \supset \sim C) \]
2. \[ A \equiv (\sim \sim C \supset \sim B) \quad 1, \text{Trans} \]
3. \[ A \equiv (C \supset B) \quad 2, \text{DN} \]

\[
\text{Material Implication (Impl)}
\]
\[ p \supset q \iff \sim p \lor q \]

This rule says:

1. If you have a subformula of the form \( p \supset q \) within a wff appearing on an accessible line, then you may replace that subformula with \( \sim p \lor q \). When you do so, cite the line number on which the wff containing \( p \supset q \) appears and write \( \text{Impl} \).

2. Similarly, if you have a subformula of the form \( \sim p \lor q \) within a wff appearing on an accessible line, then you may replace that subformula with \( p \supset q \). When you do so, cite the line number on which the wff containing \( \sim p \lor q \) appears and write \( \text{Impl} \).

\[
\text{Material Equivalence (Equiv)}
\]
\[ p \equiv q \iff (p \supset q) \land (q \supset p) \]

This rule says:

1. If you have a subformula of the form \( p \equiv q \) within a wff appearing on an accessible line, then you may replace that subformula with \( (p \supset q) \land (q \supset p) \). When you do so, cite the line number on which the wff containing \( p \equiv q \) appears and write \( \text{Equiv} \).

2. Similarly, if you have a subformula of the form \( (p \supset q) \land (q \supset p) \) within a wff appearing on an accessible line, then you may replace that subformula with \( p \equiv q \). When you do so, cite the line number on which the wff containing \( (p \supset q) \land (q \supset p) \) appears and write \( \text{Equiv} \).

3. Additionally, if you have a subformula of the form \( p \equiv q \) within a wff appearing on an accessible line, then you may replace that subformula with \( (p \land q) \lor (\sim p \land \sim q) \). When you do so, cite the line number on which the wff containing \( p \equiv q \) appears and write \( \text{Equiv} \).

4. Similarly, if you have a subformula of the form \( (p \land q) \lor (\sim p \land \sim q) \) within a wff appearing on an accessible line, then you may replace that subformula with \( p \equiv q \). When you do so, cite the line number on which the wff containing \( (p \land q) \lor (\sim p \land \sim q) \) appears and write \( \text{Equiv} \).

75
NOTE: Here, as with Disjunctive Syllogism and Simplification, the order of the disjuncts and the conjuncts matter.

**Exportation (Exp)**

\[(p \cdot q) \supset r \iff p \supset (q \supset r)\]

This rule says:

1. If you have a subformula of the form \((p \cdot q) \supset r\) within a wff appearing on an accessible line, then you may replace that subformula with \(p \supset (q \supset r)\). When you do so, cite the line number on which the wff containing \((p \cdot q) \supset r\) appears and write ‘Exp’.

2. If you have a subformula of the form \(p \supset (q \supset r)\) within a wff appearing on an accessible line, then you may replace that subformula with \((p \cdot q) \supset r\). When you do so, cite the line number on which the wff containing \(p \supset (q \supset r)\) appears and write ‘Exp’.

**Tautology (Taut)**

\[
p \iff p \lor p \\
p \iff p \cdot p
\]

This rule says:

1. If you have a subformula of the form ‘p’ within a wff appearing on an accessible line, then you may replace that subformula with ‘p \lor p’. When you do so, cite the line number on which the wff containing ‘p’ appears and write ‘Taut’.

2. Similarly, if you have a subformula of the form ‘p \lor p’ within a wff appearing on an accessible line, then you may replace that subformula with ‘p’. When you do so, cite the line number on which the wff containing ‘p \lor p’ appears and write ‘Taut’.

3. Additionally, if you have a subformula of the form ‘p’ within a wff appearing on an accessible line, then you may replace that subformula with ‘p \cdot p’. When you do so, cite the line number on which the wff containing ‘p’ appears and write ‘Taut’.

4. Similarly, if you have a subformula of the form ‘p \cdot p’ within a wff appearing on an accessible line, then you may replace that subformula with ‘p’. When you do so, cite the line number on which the wff containing ‘p \cdot p’ appears and write ‘Taut’.
A Sample Derivation

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A \equiv \sim A$</td>
</tr>
<tr>
<td>2</td>
<td>$(A \cdot \sim A) \lor (\sim A \cdot \sim A)$</td>
</tr>
<tr>
<td>3</td>
<td>$(\sim A \cdot A) \lor (\sim A \cdot \sim A)$</td>
</tr>
<tr>
<td>4</td>
<td>$\sim A \cdot (A \lor \sim A)$</td>
</tr>
<tr>
<td>5</td>
<td>$\sim A \cdot (A \lor A)$</td>
</tr>
<tr>
<td>6</td>
<td>$\sim A \cdot A$</td>
</tr>
<tr>
<td>7</td>
<td>$A \cdot \sim A$</td>
</tr>
</tbody>
</table>

Exercises

A. rules of implication, 1. Provide PL-derivations to establish that the following arguments are PL-valid.

1. $(A \equiv B) \lor (B \equiv A) / F \lor \sim (B \lor A) / \sim F / \sim (A \equiv B)$
2. $(P \cdot Q) \lor R / R \lor (U \lor V) / \sim (U \lor V) / / \sim (P \cdot Q)$

B. rules of implication, 2. Provide PL-derivations to establish that the following arguments are PL-valid.

1. $A \lor B / H \lor A / H \lor (A \lor B) / \sim (A \lor B) / / B$
2. $X / ((X \lor Y) \lor Z) \lor ((P \lor Q) \cdot R) / (X \lor W) \lor ((S \lor T) \cdot U) / P \lor S / / Q \lor T$

C. rules of replacement, 1. Provide PL-derivations to establish that the following arguments are PL-valid.

1. $(A \lor B) \lor \sim (C \cdot D) / (B \cdot E) \cdot F / C / / \sim D$
2. $\sim \sim A \cdot B / A \lor (\sim C \lor \sim D) / E \lor (C \cdot D) / / E$

D. rules of replacement, 2. Provide PL-derivations to establish that the following arguments are PL-valid.

1. $P \lor (Q \lor R) / \sim Q \cdot \sim R / T \lor P / / \sim T$
2. $P \lor (Q \cdot R) / \sim Q / (\sim P \cdot A) \lor S / / \sim A \lor S$
4.4 Four Final Rules of Inference

Last time, we covered 38 rules. We’ve just got four more to cover. However, these rules are very special—in part, because they are so powerful; and in part, because they are altogether different from the rules which preceded them.

4.4.1 Subderivations

First, we need to introduce the idea of a subderivation. A subderivation is a kind of suppositional derivation which takes place within another derivation. To indicate that the subderivation is suppositional, we indent those lines of the derivation which are taking place in the subderivation and place a scope line to the left of all those wffs which are within the scope of the supposition. For instance, the following is a derivation utilizing a subderivation.

1  \((A \supset B) \supset C\)
2  \(B \cdot D\)    \(/C\)
3  \(A\)        \(ACP\)
4  \(B\)         \(2, \text{Simp}\)
5  \(A \supset B\)  \(3-4, \text{CP}\)
6  \(C\)          \(1, 5, \text{MP}\)

The subderivation takes place from lines 3–4, as indicated by the indentation and the vertical scope line which runs from line 3 to line 4.

The intuitive idea behind a subderivation is this: even if our premises don’t tell us that \(p\), we might just want to suppose that \(p\) is true, and see what follows from this supposition. Our first two new rules tell us that we may suppose anything that we wish—bar none.

**Assumption for Conditional Proof (ACP)**
You may, at any point in a derivation, begin a new subderivation, and write any wff of PL whatsoever on the first line of that sub derivation. In the justification line, you should write ‘ACP’.

**Assumption for Indirect Proof (AIP)**
You may, at any point in a derivation, begin a new subderivation, and write any wff of PL whatsoever on the first line of that sub derivation. In the justification line, you should write ‘AIP’.

The only difference between these two rules is the justification that you provide. Those justifications will become relevant later on, as they will end up making a difference for what you get to use your subderivations to show outside of the subderivation.
You may also decide to end a subderivation whenever you wish. Now, given the way that we defined accessibility last time, these new rules threaten to make it far too easy to prove anything whatsoever. For instance, given $ACP$, there is as yet nothing to rule out the following derivation:

\[
\begin{array}{c}
1 & A \supset B & / \sim A \\
2 & \sim B & ACP \\
3 & \sim A & 1, 2, MT
\end{array}
\]

If our derivation system could be used to derive $\sim A$ from $A \supset B$, that would be disaster, since the argument $A \supset B \parallel \sim A$ is not $PL$-valid. Fortunately, we don’t allow this, since we place the following new restriction on which lines are accessible, and thus available to be legally cited, at a given line in the derivation:

At a given line in a derivation, $n$, another line of the derivation, $m$, is accessible if and only if 1) line $m$ precedes line $n$ ($m < n$), and 2) either i) line $m$ lies outside the scope of any subderivation, or ii) line $m$ lies within a subderivation whose vertical scope line extends to line $n$.

Moreover, since the two new rules below will allow us to cite, not just individual lines within a derivation, but rather entire subderivations, we will have to define which subderivations are accessible at a given line:

An entire subderivation is accessible at line $n$ so long as 1) the subderivation precedes line $n$, and 2) either i) that subderivation is outside the scope of any other subderivation, or else ii) the subderivation lines within another subderivation whose vertical scope line extends to line $n$.

Another, simpler way of putting the same point is this: while you may end a subderivation whenever you wish, once you do so, none of the lines or subderivations appearing within the scope of that subderivation are accessible any longer.

For illustration, consider the following (legal) $PL$-derivation.
On line 4, line 1 is accessible because it lies outside the scope of any subderivation and it precedes line 4. Line 2 is accessible because, even though it lies within a subderivation, that subderivation continues through to line 4 (its vertical scope line continues through to line 5). Similarly, line 3 is accessible, since, even though it lies within a subderivation, that subderivation continues through to line 4.

Once we leave the subderivation running from lines 3–4, on line 5, line 1 is still accessible, as it lies outside the scope of any subderivation. Additionally, line 2 is still accessible, since the vertical scope line of the subderivation to which it belongs continues through to line 5. However, lines 3 and 4 are no longer accessible. They occur within the scope of a subderivation which does not continue through to line 5. Nevertheless, the entire subderivation running from lines 3–4 is still accessible. It may be legitimately cited in applying a rule at line 5 (as it is here in this derivation).

Note that this changes once we end the subderivation running from lines 2–5. On line 6, the subderivation running from lines 3–4 is no longer accessible. Neither are any of the
individual lines 2, 3, 4, or 5. Nevertheless, the entire subderivation running from lines 2–5 is accessible at line 6.

Similarly, down on line 9, neither line 2, 3, 4, 5, 7, nor 8 is accessible. However, lines 1 and 6 are accessible, as are the subderivations running from lines 2–6 and from lines 7–8.

4.4.2 Conditional Proof

With that background on subderivations out of the way, here is our third new rule of inference:

\[
\text{Conditional Proof (CP)}
\]

\[
\begin{array}{c|cc}
  n & p & ACP \\
  \vdots & \vdots \\
  m & q \\
  \hline
  \d & p \supset q & n-m, CP
\end{array}
\]

This rule says: if you have an accessible subderivation whose first line, \( n \), is a wff of the form \( 'p' \)—and that wff is justified by the rule \( ACP \)—and whose last line, \( m \), is a wff of the form \( 'q' \), then you may write down \( 'p \supset q' \). When you do so, you should cite the entire subderivation running from line \( n \) to line \( m \) (\( 'n-m' \)) and write \( 'CP' \).

The intuitive thought here is this: we make a supposition that \( p \) is true. From this supposition, we are able to derive that \( q \) is true. So, it should be that case, without any supposition, that if \( p \) is true, then \( q \) is true.

4.4.3 Indirect Proof

Here is the final—and most powerful—rule of inference.

\[
\text{Indirect Proof (IP)}
\]

\[
\begin{array}{c|cc}
  n & p & AIP \\
  \vdots & \vdots \\
  m & q \cdot \sim q \\
  \hline
  \d & \sim p & n-m, IP
\end{array}
\]

This rule says: if you have a subderivation whose first line, \( n \), is a wff of the form \( 'p' \)—and that line is justified by \( AIP \)—and whose last line is an explicit contradiction of the form
§4.4. Four Final Rules of Inference

$q \cdot \sim q$, then you may write down '$\sim p'$. When you do so, you should cite the entire subderivation running from line $n$ to line $m$ ('$n$–$m$') and write 'IP'.

NOTE: the explicit contradiction must be of the form '$q \cdot \sim q'$. The following derivation is not legal.

\begin{align*}
1 & \quad A \cdot B \\
2 & \quad \sim A & \text{AIP} \\
3 & \quad A & 1, \text{Simp} \\
4 & \quad \sim A \cdot A & 2, 3, \text{Conj} \\
5 & \quad \sim \sim A & 2–4, \text{IP} \quad \triangleright \text{MISTAKE!!!}
\end{align*}

This derivation, however, is legal:

\begin{align*}
1 & \quad A \cdot B \\
2 & \quad \sim A & \text{AIP} \\
3 & \quad A & 1, \text{Simp} \\
4 & \quad A \cdot \sim A & 2, 3, \text{Conj} \\
5 & \quad \sim \sim A & 2–4, \text{IP}
\end{align*}

NOTE: what you conclude outside of the subderivation must be the negation of the thing you assumed. The following derivation is not legal.

\begin{align*}
1 & \quad \sim (A \vee \sim A) & \text{AIP} \\
2 & \quad \sim A \cdot \sim \sim A & 1, \text{DM} \\
3 & \quad A \vee \sim A & 1–2, \text{IP} \quad \triangleright \text{MISTAKE!!!}
\end{align*}

This derivation, however, is legal.

\begin{align*}
1 & \quad \sim (A \vee \sim A) & \text{AIP} \\
2 & \quad \sim A \cdot \sim \sim A & 1, \text{DM} \\
3 & \quad \sim \sim (A \vee \sim A) & 1–2, \text{IP} \\
4 & \quad A \vee \sim A & 3, \text{DN}
\end{align*}
Thus far, we’ve been showing how to use derivations to show that arguments are PL-valid. However, we can also use derivations to establish other interesting facts about the logical notions of PL that we previously defined in terms of truth-tables. Let’s just focus on one in this section (the others are summarized in the next section): that of a PL-tautology.

We defined a PL-tautology to be a wff of PL that was true in every row of the truth table. However, it turns out (in a more advanced logic course, I would ask you to prove this) that a wff of PL, \( p \), is a PL-tautology if and only if there is a legal PL-derivation without any assumptions whose final line is \( p \). In that case, let’s say that \( p \) is ‘PL-derivable’ from no assumptions.

A wff of PL is a PL-tautology if and only if it is PL-derivable without any assumptions.

This is really a fantastic fact, and we should pause momentarily to marvel at it. This tells us that if there’s some way of constructing a PL-derivation according to the 40 rules that we’ve encountered here which has no assumptions and whose final line is \( p \), then \( p \) will be true in every row of the truth-table. Isn’t it fantastic—isn’t it nothing short of amazing—that these two procedures for discovering whether something is a PL-tautology should line up so nicely?

It’s no accident, since the derivation system was specifically designed for this purpose; but isn’t it a grand accomplishment that we got a derivation system which lines up so perfectly with the truth-table method for determining both PL-validity and PL-tautology, and—as we’ll see below—all of the other logical notions of PL as well?

Now that we’ve marveled appropriately: What is it for a PL-derivation to have no assumptions? We have already seen a PL-derivation without any assumptions (it’s at the top of this page). Every line of that PL-derivation has a justification written next to it, and all of
the justifications are legal. So, it is a legal PL-derivation without any assumptions. Given the astonishing fact above, that PL-derivation tells us that ‘A ∨ ~ A’ is a PL-tautology.

Here are some more examples of PL-derivations with no assumptions establishing that certain wffs of PL are PL-tautologies.

### Sample Derivations

<table>
<thead>
<tr>
<th>Step</th>
<th>Derivation</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>~((~P ⊃ Q) ∨ (P ⊃ R))</td>
<td>AIP</td>
</tr>
<tr>
<td>2</td>
<td>~((~P ⊃ Q) • (~P ⊃ R))</td>
<td>1, DM</td>
</tr>
<tr>
<td>3</td>
<td>~(~P ⊃ Q)</td>
<td>2, Simp</td>
</tr>
<tr>
<td>4</td>
<td>~(~P ∨ Q)</td>
<td>3, Impl</td>
</tr>
<tr>
<td>5</td>
<td>~~P • ~Q</td>
<td>4, DM</td>
</tr>
<tr>
<td>6</td>
<td>~~P</td>
<td>5, Simp</td>
</tr>
<tr>
<td>7</td>
<td>~(P ⊃ R) • (~P ⊃ Q)</td>
<td>2, Com</td>
</tr>
<tr>
<td>8</td>
<td>~(P ⊃ R)</td>
<td>7, Simp</td>
</tr>
<tr>
<td>9</td>
<td>~(~P ∨ R)</td>
<td>8, Impl</td>
</tr>
<tr>
<td>10</td>
<td>~~P • ~R</td>
<td>9, DM</td>
</tr>
<tr>
<td>11</td>
<td>~~P</td>
<td>10, Simp</td>
</tr>
<tr>
<td>12</td>
<td>~~P • ~~P</td>
<td>6, 11, Conj</td>
</tr>
<tr>
<td>13</td>
<td>~~~((~P ⊃ Q) ∨ (P ⊃ R))</td>
<td>1–12, IP</td>
</tr>
<tr>
<td>14</td>
<td>(~P ⊃ Q) ∨ (P ⊃ R)</td>
<td>13, DN</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step</th>
<th>Derivation</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>~(~P ⊃ Q)</td>
<td>ACP</td>
</tr>
<tr>
<td>2</td>
<td>~(~P ∨ Q)</td>
<td>1, Impl</td>
</tr>
<tr>
<td>3</td>
<td>~~~P • ~Q</td>
<td>2, DM</td>
</tr>
<tr>
<td>4</td>
<td>~~~P</td>
<td>3, Simp</td>
</tr>
<tr>
<td>5</td>
<td>~P</td>
<td>4, DN</td>
</tr>
<tr>
<td>6</td>
<td>~P ∨ R</td>
<td>5, Add</td>
</tr>
<tr>
<td>7</td>
<td>P ⊃ R</td>
<td>6, Impl</td>
</tr>
<tr>
<td>8</td>
<td>~((~P ⊃ Q) ⊃ (P ⊃ R))</td>
<td>1–7, CP</td>
</tr>
<tr>
<td>9</td>
<td>~~~((~P ⊃ Q) ∨ (P ⊃ R))</td>
<td>8, Impl</td>
</tr>
<tr>
<td>10</td>
<td>(~P ⊃ Q) ∨ (P ⊃ R)</td>
<td>9, DN</td>
</tr>
</tbody>
</table>
4.6 PL-Derivability and the Logical Notions of PL

If it is possible to construct a legal PL-derivation whose assumptions are \( p_1, p_2, ..., p_N \) and whose final line is \( q \), then I will write

\[
p_1, p_2, ..., p_N \vdash_{PL} q
\]

This expression just means ‘there is a possible legal PL-derivation whose assumptions are \( p_1, p_2, ..., p_N \), and whose final line is \( q \)’. Or, for short ‘\( q \) is PL-derivable from \( p_1, p_2, ..., p_N \).’

We can use this notion of PL-derivability to characterize the logical notions of PL that we previously defined in terms of truth-tables. (Those notions, by the way, are still defined in terms of truth tables. The relationships I’m going to tell you about below are not mere stipulations. In more advanced logic courses, I would ask you to prove that these relationships hold.)

- A set of wffs of PL, \( \{q_1, q_2, ..., q_N\} \) is PL-inconsistent if and only if \( q_2, ..., q_N \vdash_{PL} \sim q_1 \)

  (if and only if there is no possible PL-derivation from \( q_2, ..., q_N \) to \( \sim q_1 \).)

- Two wffs of PL, \( p \) and \( q \) are PL-equivalent if and only if \( \vdash_{PL} p \equiv q \)

- Two wffs of PL, \( p \) and \( q \) are PL-contradictories if and only if \( \vdash_{PL} \sim (p \equiv q) \)

- A wff of PL, \( p \), is a PL-tautology if and only if \( \vdash_{PL} p \)

- A wff of PL, \( p \), is a PL-self-contradiction if and only if \( p \vdash_{PL} A \cdot \sim A \)
§4.7. Derivation Challenge

Exercises

A. Conditional proof. Provide PL-derivations to establish that the following arguments are PL-valid.

1. \( J \supset (K \supset L) / J \supset (M \supset L) / \sim L // J \supset (K \lor M) \)
2. \( A \supset (B \cdot C) // (Q \supset A) \supset (Q \supset C) \)
3. \( A \circ \sim (A \lor B) // A \supset Q \)

B. Indirect proof. Provide PL-derivations to establish that the following arguments are PL-valid.

1. \( (A \lor B) \supset C / (\sim A \lor D) \supset E // C \lor E \)
2. \( \sim A \supset (B \cdot C) / D \supset \sim C // D \supset A \)

C. PL-tautologies. Provide PL-derivations to establish that the following wffs are PL-tautologies.

1. \( \sim ((P \supset \sim P) \cdot (\sim P \supset P)) \)
2. \( (\sim P \lor Q) \supset ((P \lor \sim Q) \supset (P \equiv Q)) \)
3. \( P \equiv (P \lor (Q \cdot P)) \)

4.7 Derivation Challenge

Hurley’s derivation system includes 40 distinct rules of inference. However, the system involves a large amount of redundancy. Your challenge, should you choose to accept it, is to prove that we could get by with only the following 12 rules (each arrow ‘▷’ indicates a different rule of inference).

<table>
<thead>
<tr>
<th>Modus Ponens (MP)</th>
<th>Simplification (Simp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \supset q )</td>
<td>( p \cdot q )</td>
</tr>
<tr>
<td>( p )</td>
<td>( p )</td>
</tr>
<tr>
<td>( \triangleright q )</td>
<td>( \triangleright p )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Disjunctive Syllogism (DS)</th>
<th>Conjunction (Conj)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \lor q )</td>
<td>( p )</td>
</tr>
<tr>
<td>( \sim p )</td>
<td>( q )</td>
</tr>
<tr>
<td>( \triangleright q )</td>
<td>( \triangleright p \cdot q )</td>
</tr>
</tbody>
</table>
To do this, you will have to provide 28 derivation schema showing that, by utilizing only these 12 rules, you could derive anything that could be derived using these 12 rules plus any of the additional rules. To give you the idea and to get you started, here is a derivation schema demonstrating that, with only the rules given above, you could derive anything that you could derive with Modus Tollens (MT).

Thus, by just using Modus Ponens (MP), Conjunction (Conj), and Indirect Proof (IP), we were able to derive ‘¬p’ from ‘p ⊃ q’ and ‘¬q’. This shows that, any time we would want to avail ourselves of MT, we could instead simply run through the steps of the derivation schema provided above. Thus, MT is entirely unnecessary. Anything that we would want to derive with MT, we could derive instead with just MP, Conj, and IP. (The reason that the derivation schema utilizes variables ranging over the wffs of PL is that we want to show that such a derivation could always be carried out whenever MT applies—i.e., no matter what p and q happen to be. It’s for this reason that I’m calling these derivation schema, and not derivations full stop.)
Now that we’ve proven that everything we could derive with the 12 rules above plus MT could be derived with just the 12 rules above without MT, there’s no need to continue to tie our hands by not using MT. We should feel free, from here on out, to utilize MT in providing the other derivation schema. Wherever we do so, we should be understood to be making oblique reference to the derivation schema for MT that we’ve already provided. The same goes for all the other rules that we’re going to provide derivation schema for. Once we’ve shown that they follow from the first 12 rules, we are free to utilize them—but not until then! For this reason, the order in which we provide the derivation schema will matter. Though you are free to prove them in any order you wish, I found it easiest to go through the rules in the following order (going down the left-hand column on this page, then down the right-hand column on this page, next down the left-hand column on the next page, and finally down the right-hand column on the next page):

### Double Negation, 2 (DN-2)

\[ p \vdash \sim \sim p \]

### DeMorgan’s, 2 (DM-2)

\[ \sim (p \lor q) \vdash \sim p \land \sim q \]

### Hypothetical Syllogism (HS)

\[ p \supset q \\
q \supset r \\
\vdash p \supset r \]

### DeMorgan’s, 3 (DM-3)

\[ \sim (p \land q) \vdash \sim p \lor \sim q \]

### Transposition, 1 (Trans-1)

\[ p \supset q \vdash \sim q \supset \sim p \]

### DeMorgan’s, 4 (DM-4)

\[ \sim p \land \sim q \vdash \sim (p \lor q) \]

### Transposition, 2 (Trans-2)

\[ \sim q \supset \sim p \vdash p \supset q \]

### Tautology, 1 (Taut-1)

\[ p \vdash p \lor p \]

### Transposition, 3 (Trans-3)

\[ p \supset q \lor r \vdash p \supset (q \lor r) \]

### Tautology, 2 (Taut-2)

\[ p \lor p \vdash p \]

### Exportation, 1 (Exp-1)

\[ (p \land q) \supset r \vdash p \supset (q \supset r) \]

### Tautology, 3 (Taut-3)

\[ p \lor p \vdash p \land p \]

### Exportation, 2 (Exp-2)

\[ p \supset (q \supset r) \vdash (p \land q) \supset r \]

### Tautology, 4 (Taut-4)

\[ p \land p \vdash p \]

### DeMorgan’s, 1 (DM-1)

\[ \sim p \lor \sim q \vdash \sim (p \land q) \]

### Material Implication, 1 (Impl-1)

\[ p \supset q \vdash \sim p \lor q \]
Appendix

Here, I'm going to sweep up some loose ends in the proof that I'm asking you to provide. You needn't continue reading any of this in order to successfully complete the challenge and get all the credit and the glory of having done so. If, however, you're interested, you might want to read on to see why the proof I've had you provide isn't complete, and to see how to finish it up.

Providing the 28 derivation schema above does not actually suffice to show that the 12 select rules at the start are just as powerful as all the rest of the rules (in the sense that anything derivable with the full gamut is also derivable with the 12). The reason is that, within Hurley's system, for all the rules of replacement, we are not merely allowed to replace the wffs of PL that are of the appropriate form with their equivalents—we are also allowed to replace any arbitrary subformula of those wffs with their equivalents. However, the derivation schema for the rules of replacement that you're being asked to provide above...
only show that, were we to use a rule of replacement not included in the 12 select rules to replace an entire wff—and not just one of its subformulae—then we could accomplish the same thing with the rules from the select 12. What it does not show is that, were we to use a rule of replacement not included in the select 12 to replace a subformula of a wff, then we could accomplish the same thing using only the rules from the select 12.

For the purposes of completing the challenge, this needn’t worry you. However, for the sake of completeness, I’m going to offer a proof of the following theorem, which will complete the proof that the 12 select rules are every bit as powerful as the full complement of 40 that Hurley provides.

**Theorem.** If the select 12 rules allow you to replace the wff ‘p’ with the wff ‘q’ whenever ‘p’ appears on a line by itself; and if they allow you to replace the wff ‘q’ with the wff ‘p’ whenever ‘q’ appears on a line by itself, then those rules allow you to replace ‘p’ with ‘q’ when it appears as a subformula in a larger wff.

**Proof.** Suppose that we have a derivation schema allowing us to derive ‘q’ from ‘p’, and a derivation schema allowing us to derive ‘p’ from ‘q’. Let’s refer to all the intermediate steps included in the first derivation schema with ‘D<sub>p→q</sub>’ and all the intermediate steps included in the second derivation schema with ‘D<sub>q→p</sub>’.

Now, suppose that ‘p’ appears as a negand in a negation, ‘∼ p’. Then, we may utilize our derivation schema D<sub>q→p</sub> to replace ‘∼ p’ with ‘∼ q’ by simply carrying out a derivation of the following form:

```
 n  ~p
 n+1  q  AIP
   :  D<sub>q→p</sub>  :
   n+m  p  :
 n+m+1  p ~p  n, n+m, Conj
 n+m+2  ~q  n+1 — n+m+1, IP
```

Let’s call all the intermediate steps in this derivation schema ‘D<sub>∼p→∼q</sub>’.

Suppose, on the other hand, that ‘p’ appears as the second conjunct in a conjunction, ‘s • p’. Then, we may utilize our derivation schema D<sub>p→q</sub> to replace ‘s • p’ with ‘s • q’ by simply carrying out a derivation of the following form:
Let’s call all the intermediate steps in this derivation schema ‘$D_{(i \cdot p) \rightarrow (s \cdot q)}$’.

If ‘$p$’ appears as the first conjunct in a conjunction, ‘$p \cdot s$’, then we may utilize exactly the same derivation schema as the one given above, except that we swap the wffs written on lines $n$ and $n+1$, and we stop at line $n+m+1$. Call this derivation schema ‘$D_{(p \cdot s) \rightarrow (q \cdot s)}$’.

Suppose that ‘$p$’ appears as the first disjunct in a disjunction, ‘$p \lor s$’. Then, we may utilize our derivation schema $D_{p \rightarrow q}$ to replace ‘$p \lor s$’ with ‘$q \lor s$’ as follows.

Call the intermediate steps of this derivation schema ‘$D_{(p \lor s) \rightarrow (q \lor s)}$’.

If ‘$p$’ appears as the second disjunct in a disjunction, ‘$s \lor p$’, then the same derivation schema may be utilized, except that we start on line $n+1$ and finish on line $n+m+2$. Call the intermediate steps in that derivation ‘$D_{(s \lor p) \rightarrow (s \lor q)}$’.

Suppose that ‘$p$’ appears as the antecedent of a conditional, ‘$p \supset s$’. Then, we may utilize the derivation schema $D_{q \rightarrow p}$ to replace ‘$p \supset s$’ with ‘$q \supset s$’, as shown in the derivation schema below.
§4.7. Derivation Challenge

If, on the other hand, ‘p’ appears as the consequent of a conditional, ‘s ⊃ p’, then we may utilize the derivation schema $\mathcal{D}_{p \rightarrow q}$ to replace ‘s ⊃ p’ with ‘s ⊃ q’, as shown in the derivation schema below.

\[
\begin{array}{c|c|l}
\text{n} & p & s \\
\hline
\text{n+1} & q & \text{ACP} \\
\hline
\vdots & \mathcal{D}_{q \rightarrow p} & \vdots \\
\text{n+m} & p & \vdots \\
\text{n+m+1} & s & \text{n, n+m, MP} \\
\text{n+m+2} & q & \text{n+m+1, CP} \\
\end{array}
\]

We can call all the intermediate steps in the first derivation schema $\mathcal{D}_{(p \supset s) \rightarrow (q \supset s)}$, and all the intermediate steps in the second derivation schema $\mathcal{D}_{(s \supset p) \rightarrow (s \supset q)}$.

If ‘p’ appears on the left-hand-side of a biconditional ‘p ≡ s’, then let $\mathcal{D}_{(p \equiv s) \rightarrow (q \equiv s)}$ be the intermediate steps in the following derivation schema,

\[
\begin{array}{c|c|l}
\text{n} & p \equiv s \\
\hline
\text{n+1} & (p \supset s) \bullet (s \supset p) & \text{n, Equiv-1} \\
\text{n+2} & (s \supset p) \bullet (p \supset s) & \text{n+1, Com} \\
\text{n+3} & p \supset s & \text{n+2, Simp} \\
\hline
\vdots & \mathcal{D}_{(p \supset s) \rightarrow (q \supset s)} & \vdots \\
\text{n+m} & q \supset s & \vdots \\
\text{n+m+1} & s \supset p & \text{n+1, Simp} \\
\text{n+m+2} & \mathcal{D}_{(s \supset p) \rightarrow (s \supset q)} & \vdots \\
\text{n+m+k} & s \supset q & \vdots \\
\text{n+m+k+1} & (q \supset s) \bullet (s \supset q) & \text{n+m, n+m+k, Conj} \\
\text{n+m+k+2} & q \equiv s & \text{n+m+k+1, Equiv-1} \\
\end{array}
\]
and let \( \mathcal{D}_{(s \equiv p) \rightarrow (s \equiv q)} \) be the intermediate steps in the following derivation schema.

\[
\begin{array}{ll}
n & s \equiv p \\
n+1 & (s \supset p) \bullet (p \supset s) & n, \text{Equiv-1} \\
n+2 & (p \supset s) \bullet (s \supset p) & n+1, \text{Com} \\
n+3 & s \supset p & n+1, \text{Simp} \\
: & \mathcal{D}_{(s \supset p) \rightarrow (s \supset q)} & : \\
n+m & s \supset q & : \\
n+m+1 & p \supset s & n+2, \text{Simp} \\
: & \mathcal{D}_{(p \supset s) \rightarrow (q \supset s)} & : \\
n+m+k & q \supset s & : \\
n+m+k+1 & (s \supset q) \bullet (q \supset s) & n+m, n+m+k, \text{Conj} \\
n+m+k+2 & s \equiv q & n+m+k+1, \text{Equiv-1} \\
\end{array}
\]

Now, by repeated application of the derivations schemas shown above, we may replace any instance of \('p'\) with an instance of \('q'\), even if \('p'\) is a subformula of a wff in a derivation. (To show this more rigorously, we would have to appeal to a proof technique known as \textit{mathematical induction}, but I hope that the intuitive idea is clear enough even without going through the induction.)
Quantificational Logic

5

5.1 Correctness and Completeness

A recap of what’s happened so far in the course: we started with the notion of deductive validity, which we defined as follows:

An argument is **deductively valid** if and only if it is impossible for its premises to all be true while its conclusion is false.

We wanted a *theory* that would tell us, of any particular argument, whether it was deductively valid or not. The first step towards providing a theory like this came with the notion of *formal* deductive validity, which we defined as follows:

An argument is **formally deductively valid** if and only if it is a substitution instance of a deductively valid argument form.

where an argument *form* is deductively valid if and only if every substitution instance with (actually) true premises has an (actually) true conclusion as well.

An argument form is **deductively valid** if and only if every substitution instance of that form whose premises are all true has a true conclusion as well.

The insight that we could theorize about deductive validity by theorizing about deductively valid *forms* led us to start theorizing about certain types of forms—namely, those involving the following English constructions:
§5.2. Arguments that PL is Not Correct

To do this, we constructed an artificial language—propositional logic, or PL—and introduced the logical operators $\sim$, $\cdot$, $\lor$, $\Rightarrow$, and $\equiv$, which were used to translate the English expressions above. We then gave a definition of validity in PL—PL-validity—and used it to theorize about deductive validity.

How well does this theory do? Well, there are two questions we might want to ask about the relationship between PL-validity and deductive validity:

1. Are there any PL-valid arguments which are deductively invalid?
2. Are there any PL-invalid arguments which are deductively valid?

The first is a question about the correctness of PL-validity; the second is a question about the completeness of PL-validity, where the properties of correctness and completeness are as given below.

If an argument is PL-valid, then it is deductively valid. (Correctness)

If an argument is deductively valid, then it is PL-valid. (Completeness)

If PL-validity is correct, then the answer to the first question above is ‘no’. If PL-validity is complete, then the answer to the second question above is ‘no’.

Many logicians believe that PL is correct. That is, they believe that, if an English argument, translated into PL, is PL-valid, then that English argument is deductively valid. However, no logician believes that PL is complete. That is, they believe that there are English arguments which are deductively valid, but which, translated into PL, are PL-invalid. Because PL is not complete, we will need to introduce additional kinds of logical forms. This will be the task of predicate logic, or, as it is also known, quantificational logic—QL.

§5.2 Arguments that PL is Not Correct

However, before moving on to QL, it will be instructive to look at some arguments that have been given for thinking that PL is not correct. A word of warning: these arguments are controversial (some more so than others); and many logicians are not moved by them to reject the correctness of PL. But the arguments are interesting in and of themselves, even if they don’t end up motivating a rejection of PL.

In going through these arguments, it’s important to keep in mind that there’s two components to our theory PL: 1) the theory about which arguments involving the wffs of PL are valid; and 2) the translation guide from English into PL. The first two arguments below—in §§5.2.1 and 5.2.2—are really objections to the translation guide. Both of those
arguments turn on the fact that ‘⊃’ is not a perfect translation of ‘if..., then...’. Those two arguments attempt to show that the differences between ‘⊃’ and ‘if..., then...’ prevent the English ‘if..., then...’ from satisfying modus ponens and modus tollens. The second two arguments, however—those in §§5.2.3 and 5.2.3—object not to the translation guide, but rather to PL’s theory about which arguments involving the wffs of PL are valid. They attempt to show that, even within the language PL, modus ponens and/or disjunctive syllogism are invalid. This later claim is more radical than the first.

5.2.1 A Counterexample to Modus Ponens?

Here’s an argument that you’ve already seen—it was on the midterm:

1. If Mitt Romney doesn’t win, then, if a Republican wins, then Ron Paul wins.
2. Mitt Romney doesn’t win.
3. So, if a Republican wins, then Ron Paul wins.

The logician Vann McGee (1985) argues that, in the run-up to the 2012 election, this argument had true premises and a false conclusion.\(^1\) Given that there were only two Republican candidates in the 2012 general election—Mitt Romney and Ron Paul—if Mitt Romney doesn’t win, then, if a Republican wins, then Ron Paul wins. That was true. It was also true that Mitt Romney doesn’t win. However, McGee contends, it was just false that, if a Republican wins, then Ron Paul wins. Paul didn’t have a chance in hell of winning. What was true was that, if a Republican wins, then Mitt Romney wins. So this is an argument which is of the form modus ponens:

\[
\begin{align*}
\text{if } p \text{ then } q \\
p \\
\hline
\text{so, } q
\end{align*}
\]

which has true premises and a false conclusion. So that argument form must be invalid (or so says McGee).

If we accept Exportation—as PL does—then McGee’s objection can be put in an even stronger form. For the following argument is PL-valid, but appears to have true premises and a false conclusion:

1. If Mitt Romney doesn’t win and a Republican wins, then Ron Paul wins.
2. Mitt Romney doesn’t win.
3. So, if a Republican wins, then Ron Paul wins.

This argument is PL-valid, as the following PL-derivation demonstrates,

\(^1\) McGee’s example involved the 1980 U.S. Presidential election, but I’ve changed the names to make things a bit more familiar.
§5.2. Arguments that PL is Not Correct

1. \((\sim M \cdot R) \supset P\)
2. \(\sim M\)
3. \(\sim M \supset (R \supset P)\) \(1, \text{Exp}\)
4. \(R \supset P\) \(2, 3, \text{MP}\)

However, its premises seem even more obviously true than those of the first argument; and the conclusion appears false.

McGee shows that, if we accept both modus ponens and exportation for the English ‘if..., then...’, then (in the presence of some other weak assumptions), the English ‘if..., then...’ will be logically indistinguishable from the material conditional \(\supset\). And that means that we would have to accept the deductive validity of the following inference

1. Shakespeare wrote Hamlet.
2. If Shakespeare didn’t write Hamlet, then Dan Brown did.

which looks to be deductively invalid (it looks like its premise is true and its conclusion is false). So, McGee argues, we must choose between accepting exportation for the English ‘if..., then...’ and accepting modus ponens for the English ‘if..., then...’. McGee opts for Exportation and rejects Modus Ponens; others have opted for Modus Ponens and rejected Exportation; still others have accepted both and accepted the conclusion that the English ‘if..., then...’ is logically indistinguishable from the material conditional (they have stories to tell about why arguments like the ones above appear—falsely—to be invalid).

5.2.2 A Counterexample to Modus Tollens?

Imagine that we have an urn which contains 100 marbles. They are all either blue or red, and they are all either big or small. The following diagram shows how many of the marbles are blue/red/big/small.

<table>
<thead>
<tr>
<th></th>
<th>blue</th>
<th>red</th>
</tr>
</thead>
<tbody>
<tr>
<td>big</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>small</td>
<td>50</td>
<td>10</td>
</tr>
</tbody>
</table>

That is: 10 are both big and blue, 30 are both big and red, 50 are both small and blue, and 10 are both small and red.

Suppose that we have selected a marble at random from the urn, but that we do not yet know whether it is blue or red, or whether it is big or small. Yalcin (2012) contends that, in this scenario, the premises of the following argument are true; yet its conclusion may very well be false.
1. If the marble is big, then it’s likely red.
2. The marble is not likely red.
3. The marble is not big.

But this is an instance of the argument form *modus tollens*, which is *PL*-valid.

\[
\begin{align*}
  \text{if } p & \text{ then } q \\
  \text{it is not the case that } q & \\
  \text{so, it is not the case that } p
\end{align*}
\]

So, *Yalcin* contends, *modus tollens* is not deductively valid, and *PL* is not correct.

Some options: 1) We could contend that the proposed counterexample equivocates with respect to ‘likely’ (in the first premise, it means “likely given all the information that currently have, plus the information that the marble is big”; whereas, in the second premise, it means “likely, given all the information that we currently have”). 2) We could contend that the first premise is equivalent to “it’s likely that, if the marble is big, then it’s red”, so that “if..., then...” is not the main operator of the premise. (*Yalcin* has responses to both of these objections, but we don’t have the time to delve into them.)

### 5.2.3 A COUNTEREXAMPLE TO DISJUNCTIVE SYLLOGISM AND MODUS PONENTS?

The previous two counterexamples were really counterexamples to the conjunction of the correctness of *PL*-validity and our translation guide which equates *PL*’s ‘–’ with English’s ‘if..., then...’. There are those, however, who object to the correctness of *PL on its own*, even before translation into English. That is, there are those who think that *PL*-validity is incorrect even for the wffs of *PL*.

To see why some think this, consider the following statement:

This very statement is false.

This statement says, of itself, that it is false. To make things a bit clearer, let’s give this statement a name—call it ‘*L*’. Then, we can specify the content of *L* as follows.

\[
L := L \text{ is false.}
\]

Is *L* true or is it false? Graham *Priest* accepts the following argument: *L* is either true or false. Suppose that it’s true. Then, what it says must be the case. It says that it is false; so it must be false. So it must be both true and false. Suppose, on the other hand, that it is false. Well, then the thing that it says is the case is the case—namely, that it is false. So it must be true. So it must be both true and false. So, whether it is true or false, it is *both* true and false. *Priest* draws the conclusion: *L* is both true and false. So, he concludes, some statements can be both true and false. Almost everybody else in the philosophical community balks at this conclusion.
§5.2. Arguments that PL is Not Correct

If, however, Priest is right, then there are three possible ways a statement could be with respect to truth and falsity. It could be true (and not false) ‘T; it could be false (and not true) ‘F; or it could be both true and false ‘B. The logical operators told us before how to figure out the truth-value of complicated expressions in terms of the truth values of their constituents, and this doesn’t change just because we think that a single statement can have more than one truth value. We now have the following table for ‘∼’,

<table>
<thead>
<tr>
<th>p</th>
<th>∼p</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

and the following tables for ∨ and ⊃,

<table>
<thead>
<tr>
<th>p ∨ q</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>B</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>p ⊃ q</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>B</td>
<td>T</td>
</tr>
</tbody>
</table>

since a disjunction is true if and only if at least one of its disjuncts is true, and false if both of its disjuncts are false; and a conditional is false if and only if its antecedent is true and its consequent is false, and it true otherwise. (Given these tables, it still follows that ‘p ⊃ q’ is equivalent to ‘∼ p ∨ q’. Let P = ‘Pigs can fly’. Then, following the truth table, the disjunction ‘L ∨ P’ will be both true and false. And the negation ‘∼ L’ will be both true and false. Consider, then, the following arguments:

\[
\begin{align*}
L ∨ P \\
∼ L \\
\hline
p
\end{align*}
\]
\[
\begin{align*}
L ⊃ P \\
L \\
\hline
p
\end{align*}
\]

According to Priest, all of the premises of both of these arguments are true (they are also false, of course), yet the conclusions are both false (and not true). However, these arguments are of the form disjunctive syllogism and modus ponens, respectively.

\[
\begin{align*}
p ∨ q \\
∼ p \\
\hline
q
\end{align*}
\]
\[
\begin{align*}
p ⊃ q \\
p \\
\hline
q
\end{align*}
\]

So, Priest concludes, both disjunctive syllogism and modus ponens are deductively invalid.

Almost everybody else is going to get off the boat by denying that L—or any other proposition, for that matter—is both true and false. But then we have to say something about L’s truth value.

By the way, saying that it is neither true nor false doesn’t look like it’s going to help, since that move won’t help out with a claim like
If we say that $L'$ is neither true nor false, then it's not true. So, what it says of itself is correct. So it's true. But it says that it's not true, so what it says of itself is not correct. So it must be false. So $L'$ is both true and false. Most philosophers think that something has gone wrong here, but it's notoriously difficult to work out exactly what has gone wrong.

**The Sorites Paradox**

There is yet another reason to think that PL is not correct for the wffs of PL.

Suppose that we have 10,000 tiles lined up in a row. The first tile is unmistakably red. The next tile in the sequence is perceptually indistinguishable from the first, but its color has ever-so-slightly more yellow in it than the first. Similarly, the third tile has ever-so-slightly more yellow in it than the second, and so on and so forth. Any pair of sequential tiles are perceptually indistinguishable. However, by the end of the sequence, we have a tile that is unmistakably orange.

It seems undeniable that

1) The 1st tile is red.

Each of the following material conditionals also seem undeniable.

2) The 1st tile is red ⊃ the 2nd tile is red.
3) The 2nd tile is red ⊃ the 3rd tile is red.
4) The 3rd tile is red ⊃ the 4th tile is red.
   ...
10,000) The 9,999th tile is red ⊃ the 10,000th tile is red.

After all, to reject any of these conditionals, you must think that its antecedent is true while its consequent is false. But that means that you have to think that there's some pair of adjacent tiles such that, while the first one is red, the second one is not. But, by stipulation, adjacent tiles are perceptually indistinguishable. How could one of two perceptually indistinguishable tiles be red without the other being red?

However, premises (1)—(10,000), by 9,999 applications of *modus ponens*, yield the absurd conclusion that the last tile in the sequence is red.

10,001) The 10,000th tile is red.

But, by stipulation, the final tile in the sequence is orange, and not red. So what gives? Something's gone wrong with the foregoing reasoning, and some people have been tempted to point the finger at *modus ponens*.
§5.3. The Language QL

This is one of the most studied and vexed paradoxes in contemporary philosophy, and most of the popular attempts to deal with it involve logical machinery too complicated to go into here. Some of these attempts involve denying the correctness of PL—though often in subtle ways—and some of them do not. But it is, to my mind, one of the most serious challenges to the correctness of PL.

5.2.4 Why PL is Not Complete

Consider the following arguments:

1. Johann knows Filipa.

   2. So, somebody knows Filipa.

1. Everyone who owns a Ford owns a car.

   2. Rohan owns a Ford.


Both of these arguments are deductively valid, but neither is PL-valid. The first argument, translated into PL, is \( J \rightarrow S \)—with \( J \) = ‘Johann knows Filipa’ and \( S \) = ‘Somebody knows Filipa’—which is PL-invalid. And the second, translated into PL, is \( E \land F \rightarrow C \)—with \( E \) = ‘Everyone who owns a Ford owns a car’, \( F \) = ‘Rohan owns a Ford’, and \( C \) = ‘Rohan owns a car’—which is also PL-invalid.

So there are PL-invalid arguments which are deductively valid. So PL is not complete. So, even if it’s on the right track (even if, that is, we think that the counterexamples of the previous section fail to show that PL is incorrect), it’s not the end of the story. And fortunately, we can do better. In this section of the course, we’re going to learn how to extend the language PL so that it can correctly judge the above arguments to be valid.

The extension of PL that we’re going to learn about will be able to correctly classify the above arguments as valid by delving deeper into the internal structure of English statements. It will allow us to represent the subjects and the predicates of those English statements, as well as what we will come to call the quantifiers of those statements. For this reason, the theory is referred to as ‘predicate logic’, or ‘quantificational logic’. I’ll just call it ‘QL’.

5.3 The Language QL

Before getting into the nitty-gritting, some preliminary orientation: we’re going to use capital letters to denote properties that a thing might or might not have and relations things might bear to one another, and we’re going to use lowercase letters to denote the things that may or may not have those properties or may or may not bear those relations to one another. So, for instance, we could use the capital letters \( T, L, \) and \( K \) to represent the
following properties and relations:

\begin{align*}
Tx &= x \text{ was tall} \\
Lxy &= x \text{ loved } y \\
Kxy &= x \text{ killed } y
\end{align*}

and we could use \( l, b, c, \) and \( p \) to represent the following individuals:

\begin{align*}
l &= \text{Abraham Lincoln} \\
b &= \text{John Wilkes Booth} \\
c &= \text{Caesar} \\
p &= \text{Pompey}
\end{align*}

If we put the lowercase letters representing individuals in the place of ‘\( x \)’ and ‘\( y \)’ above, we get statements like the following:

\begin{align*}
Tl &= \text{Abraham Lincoln was tall} \\
Kbl &= \text{John Wilkes Booth killed Abraham Lincoln} \\
Lcp &= \text{Caesar loved Pompey}
\end{align*}

We can treat these statements the same way that we treated the statement letters of \( PL \)—they can be the negands of negations, the antecedents of conditionals, the disjuncts of disjunctions, and so on and so forth.

\begin{align*}
\sim Lbl &= \text{John Wilkes Booth didn’t love Abraham Lincoln} \\
Kc \lor \sim Lp &= \text{If Caesar killed Pompey, then he didn’t love him} \\
Tc \lor Tb &= \text{Either Caesar or John Wilkes Booth is tall}
\end{align*}

We’re also going to be able to translate claims like ‘everyone loves someone’ and ‘no one loves anyone who killed them’. They will be translated like so:

\begin{align*}
(x)(\exists y) Lxy &= \text{Everyone loves someone} \\
\sim (\exists x)(\exists y) (Kyx \land Lxy) &= \text{No one loves anyone who killed them}
\end{align*}

But in order to understand that, we’ll have to get into the nitty-gritty.

### 5.3.1 The Syntax of \( QL \)

In this section, I’m going to tell you what the vocabulary of \( QL \) is and I’m going to tell you which expressions of \( QL \) are grammatical—which are \textit{well-formed}—just as we did for \( PL \).

**Vocabulary**

The vocabulary of \( QL \) includes the following symbols:

1. for each \( n \geq 0 \), an infinite number of \( n \)-place predicates (any capital letter, along
§5.3. The Language QL

with a superscript \( n \)—perhaps with subscripts

\[
\begin{array}{ccccccccccc}
A^1 & B^1 & \ldots & Z^1 & A^1_1 & \ldots & Z^1_1 & A^1_2 & \ldots \\
A^2 & B^2 & \ldots & Z^2 & A^2_1 & \ldots & Z^2_1 & A^2_2 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
A^n & B^n & \ldots & Z^n & A^n_1 & \ldots & Z^n_1 & A^n_2 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots 
\end{array}
\]

2. An infinite number of constants (any lowercase letter between \( a \) and \( w \)—perhaps with subscripts)

\[
a, b, c, \ldots, u, v, w, a_1, b_1, \ldots, v_1, w_1, a_2, b_2, \ldots
\]

3. An infinite number of variables (lowercase \( x, y, \) or \( z \)—perhaps with subscripts)

\[
x, y, z, x_1, y_2, z_2, x_3 \ldots
\]

4. Logical operators

\[
\sim, \lor, \cdot, \supset, \equiv, \exists
\]

5. Parentheses

\[
(, )
\]

Nothing else is included in the vocabulary of QL.

Let’s call both constants and variables terms. That is, both ‘\( a \)’ and ‘\( x \)’ are terms of QL.

Grammar

Any sequence of the symbols in the vocabulary of QL is a formula of QL. For instance, all of the following are formulae of QL:

\[
\begin{align*}
V^{2800}x & \sim (\supset \exists anw \\
p^1Q^2R^3S^4T^5 & \sim \\
(\forall x)F^3xab & \supset (\exists y)P^4yst \\
N^{54}xyz & \lor \sim (\exists x)B^2x
\end{align*}
\]

However, only one—the third—is a well-formed formula (or ‘wff’) of QL. We specify what it is for a string of symbols from the vocabulary of QL to be a wff of QL with the following rules.

\( \mathcal{F} \) If ‘\( \mathcal{F}^n \)’ is an \( n \)-place predicate and ‘\( a_1, a_2, \ldots, a_n \)’ are \( n \) terms, then ‘\( \mathcal{F}^n a_1a_2\ldots a_n \)’ is a wff.

\( \sim \) If ‘\( P \)’ is a wff, then ‘\( \sim P \)’ is a wff.
If \( P \) and \( Q \) are wffs, then \( (P \land Q) \) is a wff.

\( \lor \) If \( P \) and \( Q \) are wffs, then \( (P \lor Q) \) is a wff.

\( \supset \) If \( P \) and \( Q \) are wffs, then \( (P \supset Q) \) is a wff.

\( \equiv \) If \( P \) and \( Q \) are wffs, then \( (P \equiv Q) \) is a wff.

\( x \) If \( P \) is a wff and \( x \) is a variable, then \( (x)P \) is a wff.

\( \exists \) If \( P \) is a wff and \( x \) is a variable, then \( (\exists x)P \) is a wff.

Nothing else is a wff.

Note: none of \( \mathcal{F}, \, a, \, P, \) and \( Q \) appear in the vocabulary of QL. They are not themselves wffs of QL. Rather, we are using them here as variables ranging over the formulae of QL.

All and only the strings of symbols that can be constructed by repeated application of the rules above are well-formed formulae. For instance, if we wanted to show that \( ((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx) \) is a wff of QL, we could walk through the following steps to build it up:

a) \( F^1y \) is a wff [from \( \mathcal{F} \)]

b) So, \( (y)F^1y \) is a wff [from (a) and \( x \)]

c) \( G^2zx \) is a wff [from \( \mathcal{F} \)]

d) So, \( (\exists z)G^2zx \) is a wff [from (c) and \( \exists \)]

e) So, \( (\exists x)(\exists z)G^2zx \) is a wff [from (d) and \( \exists \)]

f) So, \( \neg (\exists x)(\exists z)G^2zx \) is a wff [from (e) and \( \neg \)]

g) So, \( ((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx) \) is a wff [from (b), (f), and \( \exists \)]

As before, we will adopt the convention of dropping the outermost parentheses in a wff of QL. We will additionally adopt the convention of dropping the superscripts on the predicates of QL. So, abiding by our informal conventions, we would write the wff of QL \( ((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx) \) as:

\[ (y)Fy \supset \sim (\exists x)(\exists z)Gzx \]

I’ll adopt these conventions from here on out.

We could, just as before, use syntax trees to represent the way that a wff of QL is built up according to the rules for wffs given above. For instance, the syntactic structure of \( ((y)Fy \supset \sim (\exists x)(\exists z)Gzx) \) is:
§5.3. The language QL

\[(y)Fy \supset (\exists x)(\exists z)Gzx\]

\[(y)Fy \quad \supset (\exists x)(\exists z)Gzx\]
\[\quad (\exists x)(\exists z)Gzx\]
\[\quad (\exists x)(\exists z)Gzx\]
\[\quad (\exists x)(\exists z)Gzx\]
\[\quad (\exists z)Gzx\]
\[\quad (\exists z)Gzx\]
\[\quad (\exists z)Gzx\]

Free and Bound Variables

Our rules for wffs count ‘Fx’ and ‘AyC’ as well-formed formulae. However, the variables that appear in these wffs are free. On the other hand, the variables appearing in ‘(x)(y)Fxy’ are bound. In ‘(x)Fx ⊃ Qx’, the first occurrence of the variable ‘x’ is bound, whereas the second occurrence is free.

To make these ideas precise, let’s introduce the idea of a quantifier. For any variable \(x\), both ‘(x)’ and ‘(∃x)’ are quantifiers. We call ‘(x)’ the universal quantifier, and we call ‘(∃x)’ the existential quantifier. These quantifiers are logical operators. They can be the main operator of a wff of QL or they can be the main operator of a wff’s subformulae. Each quantifier has one and only one associated variable. For instance, the variable associated with the quantifier ‘(x)’ is ‘x’. The variable associated with the quantifier ‘(∃y)’ is ‘y’.

As before, we can define the main operator of a wff of QL to be the logical operator whose associated rule is last appealed to when building the wff up according to the rules given above. So, for instance, the main operator of ‘(y)Fy ⊃ (∃x)(∃z)Gzx’ is the horseshoe ‘⊃’. The main operator of ‘(x)Fx’, on the other hand, whose syntax tree is shown below, is the universal quantifier ‘(x)’.

\[(x)Fx\]
\[\quad Fx\]

Similarly, the main operator of ‘(∃y)(Fy • Ga)’ is ‘(∃y)’.

\[(∃y)(Fy • Ga)\]
\[\quad (Fy • Ga)\]
\[\quad Fy • Ga\]

We can define subformula in the same way that we defined it before: P is a subformula of Q if and only if P must show up on a line during the proof that Q is a wff of QL. In terms of the syntax trees: P is a subformula of Q if and only if P lies somewhere on Q’s syntax tree.

Similarly, we can define immediate subformula in precisely the same way as before: P is an immediate subformula of Q iff a line asserting that P is a wff must be appealed to in the final line of a proof showing that Q is a wff, according to the rules for wffs given above.
In terms of the syntax tree: \( P \) is an immediate subformula of \( Q \) iff \( P \) lies immediately below \( Q \) on the syntax tree. Then, the immediate subformula of \( (x)Fx \) is \( Fx \), and the immediate subformula of \( (\exists y)(Fy \cdot Ga) \) is \( Fy \cdot Ga \).

The scope of a quantifier is its immediate subformula. So, for instance, in the wff \( (\exists y)Lyy \supset (\exists x)(\exists y)Lxy \), whose syntax tree is shown below,

\[
\begin{array}{c}
(\exists y)Lyy \\
\quad (\exists x)(\exists y)Lxy \\
\quad \quad Lyy \\
\quad \quad \quad Lxy
\end{array}
\]

The scope of the very first existential quantifier \((\exists y)\) is the formula \( Lyy \). The scope of the second existential quantifier \((\exists x)\) is \((\exists y)Lxy\). And the scope of the final existential quantifier \((\exists y)\) is \( Lxy \).

Now, we can define the notions of a free and a bound variable.

\[
\text{A variable } x \text{ in a wff of PL is bound if and only if it occurs within the scope of a quantifier, } (x) \text{ or } (\exists x), \text{ whose associated variable is } x. \\
\]

\[
\text{A variable } x \text{ in a wff of PL is free if and only if it does not occur within the scope of a quantifier, } (x) \text{ or } (\exists x), \text{ whose associated variable is } x. \\
\]

For instance, in the wff

\[
(x)(y)Fy \supset (\exists z)Gzx
\]

The final occurrence of \( x \) is free. Even though there is a universal quantifier \((x)\) in the wff, the final \( x \) does not occur within the scope of this universal quantifier, so it is not bound by it.

We can similarly define the notion of what it is for a quantifier to bind a variable.

\[
\text{In a wff of QL, a quantifier } (x) \text{ or } (\exists x) \text{ binds a variable } x \text{ if and only if } x \text{ occurs free within that quantifier’s scope.} \\
\]

This means that a variable can only be bound by a single quantifier. So, for instance, in the following wff of QL,

\[
(\exists x)(x)Fx
\]

The variable \( x \) is bound by the universal quantifier \((x)\). It is not bound by the existential quantifier \((\exists x)\).
5.3.2 Semantics for QL

The meaning of ‘∼’, ‘∨’, ‘•’, ‘¬’, and ‘≡’ are the same as before—they are just functions from truth-values to other truth-values, as specified by the truth-tables. In order to explain the meaning of the other expressions in QL, we’ll introduce the idea of a QL-interpretation for a wff or set of wffs of QL.

A QL-interpretation, \( \mathcal{I} \), of the language QL provides:

1. A specification of which things fall in the domain, \( D \), of the interpretation.
2. A specification of which things in the domain the terms of QL represent (every term must represent something).
3. For every \( n \)-place predicate of QL, a specification of the \( n \)-place property it represents.

That’s a full interpretation. However, we can get by often enough with just providing a partial interpretation—or an interpretation just for a single wff or a set of wffs of QL. A merely partial interpretation provides enough of the full interpretation for the purposes at hand. For instance,

A partial QL-interpretation of a wff or set of wff of QL provides:

1. A specification of which things fall in the domain, \( D \), of the interpretation.
2. A specification of which things in the domain the constants appearing in the wff or wffs represent.
3. A specification of which things in the domain the free variables appearing in the wff or wffs represent.
4. For every \( n \)-place predicate appearing in the wff or wffs, a specification of the \( n \)-place property it represents.

For instance, suppose that we have the following wff of QL,

\[(y) Lya \lor (\exists y) \sim Lya\]

Here is a partial interpretation of this wff:

\[
\begin{align*}
D & = \text{the set of all people} \\
a & = \text{Steve} \\
Lxy & = x \text{ loves } y
\end{align*}
\]

There are no free variables, so the interpretation need not say which things in the domain they represent. There is a single constant, ‘\( a \)’, so the interpretation must say which thing
in the domain this constant represents. And there is a single 2-place predicate, \( L \), so the interpretation must say which 2-place relation this predicate represents. Once we've done this, we've provided an interpretation for \( \forall y \, Ly a \lor (\exists y) \, Ly a' \)

Truth on an Interpretation

Here's how we're going to specify the meaning of the wffs of \( QL \): we're going to say what it is for them to be true on an interpretation. You should think of these interpretations as analogous to the rows of the truth-table in \( PL \). We'll ultimately end up saying that an argument is \( QL \)-valid if and only if every interpretation which makes the premises true makes the conclusion true; and we'll say that a wff is a \( QL \)-tautology if and only if it is true on every interpretation; and so on and so forth. So an interpretation will play the same role in our theory as the rows of the truth-tables played in the theory \( PL \).

Suppose that we've got an interpretation \( I \). Then, we can lay down the following rules which tell us what the wffs of \( QL \) mean on that interpretation—that is, under which conditions they are true on that interpretation. (Rules 2–6 should be familiar from \( PL \).)

1. A wff of the form \( \forall^n a_1 \ldots a_n \) is true on the interpretation \( I \) if the things in the domain represented by \( a_1 \ldots a_n \) have the property represented by \( \forall^n \). Otherwise, it is false on the interpretation \( I \).

2. A wff of the form \( \sim P \) is true on the interpretation \( I \) if \( P \) is false on the interpretation \( I \). Otherwise, it is false on the interpretation \( I \).

3. A wff of the form \( P \lor Q \) is true on the interpretation \( I \) if either \( P \) is true on the interpretation \( I \) or \( Q \) is true on the interpretation \( I \). Otherwise, it is false on the interpretation \( I \).

4. A wff of the form \( P \land Q \) is true on the interpretation \( I \) if both \( P \) is true on the interpretation \( I \) and \( Q \) is true on the interpretation \( I \). Otherwise, it is false on the interpretation \( I \).

5. A wff of the form \( P \supset Q \) is true on the interpretation \( I \) if either \( P \) is false on the interpretation \( I \) or \( Q \) is true on the interpretation \( I \). Otherwise, it is false on the interpretation \( I \).

6. A wff of the form \( P \equiv Q \) is true on the interpretation \( I \) if both \( P \) and \( Q \) have the same truth value on the interpretation \( I \). Otherwise, it is false on the interpretation \( I \).

Before getting to the rules for the quantifiers, \( (x) \) and \( (\exists x) \), we have to introduce one more idea—but it's one we've seen several times already in the course: that of a substitution instance. A substitution instance of a quantified wff of the form \( (x)P \) or \( (\exists x)P \) is the wff that you get by removing the quantifier, leaving behind just its immediate subformula, and uniformly replacing every instance of the variable \( x \) which is bound by the quantifier with some one constant \( a \)—note: it must be the same constant throughout.
§5.3. The Language QL

For instance, all of the following are substitution instances of the quantified wff \((\exists y)((Ay \cdot \sim Lay) \supset (x) \sim Lax))\):

\[
\begin{align*}
(Ab \cdot \sim Lab) & \supset (x) \sim Lax \\
(Ac \cdot \sim Lac) & \supset (x) \sim Lax \\
(Ak \cdot \sim Lak) & \supset (x) \sim Lax \\
(Aa \cdot \sim Laa) & \supset (x) \sim Lax
\end{align*}
\]

However, the following are not substitution instances of \((\exists y)((Ay \cdot \sim Lay) \supset (x) \sim Lax))\):

\[
\begin{align*}
(Ax \cdot \sim Lax) & \supset (x) \sim Lax \\
(Aa \cdot \sim Lab) & \supset (x) \sim Lax \\
(Ab \cdot \sim Lbb) & \supset (x) \sim Lcx \\
(Ar \cdot \sim Lra) & \supset (x) \sim Lax
\end{align*}
\]

We’re now in a position to give the rules for quantified statements being true on an interpretation, \(\mathcal{I}\):

7. A wff of the form \((x)P\) is true on the interpretation \(\mathcal{I}\) if every substitution instance of \((x)P\) is true on the interpretation \(\mathcal{I}\). Otherwise, it is false on the interpretation \(\mathcal{I}\).

8. A wff of the form \((\exists x)P\) is true on the interpretation \(\mathcal{I}\) if there is some substitution instance of \((x)P\) which is true on the interpretation \(\mathcal{I}\). Otherwise, it is false on the interpretation \(\mathcal{I}\).

Exercises

A. well formed formulae. Which of the following are well-formed formulae of QL?

1. \((x, y)Fx \lor (\exists z)Gz\)
2. \((a)Ga\)
3. \(Rxyzw\)
4. \((\exists x).Fx\)
5. \((x)(\exists x)(x)Fax\)
6. \((x)(\exists y)Lxy \supset (\exists x)(y)Lyx\)

B. quantifier scope, free and bound variables. For each of the following wffs of QL, say what its main operator is. Then, for each quantifier appearing in the wff, say what the scope of the quantifier is. Then, for each variable in the scope of the quantifier, say whether it is free or bound, and if it is bound, say which quantifier binds it.

1. \((x)(\exists x)Fx \supset (y)Gy\)
2. \((x)((y)Fx \supset (\exists z) \sim Gxz)\)
c. Truth in an interpretation. Consider the following interpretation:

\[ \mathcal{D} = \{ \text{Barack Obama, George W. Bush, Bill Clinton, George H.W. Bush} \} \]

\[ o = \text{Barack Obama} \]

\[ c = \text{Bill Clinton} \]

\[ w = \text{George W. Bush} \]

\[ h = \text{George H.W. Bush} \]

\[ Dx = x \text{ is a Democrat} \]

\[ Rx = x \text{ is a Republican} \]

\[ Sxy = x \text{ succeeded } y \]

which of the following wffs of QL are true on this interpretation?

1. \[ Do \land Rb \]

2. \[ (x)(\exists y)Sxy \]

3. \[ (z)(Dz \lor Rz) \]

4. \[ (x)(Rx \supset (\exists y)(Dy \land Syx)) \]

5.4 Notation

Let's start off by introducing some notation. We'll use expressions like

\[ P[x], Q[x] \]

as variables ranging over the wffs of QL in which the variable \( x \) occurs freely (\( x \) is itself a variable ranging over the variables of QL; it is not itself a part of the language QL). And we'll use expressions like

\[ P[x \rightarrow a], Q[x \rightarrow a] \]

to refer to the wffs of QL that you get when you replace every free occurrence of \( x \) in \( P[x] \) and \( Q[x] \) with the term \( a \). That is: given a wff \( P[x] \), you get the wff \( P[x \rightarrow a] \) by going through \( P[x] \), and every time \( x \) appears free, you swap it out for the term \( a \).

Using this notation,

\[ P[x \rightarrow a] \]

refers to a substitution instance of the quantified formulae

\[ (x)P[x] \]

and

\[ (\exists x)P[x] \]
§5.5  Translations from QL into English

In order to translate from QL into English, we need a QL-interpretation. To refresh your memory, a QL-interpretation will provide the following things:

A QL-interpretation, \( \mathcal{I} \), of a wff or set of wffs of QL provides:

1. A specification of which things fall in the domain, \( \mathcal{D} \), of the interpretation.
2. A constant of QL to name every thing in the domain.\(^a\)
3. A specification of which things in the domain the constants appearing in the wff or wffs of QL represent.
4. A specification of which things in the domain the free variables appearing in the wff or wffs of QL represent.
5. For every \( n \)-place predicate appearing in the wff or wffs of QL, a specification of the \( n \)-place property it represents.

\(^a\) You might worry about this, since our domain might contain uncountably many things, yet QL only has countably many constants. That's something to worry about, but I won't worry about it here. Dealing with that worry would complicate things too much for present purposes.

5.5.1  Translating Simple Quantified wffs of QL

We already know how to translate expressions involving the operators \( \sim, \cdot, \lor, \lor, \) and \( \equiv \) into English. What's needed is a method for translating the quantifiers \((x)\) and \((\exists x)\) into English. The following will do as a good translation guide in the simple case where the quantifier scopes over a wff consisting of just an \( n \)-place predicate followed by \( n \) terms.

\[
(x)P(x) \quad \rightarrow \quad \text{Everything is } P.
\]

\[
(\exists x)P(x) \quad \rightarrow \quad \text{Something is } P.
\]

For instance, given the partial interpretation \( \mathcal{I} \) (the interpretation is only partial because I haven't given a name to every thing in the domain),

\[
\mathcal{I} = \begin{cases} 
\mathcal{D} = \{ \text{the set of all actually existing things} \} \\
e = \text{The Eiffel Tower} \\
m = \text{the Mona Lisa} \\
Bxy = x \text{ is more beautiful than } y
\end{cases}
\]

We can give the following translations from QL into English:

\[
(y)Bxy \quad \rightarrow \quad \text{Everything is more beautiful than the Eiffel Tower.}
\]

\[
(\exists z)Bmz \quad \rightarrow \quad \text{The Mona Lisa is more beautiful than something.}
\]

Here's a good general rule for translating out of QL into English. Start by translating
universally quantified sentences like \((x)P[x]\) as ‘Everything is such that \(P\);’ and translate existentially quantified sentences like \((\exists x)P[x]\) as ‘Something is such that \(P\);’ then, find a more colloquial English sentence that expresses the same thought. So we could go through the following steps to arrive at the first translation above:

1. Everything is such that it is more beautiful than the Eiffel Tower.
2. Everything is more beautiful than the Eiffel Tower.

And we could go through the following steps to arrive at the second translation above:

1. Something is such that the Mona Lisa is more beautiful than it.
2. The Mona Lisa is more beautiful than something.

### 5.5.2 Translating More Complicated Quantified wffs of QL

Often, a quantified wff of QL will have a more complicated wff in its scope. There are four kinds of quantified wffs of QL that you should be familiar with, and which you should be able to translate from QL to English (and vice versa). They are the ones given below.

\[
(\exists x)(P[x] \cdot Q[x]) \quad \rightarrow \quad \text{Some } P \text{ is } Q.
\]

\[
(\exists x)(P[x] \cdot \neg Q[x]) \quad \rightarrow \quad \text{Some } P \text{ is not } Q.
\]

\[
(x)(P[x] \supset Q[x]) \quad \rightarrow \quad \text{All } P \text{s are } Q\text{s}.
\]

\[
(x)(P[x] \supset \neg Q[x]) \quad \rightarrow \quad \text{No } P \text{ is } Q.
\]

**Some P is Q**

To see why these wffs of QL translate into these English sentences, we should think about the Venn diagrams that we learned about earlier in the course. There, we saw that the way to represent a sentence of the form ‘Some \(P\) is \(Q\)’ with a Venn Diagram is as shown in figure 5.1. That is, ‘Some \(P\) is \(Q\)’ is true if and only if there is something which is both \(P\) and \(Q\)—i.e., if and only if there is something which is inside both of the circles \(P\) and \(Q\). There will be something like that—call it ‘a’—if and only if there is some true substitution instance of \((\exists x)(P[x] \cdot Q[x])\), namely, \(P[x \rightarrow a] \cdot Q[x \rightarrow a]\). But there will be a true substitution instance of \((\exists x)(P[x] \cdot Q[x])\) if and only if \((\exists x)(P[x] \cdot Q[x])\) is true, since (from last time):
§5.5. Translations from QL into English

8. A wff of the form ‘(∃x)P’ is true on the interpretation \( \mathcal{I} \) if there is some substitution instance of ‘(x)P’ which is true on the interpretation \( \mathcal{I} \). Otherwise, it is false on the interpretation \( \mathcal{I} \).

So ‘Some P is Q’ is a good translation of ‘(∃x)(P[x] • Q[x])’.

**Some P is not Q**

Next, consider ‘Some P is not Q’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 5.2. That is, ‘Some P is not Q’ is true if and only if there is something which is P but not Q—i.e., if and only if there is something which is inside the circle P yet outside of the circle Q. There will be something like that—call it ‘a’—if and only if there is some true substitution instance of (∃x)(P[x] • Q[x]), namely, P[x → a] • Q[x → a]. But there will be a true substitution instance of (∃x)(P[x] • Q[x]) if and only if (∃x)(P[x] • Q[x]) is true, since (again):

8. A wff of the form ‘(∃x)P’ is true on the interpretation \( \mathcal{I} \) if there is some substitution instance of ‘(x)P’ which is true on the interpretation \( \mathcal{I} \). Otherwise, it is false on the interpretation \( \mathcal{I} \).

So ‘Some P is not Q’ is a good translation of ‘(∃x)(P[x] • Q[x])’.

**All Ps are Qs**

Next, consider ‘All Ps are Qs’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 5.3. That is, ‘All Ps are Qs’ is true if and only if there is nothing which is P but not Q—i.e., if and only if there is nothing which is inside the
circle $P$ yet outside of the circle $Q$. Think about what it would take for this claim to be false. This claim would be false if and only if there were something which were $P$ but not $Q$. Otherwise, it would be true. Suppose that there were something—call it ‘$a$’—which were $P$ but not $Q$. Then, the wff of $QL P[x \rightarrow a] \supset Q[x \rightarrow a]$ would be false—since its antecedent is true, yet its consequent is false. On the other hand, if anything $b$ in the domain is either both $P$ and $Q$ or not $P$, then $P[x \rightarrow b] \supset Q[x \rightarrow b]$ would still be true (by the definition of ‘$\supset$’).

So, there is something which is $P$ and not $Q$ if and only if there is some $a$ such that $P[x \rightarrow a] \supset Q[x \rightarrow a]$ is false.

So, there is something which is $P$ and not $Q$ if and only if $(x)(P[x] \supset Q[x])$ is false, since (from last time):

7. A wff of the form ‘$(x)P$’ is true on the interpretation $I$ if every substitution instance of ‘$(x)P$’ is true on the interpretation $I$. Otherwise, it is false on the interpretation $I$.

By the same token, if there is nothing which is $P$ and not $Q$, then $(x)(P[x] \supset Q[x])$ will be true, since all of its substitution instances will be true.

So ‘All $P$s are $Q$s’ is true in exactly the same circumstances as ‘$(x)(P[x] \supset Q[x])$’. So the former provides a good translation of the latter.

No $P$ is $Q$

Finally, consider ‘No $P$ is $Q$’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 5.4. That is: the claim ‘No $P$ is $Q$’ is true if and only

\[
\setminus
\]

Figure 5.4

if there is nothing which is both $P$ and $Q$. Think about the circumstances under which this claim would be false. It would be false if and only if there were something—call it ‘$a$’—which were both $P$ and $Q$. Then, $P[x \rightarrow a]$ would be true and $\sim Q[x \rightarrow a]$ would be false. So $P[x \rightarrow a] \supset Q[x \rightarrow a]$ would be false (by the definition of ‘$\supset$’). So $(x)(P[x] \supset \sim Q[x])$ would be false (since it has a false substitution instance).

If there were nothing which were both $P$ and $Q$, then ‘No $P$ is $Q$’ would be true. And, similarly, $(x)(P[x] \supset \sim Q[x])$ would be true, since the only way that could be false would be if it had a false substitution instance,

$P[x \rightarrow a] \supset \sim Q[x \rightarrow a]$
but the above wff would be false only if \( a \) were both \( P \) and \( Q \)—since that is the only thing that would make its antecedent true and its consequent false.

So ‘No \( P \) is \( Q \)’ is true in exactly the same circumstances as ‘\((x)(P[x] \supset Q[x])\)’. So the former provides a good translation of the latter.

## §5.6 Translations from English into QL

The English expressions appearing in the translation guides from the previous section constitute the *canonical logical form* of English. In general, if we have an English expression in canonical logical form, we may translate it into QL directly according to that translation schema:

<table>
<thead>
<tr>
<th>English Expression</th>
<th>QL Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everything is ( P )</td>
<td>((x)P[x])</td>
</tr>
<tr>
<td>Something is ( Q )</td>
<td>((\exists x)Q[x])</td>
</tr>
<tr>
<td>Some ( P ) is ( Q )</td>
<td>((\exists x)(P[x] \land Q[x]))</td>
</tr>
<tr>
<td>Some ( P ) is not ( Q )</td>
<td>((\exists x)(P[x] \land \sim Q[x]))</td>
</tr>
<tr>
<td>All ( P )s are ( Q )</td>
<td>((x)(P[x] \supset Q[x]))</td>
</tr>
<tr>
<td>No ( P ) is ( Q )</td>
<td>((x)(P[x] \supset \sim Q[x]))</td>
</tr>
</tbody>
</table>

There are, however, other ways of translating these English sentences into QL. For instance, given the following (partial) interpretation,

\[
\mathcal{I} = \begin{cases} 
\mathcal{D} & = \text{the set of all people} \\
Rx & = x \text{ is a Republican} \\
Sx & = x \text{ is socially liberal}
\end{cases}
\]

each of the following wffs of QL correctly translate the English sentence ‘Some Republican is socially liberal’.

\[
\text{Some Republican is socially liberal} \rightarrow \begin{cases} 
(\exists x)(Rx \land Sx) \\
\sim (x)(Rx \supset \sim Sx)
\end{cases}
\]

(这些wffs are QL-equivalent—they are true in all the same QL-interpretations and false in all the same QL-interpretations).

Likewise, each of the following wffs correctly translate the English ‘Some Republican is not socially liberal’:

\[
\text{Some Republican is not socially liberal} \rightarrow \begin{cases} 
(\exists x)(Rx \land \sim Sx) \\
\sim (x)(Rx \supset Sx)
\end{cases}
\]

Also, each of the following wffs of QL correctly translate the English sentence ‘No Repub-
licans are socially liberal'.

\[
\text{No Republicans are socially liberal} \rightarrow \left\{ \begin{array}{l}
(x)(Rx \supset \sim Sx) \\
\sim (\exists x)(Rx \cdot \sim Sx)
\end{array} \right.
\]

(theswffs are QL-equivalent).

Similarly, each of the following wffs correctly translate the English 'All Republicans are socially liberal':

\[
\text{All Republicans are socially liberal} \rightarrow \left\{ \begin{array}{l}
(x)(Rx \supset Sx) \\
\sim (\exists x)(Rx \cdot \sim Sx)
\end{array} \right.
\]

(theswffs are QL-equivalent).

If the English sentence does not appear in canonical logical form, then we should attempt to find some expression of English which is in canonical logical form and which has the same meaning as it. For instance, suppose that we have the following (partial) interpretation,

\[
\mathcal{I} = \begin{cases} 
  \mathcal{D} & = \text{the set of all people} \\
  j & = \text{Jon} \\
  d & = \text{Damian} \\
  Bxy & = x \text{ is more beautiful than } y \\
  Lxy & = x \text{ loves } y
\end{cases}
\]

And we want to translate the following sentence into QL:

Jon loves anybody who is more beautiful than Damian.

We should go through the following steps, recognizing that at each step, we get a sentence which has the same meaning as the one that preceded it.

1. If anybody is more beautiful than Damian, then Jon loves them.

2. If anybody is more beautiful than Damian, then they are loved by Jon.

3. Everybody more beautiful than Damian is loved by Jon.

But this is a sentence of the form 'All \( P \)s are \( Q \)'s, which we know may be translated as either '(x)(P[x] \supset Q[x])', or '\( \sim (\exists x)(P \cdot \sim Q) \)'. Thus, we may translate the sentence into QL is as either

\[
(x)(Bxd \supset Ljx)
\]

or

\[
\sim (\exists x)(Bxd \cdot \sim Ljx)
\]

(There are other translation options as well, but either of these will be correct.)
5.7  QL-Validity

In PL, we defined validity in terms of truth-preservation on rows of the truth table—that is, an argument was PL-valid if and only if every row of the truth table which made all of the premises true was a row of the truth table which made the conclusion true also.

In QL, we’re going to define validity in precisely the same way, except that we’re going to exchange ‘row of the truth table’ for ‘QL-interpretation’.

For instance, let’s show that the following inference, called *universal instantiation* (UI), is QL-valid:

\[(x)P \quad \frac{\text{P}[x \rightarrow a]}{\text{P} \rightarrow x}
\]

Pick any interpretation, \(\mathcal{I}\), which makes the premise, \((x)P\), true. A wff of the form \((x)P\) is true on an interpretation \(\mathcal{I}\) only if every substitution instance of the wff is true. But \(P[x \rightarrow a]\) is a substitution instance of \((x)P\). So \(P[x \rightarrow a]\) must be true on the interpretation \(\mathcal{I}\) too.

Therefore, every substitution instance which makes a wff of the form \((x)P\) true makes a wff of the form \(P[x \rightarrow a]\) true as well. So UI is QL-valid.

5.8  Proving QL-Invalidity

An argument is QL-invalid iff there is some interpretation on which all of the premises of the argument are true and the conclusion is false.

Therefore, to show that an argument is QL-invalid, you can provide a QL-interpretation on which the premises of the argument are true, yet the conclusion is false.
For instance, to show that the following argument is QL-invalid:

\[(\exists x)Ax \cdot (\exists x)Bx \quad \frac{}{(\exists x)(Ax \cdot Bx)}\]

It suffices to provide the following QL-interpretation:

\[\mathcal{I} = \begin{cases} 
  \mathcal{D} &= \{1, 2\} \\
  a &= 1 \\
  b &= 2 \\
  Ax &= x \text{ is odd} \\
  Bx &= x \text{ is even}
\end{cases}\]

Given this interpretation, the premise of the above argument is true, yet its conclusion is false.

5.9 QL-Derivations

It’s not nearly as easy to prove that arguments are QL-valid using interpretations as it was to show that arguments were PL-valid using truth-tables. So rather than dwell on establishing validity, we’re going to march right into the QL-derivation system. This derivation system is an extension of the derivation system for PL. We just add to it eight new rules of replacement and six new rules of implication. The resulting system has the following property: if there is a legal derivation with assumptions \(P_1, P_2, \ldots, P_N\) and which has \(C\) on its final line, then the argument \(P_1 / P_2 / \ldots / P_N // C\) is QL-valid.

5.9.1 New Rules of Replacement

The new rules of replacement all trade on the fact that, given our semantics, ‘(x)’ has the same meaning as ‘\(\sim (\exists x)\)’ and ‘(\exists x)’ has the same meaning as ‘\(\sim (x)\)’.

<table>
<thead>
<tr>
<th>Quantifier Negation (QN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)P (\leftrightarrow) (\sim (\exists x)) (\sim P)</td>
</tr>
<tr>
<td>(\sim (x))P (\leftrightarrow) ((\exists x)) (\sim P)</td>
</tr>
<tr>
<td>((\exists x))P (\leftrightarrow) (x) (\sim P)</td>
</tr>
<tr>
<td>(\sim (\exists x))P (\leftrightarrow) (x) (\sim P)</td>
</tr>
</tbody>
</table>

Because QN is a rule of replacement, it may be applied to subformulae of a wff. For instance, the following is a legal derivation.
## §5.9. QL-Derivations

\[ \sim (\exists z) Fz \equiv (x)(\exists y) Lxy \]

\[ (z) \sim Fz \equiv (x)(\exists y) Lxy \quad 1, \text{QN} \]

\[ (z) \sim Fz \equiv (\exists x) (\exists y) Lxy \quad 2, \text{QN} \]

\[ (z) \sim Fz \equiv (\exists x)(y) \sim Lxy \quad 3, \text{QN} \]

### 5.9.2 New Rules of Implication

The first new rule of replacement says that, whenever you have a universally quantified formula, you can write down either a substitution instance of it, or, alternatively, you can remove the universal quantifier and replace every bound variable with a(nother) variable (leaving behind a wff in which that variable now occurs free).

![Universal Instantiation (UI)]

\[
\begin{align*}
(x)P \\
\downarrow \\
P[x \rightarrow a]
\end{align*}
\]

where ‘a’ is a constant; or:

\[
\begin{align*}
(x)P \\
\downarrow \\
P[x \rightarrow y]
\end{align*}
\]

where ‘y’ is a variable.

**NOTE:** when you use *UI*, you must be sure that you replace every occurrence of the bound variable with the same constant or the same variable.

For instance, the following QL-Derivations are *not* legal.

\[ (y)(Fy \supset Gy) \]

\[ Fa \supset Gb \quad 1, UI \quad \leftarrow \text{MISTAKE!!} \]

\[ (z)(Az \equiv Bz) \]

\[ Aa \equiv Bx \quad 1, UI \quad \leftarrow \text{MISTAKE!!} \]

These QL-Derivations, on the other hand, *are* legal.

\[ (y)(Fy \supset Gy) \]

\[ Fa \supset Ga \quad 1, UI \]
The second new rule of implication tells us that, if you have a wff of \( QL \) according to which some particular thing \( a \) has a certain property, then you may infer that something has that property. That is, if you have a substitution instance of an existentially quantified formula, then you may write down that existentially quantified formula.

### Existential Generalization (EG)

\[
\begin{align*}
P[x \to a] & \quad \Rightarrow (\exists x)P \\
\end{align*}
\]

where ‘\( a \)’ is a constant; or:

\[
\begin{align*}
P[x \to y] & \quad \Rightarrow (\exists x)P \\
\end{align*}
\]

where ‘\( y \)’ is a variable.

For instance, the following are legal \( QL \)-derivations.

1. \((x)(Ay \equiv Bz)\)
2. \(Ax \equiv Bx\) \(1, UI\)

This final proof is legal because line 2 is a substitution instance of line 1.

The third new rule of implication says that, if you have an existentially quantified wff, \((\exists x)P\), then you may write down a substitution instance of that wff, \(P[x \to a]\) — provided that the constant that you introduce is entirely new (it doesn’t appear on any previous line),
and provided that you get rid of it before you're done (that is, provided that it doesn't appear on the conclusion line).

Existential Instantiation (EI)

\[(\exists x)P \Rightarrow P[x \rightarrow a]\]

where 'a' is a constant.

provided that:

1. a does not appear on any previous line (or the conclusion line)

It is important to keep this provision in mind. The idea behind this rule is that, if you know that there is something which is P, then it's o.k. to give that thing a name. However, you don't want to assume anything about this thing other than that it's P. So you'd better give it an entirely new name; otherwise, you'd be assuming more about the thing than that it is P.

The following derivation is not legal:

1. \((\exists y)(Dy \equiv (He \lor Jy))\)
2. \(De \equiv (He \lor Je)\) 1, EI ← MISTAKE!!!

The constant 'e' appears on line 1, so it cannot be instantiated on line 2 by EI.

This derivation, however, is legal:

1. \((\exists y)(Dy \equiv (He \lor Jy))\)
2. \(Da \equiv (He \lor Ja)\) 1, EI
A Sample Derivation

1. \((x)(Fx \cdot (\exists y)Gy)\) \(\Rightarrow ((\exists z)(Fz \cdot Gz))\)
2. \(Fa \cdot (\exists y)Gy\) 1, UI
3. \((\exists y)Gy \cdot Fa\) 2, Com
4. \((\exists y)Gy\) 3, Simp
5. \(Gc\) 4, EI
6. \(Fc \cdot (\exists y)Gy\) 1, UI
7. \(Fc\) 6, Simp
8. \(Fc \cdot Gc\) 5, 7, Conj
9. \((\exists z)(Fz \cdot Gz)\) 8, EG

The final new rule of implication says that, if you have a wff of QL in which a variable occurs freely, then you may replace it with another variable and tack on a quantifier out front—provided that the freely occurring variable does not occur free in either the assumptions or the first line of any accessible subderivation, and provided that it does not occur freely in any line which is justified by EI.

\textbf{Universal Generalization (UG)}

\[
P[x \rightarrow y] \Rightarrow (x)P
\]

provided that:

1. \(y\) does not occur free in the assumptions or on the first line of an accessible subderivation.
2. \(y\) does not occur free in any accessible line justified by EI.

The following derivation is not legal:

1. \(Fx\) ACP
2. \((y)Fy\) 1, UI \(\Rightarrow\) MISTAKE!!!
3. \(Fx \supset (y)Fy\) 1–2, CP
4. \((z)(Fz \supset (y)Fy)\) 3, UG

Line 2 does not follow from line 1, since the free variable \(x\) appears free in the assumption of that accessible subderivation. (Good thing, too, since \((z)(Fz \supset (y)Fy)\) is false on any
interpretation in which one thing is $F$ and another is not $F$—so it is not a QL-tautology.)

This derivation is not legal either:

1. $(x)(\exists y)Axy$
2. $(\exists y)Azy$  $1, UI$
3. $Azc$  $2, EI$
4. $(x)Axc$  $3, UG$  $\leftarrow$ MISTAKE!!!
5. $(\exists y)(x)Axy$  $4, EG$

Line 4 does not follow from line 3, since the variable $z$ appears free on a line of the derivation which is justified by ‘$EI$’—namely, line 3. (It’s a good thing, too, since $(x)(\exists y)Axy$ doesn’t follow from $(x)(\exists y)Axy$—there are QL-interpretations on which the first is true but the second false.)

This derivation, however, is legal.

1. $(x)(Yx \cdot Zx)$  $(x)Yx \cdot (x)Zx$
2. $Yy \cdot Zy$  $1, UI$
3. $Yy$  $2, Simp$
4. $Zy \cdot Yy$  $2, Com$
5. $Zy$  $4, Simp$
6. $(x)Yx$  $3, UG$
7. $(x)Zx$  $5, UG$
8. $(x)Yx \cdot (x)Zx$  $6, 7, Conj$

### Sample Derivations

1. $\sim (x)(Ax \cdot Bx)$  $(\exists y)(Ay \supset \sim By)$
2. $(\exists x) \sim (Ax \cdot Bx)$  $1, QN$
3. $\sim (Ac \cdot Bc)$  $2, EI$
4. $\sim Ac \lor \sim Bc$  $3, DM$
5. $Ac \supset \sim Bc$  $4, Impl$
6. $(\exists y)(Ay \supset \sim By)$  $5, EG$
Exercises

A. TRANSLATIONS FROM QL INTO ENGLISH. Given the (partial) interpretation below, translate the following wffs of QL into English.

\[ I = \{ \mathcal{D} = \text{the set of all people} \]

\[ \mathcal{J} = \{ \]

\[ \mathcal{R} = \{ a = \text{Abelard} \]

\[ h = \text{Heloise} \]

\[ Lxy = x \text{ loves } y \]

\[ Px = x \text{ is a philosopher} \]

1. Lah

2. Lha

3. \((\exists z)(Pz \cdot Lza)\)

4. \((x)(Px \supset Lax)\)

5. \((\exists y)(Py \cdot (x)Lyx)\)

6. \((x)(\exists y)Lxy\)

7. \((\exists y)(x)Lxy\)

B. TRANSLATIONS INTO QL. Using the same interpretation, translate the following English sentences into QL.

1. Abelard loves any philosopher who loves him.

2. No philosopher loves Heloise.

3. Only philosophers love philosophers.
4. No philosopher loves a philosopher.
5. Somebody loves everybody.

C. PROVING QL INVALIDITY. Provide an interpretation to show that the following argument is QL-invalid.

\[(\forall x)(\exists y)A_{xy} \quad \Rightarrow \quad (\exists y)(\forall x)A_{xy}\]

D. QL-DERIVATIONS. Provide a QL-derivation to establish that the following arguments are QL-valid.

1. \(~(\forall x)L_{xx} \quad \Rightarrow \quad (\exists x)(\exists y)~L_{xy}\)
2. \((\forall x)(\forall y)P_{xy} \quad \Rightarrow \quad (\forall x)P_{xx}\)
3. \((\exists x)(\exists y)P_{xy} \quad \Rightarrow \quad (\exists y)(\exists x)P_{xy}\)
4. \((\forall x)P_{x} \cdot (\forall y)Q_{y} \quad \Rightarrow \quad (\forall x)(P_{x} \cdot Q_{x})\)
6.1 Inductive Strength

Let's return to something that I said at the very beginning of the course:

The inductive strength of an argument is given by how probable its conclusion is, given the truth of its premises.

Inductive strength, then, is a matter of degree. Two arguments can both be inductively strong, but one can be stronger than the other. Similarly, two arguments can both be inductively weak, yet one can be weaker than the other. We can measure the inductive strength of an argument by looking at how probable the premises make the conclusion.

The inductive strength of an argument is \( x \) if and only if

\[
Pr(\text{conclusion} \mid \text{premises}) = x
\]

By setting some arbitrary threshold—let's say '0.5'—we can say what it is for an argument to be inductively strong—full stop.

An argument is inductively strong if and only if

\[
Pr(\text{conclusion} \mid \text{premises}) > 0.5
\]

We're going to spend the rest of this course thinking about what the theory of probability can tell us about inductive strength.
Here, I’m going to present the theory of probability roughly as it has been handed down to us by \(\text{Plato}\). A \textit{probability function} \(\Pr\) is a function which takes as inputs \textit{wffs} of \(\text{PL}\) and gives as outputs real numbers. Beyond this, the only constraints that a function has to satisfy in order to count as a \textit{probability} function are the following three.

For any \textit{wff} of \(\text{PL}\), \(p\),

\[\Pr(p) \geq 0\]  \hspace{1cm} (6.1)

For any \(\text{PL}\)-tautology, \(\top\),

\[\Pr(\top) = 1\]  \hspace{1cm} (6.2)

For any \textit{wffs} of \(\text{PL}\), \(p\) and \(q\), which are mutually exclusive

\[\Pr(p \lor q) = \Pr(p) + \Pr(q)\]  \hspace{1cm} (6.3)

\(p\) and \(q\) are mutually exclusive if and only if they cannot both be true at the same time. Two \textit{wffs} which are mutually exclusive could still both be \textit{false} at the same time; however, they can’t be both be true at the same time.

Together with these three axioms, we’ll introduce the following stipulative definition: for any \textit{wffs} of \(\text{PL}\), \(p\) and \(q\),

\[\Pr(p \mid q) \overset{\text{def}}{=} \frac{\Pr(p \cdot q)}{\Pr(q)}\]  \hspace{1cm} (6.4)

The expression ‘\(\Pr(p \mid q)\)’ is read as ‘the probability of \(p\), \textit{given that} \(q\)’. Or, alternatively, ‘the probability of \(p\), \textit{conditional on} \(q\)’. For that reason, expressions of the form ‘\(\Pr(p \mid q)\)’ are known as \textit{conditional probabilities}.

Additionally, we’ll stipulatively define the notion of \textit{probabilistic independence}. We’ll say that two \textit{wffs} of \(\text{PL}\), \(p\) and \(q\), are \textit{probabilistically independent} if and only if the probability of \(p \cdot q\) is the probability of \(p\) multiplied by the probability of \(q\).

\[p\text{ and } q\text{ are probabilistically independent } \overset{\text{def}}{=} \Pr(p \cdot q) = \Pr(p) \times \Pr(q)\]  \hspace{1cm} (6.5)

### §6.3 Rules for the Probability Calculus

All of the other rules we’ll want to use when reasoning about probability follow from axioms (6.1–6.3), together with the definition of conditional probability (6.4) and the definition of probabilistic independence (6.5).

#### 6.3.1 Restricted Conjunction Rule

The first rule that Hurley provides is what he calls the ‘restricted conjunction rule’. This rule follows straightaway from the definition of probabilistic independence given above. It says that, if \(p\) is probabilistically independent of \(q\), then the probability of \(p \cdot q\) is the probability of \(p\) multiplied by the probability of \(q\).
6.3.2 Restricted Disjunction Rule

Hurley's Restricted Disjunction Rule follows immediately from the axioms. It is just a re-statement of axiom (6.3).

\[ \text{Restricted Disjunction Rule} \]

If \( p \) and \( q \) are mutually exclusive, then

\[ \Pr(p \lor q) = \Pr(p) + \Pr(q) \]

6.3.3 Negation Rule

The next rule, called the negation rule, takes some work to get out of the axioms. Take any wff of PL, \( p \). \( p \lor \neg p \) will be a PL-tautology, as the following truth-table demonstrates.

\[
\begin{array}{c|cc|c|c}
\neg & T & T & T & F \\
\lor & F & T & T & F \\
\end{array}
\]

Therefore, by axiom (6.2), it has probability 1.

\[ \Pr(p \lor \neg p) = 1 \quad (6.6) \]

Moreover, \( p \) and \( \neg p \) are mutually exclusive, so, by the restricted disjunction rule,

\[ \Pr(p \lor \neg p) = \Pr(p) + \Pr(\neg p) \quad (6.7) \]

Putting (6.6) and (6.7) together, we get

\[ \Pr(p) + \Pr(\neg p) = 1 \quad (6.8) \]

\[ \Pr(\neg p) = 1 - \Pr(p) \quad (6.9) \]

Thus, we have the negation rule:
§6.3.  **Rules for the Probability Calculus**

### Negation Rule

For any wff of PL $p$, 
\[ \Pr(\neg p) = 1 - \Pr(p) \]

6.3.4  **Self-Contradiction Rule**

From the *negation rule*, it follows that any PL-self-contradiction will have probability 0. For, if $p$ is a PL-self-contradiction, then $\neg p$ will be a PL-tautology. Then, axiom (6.2) tells us that
\[ \Pr(\neg p) = 1 \quad (6.10) \]

And the *negation rule* tells us that
\[ \Pr(\neg p) = 1 - \Pr(p) \quad (6.11) \]

Thus, putting together (6.10) and (6.11), we get
\[
\begin{align*}
1 - \Pr(p) &= 1 \\
\Pr(p) &= 0
\end{align*}
\]

Thus, any PL-self-contradiction will have probability zero.

### Self-Contradiction Rule

If $p$ is a PL-self-contradiction, then 
\[ \Pr(p) = 0 \]

6.3.5  **Equivalence Rule**

Though Hurley doesn't mention this, it is quite important that it follows from axioms (6.1), (6.2), and (6.3) that any two wffs of PL which are PL-equivalent are assigned the same probability.

To see this, note that, if $p$ and $q$ are PL-equivalents, then they will have the same truth-value in every row of the truth-table, so $p \lor \neg q$ will be a PL-tautology.

\[
\begin{array}{c|c|c|c|c|c}
 p & q & p & \lor & \neg & q \\
 T & T & T & T & T & T \\
 F & F & F & T & T & F \\
\end{array}
\]
Thus, by axiom (6.2),
\[ \Pr(p \lor \sim q) = 1 \] (6.14)
And, since \( p \) and \( \sim q \) are mutually exclusive, by the restricted disjunction rule,
\[ \Pr(p \lor \sim q) = \Pr(p) + \Pr(\sim q) \] (6.15)
Putting (6.14) and (6.15) together, then
\[ \Pr(p) + \Pr(\sim q) = 1 \] (6.16)
\[ \Pr(p) = 1 - \Pr(\sim q) \] (6.17)
From the negation rule,
\[ \Pr(\sim q) = 1 - \Pr(q) \] (6.18)
\[ \Pr(q) = 1 - \Pr(\sim q) \] (6.19)
Substituting (6.19) into (6.17), we get
\[ \Pr(p) = \Pr(q) \] (6.20)

**Equivalence Rule**
If two wffs of PL, \( p \) and \( q \), are PL-equivalent, then
\[ \Pr(p) = \Pr(q) \]

### 6.3.6 General Conjunction Rule
The next rule follows almost immediately from our definition of conditional probability.
\[ \frac{\Pr(q \cdot p)}{\Pr(p)} = \Pr(q \mid p) \] (6.21)
\[ \Pr(q \cdot p) = \Pr(p) \times \Pr(q \mid p) \] (6.22)
\[ \Pr(p \cdot q) = \Pr(p) \times \Pr(q \mid p) \] (6.23)
(6.23) follows from (6.22) because ‘\( p \cdot q \)’ is PL-equivalent to ‘\( q \cdot p \)’.
Thus, we have the **general conjunction rule**.

**General Conjunction Rule**
For any wffs of PL, \( p \) and \( q \),
\[ \Pr(p \cdot q) = \Pr(p) \times \Pr(q \mid p) \]
The restricted conjunction rule now follows from the general conjunction rule, together with the fact that, if \( p \) and \( q \) are probabilistically independent, then \( \Pr(q \mid p) = \Pr(q) \), since

\[
\Pr(q \mid p) = \frac{\Pr(q \cdot p)}{\Pr(p)} \quad (6.24)
\]

\[
\Pr(q \mid p) = \frac{\Pr(q) \times \Pr(p)}{\Pr(p)} \quad (6.25)
\]

\[
\Pr(q \mid p) = \Pr(q) \quad (6.26)
\]

So, if \( p \) and \( q \) are probabilistically independent, then

\[
\Pr(p \cdot q) = \Pr(p) \times \Pr(q \mid p) \quad \| \quad \Pr(p \cdot q) = \Pr(p) \times \Pr(q)
\]

### 6.3.7 Total Probability Rules

\( p \) is logically equivalent to \((p \cdot q) \lor (p \cdot \lnot q)\), as the following truth-table shows:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>((p \cdot q)) \lor ((p \cdot \lnot q))</th>
<th>((p \cdot q)\lor(p \cdot \lnot q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Therefore, by the *Equivalence Rule*,

\[
\Pr(p) = \Pr((p \cdot q) \lor (p \cdot \lnot q)) \quad (6.27)
\]

Moreover, the same truth-table shows that \('p \cdot q'\) and \('p \cdot \lnot q'\) are mutually exclusive. Therefore, by the *restricted disjunction rule*,

\[
\Pr((p \cdot q) \lor (p \cdot \lnot q)) = \Pr(p \cdot q) + \Pr(p \cdot \lnot q) \quad (6.28)
\]

Putting together (6.27) and (6.28), we get

\[
\Pr(p) = \Pr(p \cdot q) + \Pr(p \cdot \lnot q) \quad (6.29)
\]

And, by the *general conjunction rule*, from (6.29) we can get

\[
\Pr(p) = \Pr(p \mid q) \times \Pr(q) + \Pr(p \mid \lnot q) \times \Pr(\lnot q) \quad (6.30)
\]

(6.29) and (6.30) give us the **total probability rules**.
**Total Probability Rules**

For any wffs of PL, \( p \) and \( q \),

\[
Pr(p) = Pr(p \cdot q) + Pr(p \cdot \sim q)
\]

and

\[
Pr(p) = Pr(p \mid q) \times Pr(q) + Pr(p \mid \sim q) \times Pr(\sim q)
\]

6.3.8 **General Disjunction Rule**

The following truth table shows that \( p \lor q \) is logically equivalent to \( (p \cdot q) \lor ((p \cdot \sim q) \lor (\sim p \cdot q)) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( (p \cdot q) \lor ((p \cdot \sim q) \lor (\sim p \cdot q)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T T T T T F F T T F F T T F F T T F F T T F F T F T F T T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T F T F T F T T T T T F T T F T F F T T F T F T T F F T F T T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T F F T T F F T T T T F F T T F T F F T T F T F T T F F T F T T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F F F F F F F T T F T F F T T F F T T F T F T T T T F F F F F F</td>
</tr>
</tbody>
</table>

Therefore, by the *equivalence rule*,

\[
Pr(p \lor q) = Pr((p \cdot q) \lor (p \cdot \sim q) \lor (\sim p \cdot q)) \tag{6.31}
\]

The same truth-table shows that \( p \cdot q \) and \( (p \cdot \sim q) \lor (\sim p \cdot q) \) are mutually exclusive, so by the *restricted disjunction rule* and (6.31)

\[
Pr(p \lor q) = Pr(p \cdot q) + Pr((p \cdot \sim q) \lor (\sim p \cdot q)) \tag{6.32}
\]

Moreover, the truth table also shows that \( p \cdot \sim q \) and \( \sim p \cdot q \) are mutually exclusive, so by the *restricted disjunction rule* and (6.32)

\[
Pr(p \lor q) = Pr(p \cdot q) + Pr(p \cdot \sim q) + Pr(\sim p \cdot q) \tag{6.33}
\]

The *total probability rule* tells us that \( * \) is \( Pr(p) \), so (6.33) entails

\[
Pr(p \lor q) = Pr(p) + Pr(\sim p \cdot q) \tag{6.34}
\]

\[
Pr(p \lor q) = Pr(p) + Pr(q \cdot \sim p) \tag{6.35}
\]

\[
Pr(p \lor q) = Pr(p) + Pr(q \cdot \sim p) + Pr(p \cdot q) - Pr(p \cdot q) \tag{6.36}
\]

\[
Pr(p \lor q) = Pr(p) + Pr(q \cdot \sim p) + Pr(q \cdot p) - Pr(p \cdot q) \tag{6.37}
\]

\[
Pr(p \lor q) = Pr(p) + Pr(q \cdot p) + Pr(q \cdot \sim p) - Pr(p \cdot q) \tag{6.38}
\]
§6.3. Rules for the Probability Calculus

The *total probability rule* tells us that \( \dagger \) is \( \text{Pr}(q) \), so \((6.38)\) entails

\[
\text{Pr}(p \lor q) = \text{Pr}(p) + \text{Pr}(q) - \text{Pr}(p \cdot q) \tag{6.39}
\]

This gives us the **general disjunction rule**.

---

**General Disjunction Rule**
For any wffs of PL, \( p \) and \( q \),

\[
\text{Pr}(p \lor q) = \text{Pr}(p) + \text{Pr}(q) - \text{Pr}(p \cdot q)
\]

---

The restricted disjunction rule now follows from the general disjunction rule, together with the fact that, when \( p \) and \( q \) are mutually exclusive, then \( p \cdot q \) will be PL-self-contradictory. Since PL-self-contradictions have probability zero, if \( p \) and \( q \) are mutually exclusive,

\[
\text{Pr}(p \lor q) = \text{Pr}(p) + \text{Pr}(q) - \text{Pr}(p \cdot q) = 0 \tag{6.40}
\]

\[
\text{Pr}(p \lor q) = \text{Pr}(p) + \text{Pr}(q) \tag{6.41}
\]

---

6.3.9 Bayes’s Theorem

From the definition of conditional probability, we have that

\[
\text{Pr}(p \mid q) = \frac{\text{Pr}(p \cdot q)}{\text{Pr}(q)} \tag{6.42}
\]

From the *general conjunction rule*, we have that

\[
\text{Pr}(p \cdot q) = \text{Pr}(p) \times \text{Pr}(q \mid p) \tag{6.43}
\]

Putting together \((6.42)\) and \((6.43)\), we get

\[
\text{Pr}(p \mid q) = \frac{\text{Pr}(p) \times \text{Pr}(q \mid p)}{\text{Pr}(q)} \tag{6.44}
\]

This gives us the first version of Bayes’s Theorem.

---

**Bayes’s Theorem, 1**
For any wffs of PL, \( p \) and \( q \),

\[
\text{Pr}(p \mid q) = \frac{\text{Pr}(p) \times \text{Pr}(q \mid p)}{\text{Pr}(q)}
\]
Moreover, the total probability rule tells us that, for any wff \( q \),
\[
\Pr(q) = \Pr(q \mid p) \times \Pr(p) + \Pr(q \mid \neg p) \times \Pr(\neg p)
\]  
(6.45)
Substituting (6.45) into Bayes’s Theorem, we get
\[
\Pr(p \mid q) = \frac{\Pr(p) \times \Pr(q \mid p)}{\Pr(q \mid p) \times \Pr(p) + \Pr(q \mid \neg p) \times \Pr(\neg p)}
\]  
(6.46)
(6.46) gives us the second version of Bayes’s Theorem, 1.
they are both false. For instance, the Venn diagram in figure 6.1 tells us that each of these
four possibilities is equally likely; they are each assigned probability 1/4.

\[
\begin{align*}
\Pr_1(A \land B) &= 1/4 & \Pr_1(A \land \sim B) &= 1/4 \\
\Pr_1(\sim A \land \sim B) &= 1/4 & \Pr_1(\sim A \land B) &= 1/4
\end{align*}
\]

One way of thinking about a probability function, that as far as I know, originates with the
philosopher Bas, is as a muddy Venn diagram. That is, we take our original, un-muddied
Venn diagrams, where the domain of the diagram \(\mathcal{D}\), is the set of all possibilities, and we
spread a heap of mud on top of them, in the following way: if we think that a particular
area of the Venn diagram is more likely to contain the actual world, then we put more mud
on top of it. If we think that a particular area of the Venn diagram is less likely to contain
the actual world, then we put less mud on top of it. Then, if we want to know how likely
it is that any particular area of the Venn diagram contain the actual world, then we just
measure the total amount of mud on top of that area of the diagram, and we divide it by
the total amount of mud on the table.

Thus, we can figure out the probability of propositions like \(A, B, A \lor B, \sim (A \land B)\), and
so on, by just looking at the Venn diagram in figure 6.1, and counting up all the numbers
which fall within the areas of the diagram in which \(A, B, A \lor B, \sim (A \land B)\) are true,

\[
\begin{align*}
\Pr_1(A) &= 1/2 & \Pr_1(A \lor B) &= 3/4 \\
\Pr_1(B) &= 1/2 & \Pr_1(\sim (A \land B)) &= 3/4
\end{align*}
\]

Suppose, on the other hand, that we have a muddy Venn diagram like the one shown in
figure 6.2 (where we’re using the same interpretation). According to that muddy Venn
diagram,

\[
\begin{align*}
\Pr_2(A \land B) &= 2/10 & \Pr_2(A \land \sim B) &= 3/10 \\
\Pr_2(\sim A \land \sim B) &= 1/10 & \Pr_2(\sim A \land B) &= 4/10
\end{align*}
\]

and

\[
\begin{align*}
\Pr_2(A) &= 1/2 & \Pr_2(A \lor B) &= 9/10 \\
\Pr_2(B) &= 3/5 & \Pr_2(\sim (A \land B)) &= 8/10
\end{align*}
\]
This metaphor also gives us a nice, intuitive way of thinking about conditional probability. When we want to figure out that conditional probability of \( p \) given that \( q \), \( \Pr(p \mid q) \) we just wipe all of the mud outside of \( q \) off of our diagram, measure how much is left sitting on top of \( p \), and divide that by the total amount of mud left on the diagram. For instance, if we’re given the muddy Venn diagram in figure 6.1, and we want to figure out the probability of \( B \) given that \( A \), \( \Pr_1(B \mid A) \), we just remove all of the mud laying outside of \( A \) and see how much of the remaining mud lies on top of \( B \), as shown in figure 6.3. Once we’ve swept all the mud outside of \( A \) off the diagram, we just figure out how much of the remaining mud lies on top of \( A \cdot B \) and half of the remaining mud on \( A \cdot B \). So,

\[
\Pr_1(B \mid A) = \frac{\Pr_1(B \cdot A)}{\Pr_1(A)} = \frac{1/4}{2/4} = \frac{1}{2}
\]

Similarly, if we’re given the muddy Venn diagram shown in figure 6.2, and we want to know the probability of \( B \) given that \( A \), we just wipe all the mud lying outside of \( A \) off of the diagram, leaving behind \( 3/10 \) of the original mud on \( A \cdot \sim B \) and \( 2/10 \) of the original mud on \( A \cdot B \). Thus, there is \( 3/5 \) of the remaining mud on \( A \cdot \sim B \) and \( 2/5 \) of the remaining mud on \( A \cdot B \). (As shown on the left-hand-side of figure 6.4.) Thus,

\[
\Pr_2(B \mid A) = \frac{\Pr_2(B \cdot A)}{\Pr(A)} = \frac{2/10}{5/10} = \frac{2}{5}
\]

Or, if we’re given the muddy Venn diagram shown in figure 6.2, and we want to know that probability of \( B \), given that \( \sim A \), we just wipe all the mud lying outside of \( \sim A \) (that is, all the mud lying within \( A \), off of the diagram, leaving behind \( 1/10 \) of the original mud lying on \( \sim A \cdot \sim B \) and \( 4/10 \) of the original mud lying on \( \sim A \cdot B \). Thus, there is \( 1/5 \) of the remaining mud on \( \sim A \cdot \sim B \) and \( 4/5 \) of the remaining mud lying on \( \sim A \cdot B \). (As shown on the right-hand-side of figure 6.4.) So

\[
\Pr_2(B \mid \sim A) = \frac{\Pr_2(B \cdot \sim A)}{\Pr_2(\sim A)} = \frac{4/10}{5/10} = \frac{4}{5}
\]
Figure 6.3: How to attain the function $\Pr_1(\cdot \mid A)$ from the probability function $\Pr_1(\cdot)$ shown in figure 6.1.
Figure 6.4: How to attain the functions $\Pr_2(\cdot | A)$ and $\Pr_2(\cdot | \sim A)$ from the probability function $\Pr_2(\cdot)$ shown in figure 6.2.
§6.5. Deductive Validity and Inductive Strength

Now, we can begin to better see the close relationship between deductive validity and inductive strength. We saw before that an argument from the premises $p_1$ and $p_2$ to the conclusion $c$, $p_1 / p_2 \vdash c$, was deductively valid if and only if there were no possibilities in which both of the premises were true, yet the conclusion was simultaneously false. That is: the claim that $p_1 / p_2 \vdash c$ is deductively valid is shown in the Venn diagram on the left-hand-side of figure 6.5, where we are using the interpretation given below:

\[ D = \text{the set of all possibilities} \]
\[ P_1 = \text{the set of possibilities in which } p_1 \text{ is true} \]
\[ P_2 = \text{the set of possibilities in which } p_2 \text{ is true} \]
\[ C = \text{the set of possibilities in which } c \text{ is true} \]

And $p_1 / p_2 \vdash c$ is deductively invalid if and only if there is some possibility in which the premises $p_1$ and $p_2$ are both true, yet the conclusion $c$ is false. That is: the claim that $p_1 / p_2 \vdash c$ is deductively invalid is shown on the right-hand-side of figure 6.5.

However, just because it is possible for the premises of an argument to be true while its conclusion is false, this doesn’t tell us anything about how likely it is that the conclusion is false while the premises are true. As I said at the start of the course, there are some deductively invalid arguments that are, nevertheless, astonishingly good arguments. For instance,

1. Every human born before 1880 has died.
2. So, I will die.

is deductively invalid; nevertheless, its premise gives fantastic reason to believe its conclusion. If we find ourselves with an invalid argument—that is, if we find ourselves with an argument like the one represented on the right-hand-side of figure 6.5—and we want to know how likely the premises of the argument make its conclusion, we can muddy the Venn diagram up by laying a probability distribution over top it, as shown on the left-hand-side of figure 6.6. We can then ask how likely the conclusion is given the truth of the premises, by looking at the conditional probability $\Pr(\text{conclusion} \mid \text{premises})$, as shown on the

Figure 6.5: If the Venn diagram on the left is correct, then $p_1 / p_2 \vdash c$ is deductively valid. If the Venn diagram on the right is correct, then $p_1 / p_2 \vdash c$ is deductively invalid.
right-hand-side of figure 6.6. That muddy Venn diagram tells us that

$$\Pr(c \mid p_1 \cdot p_2) = \frac{3}{4}$$

So the argument is inductively strong—it’s inductive strength is 0.75.

From this standpoint, we can see that a deductively valid argument is just the inductively strongest argument possible. That is: it is an inductively strong argument with a strength of 1.

An argument is **deductively valid** only if

$$\Pr(\text{conclusion} \mid \text{premises}) = 1$$

So the notion of inductive strength is a generalization of the notion of deductive validity.

## 6.6 Examples

### 6.6.1 The Gambler’s Fallacy

Suppose that I have a fair die—*i.e.*, it has a probability of $1/6$ of landing 1, a probability of $1/6$ of landing 2, and so on and so forth. I roll the die five times. Each die roll is probabilistically independent of all of the others. It lands 1 the first four times. Consider the following argument.

1. The die landed 1 the first four rolls
2. The die will land 1 on the fifth roll

Let ‘$O_1$’ be the statement that the die lands 1 on the first roll, let ‘$O_2$’ be the statement that the die lands 1 on the second roll, and so on and so forth.
To determine the inductive strength of the argument, we need to determine

$$\Pr(O_5 \mid O_1 \cdot (O_2 \cdot (O_3 \cdot O_4)))$$

By the definition of conditional probability, we get

$$\Pr(O_5 \mid O_1 \cdot (O_2 \cdot (O_3 \cdot O_4))) = \frac{\Pr(O_5 \cdot (O_1 \cdot (O_2 \cdot (O_3 \cdot O_4))))}{\Pr(O_1 \cdot (O_2 \cdot (O_3 \cdot O_4)))}$$

And, because the outcomes of the die rolls are probabilistically independent,

$$\Pr(O_5 \mid O_1 \cdot (O_2 \cdot (O_3 \cdot O_4))) = \frac{\Pr(O_5) \times \Pr(O_1 \cdot (O_2 \cdot (O_3 \cdot O_4)))}{\Pr(O_1 \cdot (O_2 \cdot (O_3 \cdot O_4)))} = \Pr(O_5) = 1/6$$

So this argument is inductively quite weak. The fact that the die roll has landed 1 several times in a row gives us no reason to think that it will land 1 on the next roll, given that the outcomes of the die rolls are probabilistically independent. The impulse to say otherwise is known as 'the gambler's fallacy'.

