Inverse Born series for the Calderon problem

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Abstract

We propose a direct reconstruction method for the Calderon problem based on inversion of the Born series. We characterize the convergence, stability and approximation error of the method and illustrate its use in numerical reconstructions.

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1980, Calderon published a widely influential paper on the inverse problem of recovering the spatially varying electrical conductivity of a medium from boundary measurements [5]. More precisely, he posed the following problem. In a bounded domain with conductivity $\sigma$ and Dirichlet-to-Neumann map $\Lambda_{\sigma}$, decide whether $\sigma$ is uniquely determined by $\Lambda_{\sigma}$ and, assuming this is true, reconstruct $\sigma$ from $\Lambda_{\sigma}$. There are numerous applications of this problem to biomedical imaging, geophysics and non-destructive testing. See [3, 20] for reviews of the massive body of work in this field. We note, in particular, that a variety of reconstruction algorithms have been developed. Many of them have also been analyzed theoretically in recent years. These include the $\tilde{\delta}$-method [10, 12, 18], regularized Newton-type schemes [4, 9, 13] and linearization methods [6, 7, 19]. The Calderon problem is closely related to the inverse problem of optical tomography, where similar algorithms have been explored [1].

In previous work, we have proposed a direct method to solve the inverse problem of optical tomography that is based on inversion of the Born series [15–17]. In this approach, the solution to the inverse problem is expressed as an explicitly computable functional of the scattering data. In combination with a spectral method for solving the linear inverse problem, the inverse Born series leads to a fast image reconstruction algorithm with analyzable convergence, stability and error.

In this paper, we apply the inverse Born series to the Calderon problem. We characterize the convergence, stability and approximation error of the method. We also illustrate its use in numerical simulations and compare the results to reconstructions performed with
an optimization method. We find that the series appears to converge quite rapidly for low-contrast objects. As the contrast is increased, the higher order terms systematically improve the reconstructions until, at sufficiently large contrast, the series diverges.

The remainder of this paper is organized as follows. In section 2, we construct the Born series for the Calderon problem. We then derive various estimates that are later used to study the convergence of the inverse Born series. The inversion of the Born series is taken up in section 3, where we also obtain our main results on the convergence, stability and approximation error of the method. In section 4, we present the results of numerical reconstructions of a two-dimensional medium.

2. Forward problem

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$, for $d \geq 2$. We consider a scalar field $u$ that obeys the equation

$$\nabla \cdot \sigma(x) \nabla u = 0 \quad \text{in} \quad \Omega,$$

where the coefficient $\sigma(x) > 0$ for all $x \in \Omega$. The field is also taken to satisfy the Robin boundary condition

$$u + \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega,$$

where $\sigma$ and $z$ are non-negative and constant on $\partial\Omega$. In electrical impedance tomography, the field $u$ is identified with the electric potential and the coefficient $\sigma$ with the conductivity [7]. The coefficient $z$ in (2) is the surface impedance and $g$ is the current density. A typical choice for $g$ is a dipole source of unit strength:

$$g = \delta_{x_1} - \delta_{x_2}, \quad x_1, x_2 \in \partial\Omega,$$

where $\delta_{x_1}$ and $\delta_{x_2}$ are Dirac delta functions at $x_1$ and $x_2$. Likewise, in optical tomography, $u$ is the energy density of a diffuse wave in a non-absorbing medium and $\sigma$ is the optical diffusion coefficient.

The forward problem is to determine the field $u$ for a given coefficient $\sigma$. To proceed, we assume that the conductivity is of the form $\sigma(x) = \sigma_0 (1 + \eta(x))$, where the background coefficient $\sigma_0 = \sigma|_{\partial\Omega}$ is constant and $\eta \in L^\infty(B_a)$ is assumed to be supported in a closed ball $B_a$ of radius $a$ centered at the origin. We thus find that (1) becomes

$$-\Delta u = \nabla \cdot \eta(x) \nabla u \quad \text{in} \quad \Omega.$$

The solution to (4) obeys the integral equation

$$u(x) = u_0(x) + \int_\Omega G(x, y) \nabla \cdot \eta(y) \nabla u(y) \, dy, \quad x \in \Omega,$$

where $u_0$ obeys (4) with $\eta = 0$ and satisfies the boundary condition (2). Here $G$ is Green’s function for the operator $-\Delta$, which obeys the boundary condition (2) with zero right-hand side. It is easily seen that $u_0$ is given by the formula

$$u_0(x) = \frac{1}{\sigma_0} \int_\Omega G(x, y) g(y) \, dy, \quad x \in \Omega.$$

Finally, upon integrating (5) by parts, we see that the solution to the forward problem obeys the integral equation

$$u(x) = u_0(x) - \int_\Omega \nabla G(x, y) \cdot \nabla u(y) \eta(y) \, dy, \quad x \in \Omega.$$
The integral equation (7) has a unique solution. If we apply fixed point iteration beginning with \( u = u_0 \), we obtain an infinite series for \( u \) of the form
\[
a(x) = u_0(x) + u_1(x) + u_2(x) + \cdots , \tag{8}
\]
where
\[
u_{n+1}(x) = - \int_{\Omega} \nabla_x G(x, y) \cdot \nabla u_0(y) \eta(y) \, dy, \quad n = 0, 1, \ldots . \tag{9}
\]
We will refer to (8) as the Born series and the approximation to \( u \) that results from retaining only the linear term in the series as the Born approximation.

It will prove useful to express the Born series as a formal power series in tensor powers of \( \eta \) of the form
\[
\phi = K_1 \eta + K_2 \eta \otimes \eta + K_3 \eta \otimes \eta \otimes \eta + \cdots , \tag{10}
\]
where \( \phi = u_0 - u \). Each term in the series is multilinear in \( \eta \) and the forward operators \( K_\alpha \) are defined by
\[
(K_1 \eta)(x) = - \int_{\Omega} \eta(y_1) \nabla_y G(y_1, x) \cdot \nabla u_0(y_1) \, dy_1, \tag{11}
\]
\[
(K_2 \eta \otimes \eta)(x) = \int_{\Omega} \eta(y_1) \nabla_y G(y_1, x) \cdot \nabla \eta \int_{\Omega} \eta(y_2) \nabla_y G(y_2, y_1) \cdot \nabla u_0(y_2) \, dy_1 \, dy_2, \tag{12}
\]
and in general
\[
(K_\alpha \eta \otimes \cdots \otimes \eta)(x) = (-1)^\alpha \int_{\Omega} \eta(y_1) \nabla_y G(y_1, x) \cdot \nabla \eta \int_{\Omega} \eta(y_2) \nabla_y G(y_2, y_1) \cdot \nabla \eta \int_{\Omega} \cdots \int_{\Omega} \eta(y_\alpha) \nabla_y G(y_\alpha, y_{\alpha-1}) \cdot \nabla \eta u_0(y_\alpha) \, dy_1 \cdots \, dy_\alpha. \tag{13}
\]
In order to simplify the discussion of the Born series, we introduce the operators
\[
S : [L^2(\Omega)]^d \rightarrow H^1(\Omega), \tag{14}
\]
defined by
\[
(Sf)(x) = \int_{\Omega} \nabla_x G(x, y) \cdot f(y) \, dy, \tag{15}
\]
and
\[
T : [L^2(\Omega)]^d \rightarrow [L^2(\Omega)]^d, \tag{16}
\]
defined by
\[
Tf = \nabla (Sf). \tag{17}
\]
Note that \( S \) is a bounded operator (it has a weakly singular kernel) and is smoothing by one order [8]. The boundedness of \( T \) follows automatically. Using the above definitions, we can rewrite the forward operators in the form
\[
K_1 \eta = - S(\eta \nabla u_0), \tag{18}
\]
\[
K_2 \eta \otimes \eta = S(\eta T(\eta \nabla u_0)) \tag{19}
\]
and
\[
K_\alpha \eta \otimes \cdots \otimes \eta = (-1)^\alpha S(\eta(T(\cdots \eta(T(\eta \nabla u_0))))). \tag{20}
\]
We observe that \( \| \eta \nabla u_0 \|_{L^p(\Omega)} \) is bounded for \( p = 2, \infty \), since \( \eta \) is compactly supported in \( \Omega \) and \( u_0 \) is smooth away from \( \partial \Omega \). Note that \( K_\alpha \) may be extended to act naturally as a multilinear form on \( L^n(\Omega) \):
\[
K_\alpha (\eta_1 \otimes \cdots \otimes \eta_\alpha) = (-1)^\alpha S(\eta_1(T(\cdots \eta_\alpha T(\eta_\alpha \nabla u_0)))) \tag{21}
\]
The following lemma establishes the boundedness of \( K_\alpha \) as a multilinear form on \( L^n \).
Lemma 2.1. Suppose that \( \eta_1, \eta_2, \ldots, \eta_n \in L^\infty(B_a) \) are compactly supported in a closed ball \( B_a \) of radius \( a \). Then if \( K_n \) is defined by (21), we have
\[
\| K_n \eta_1 \otimes \cdots \otimes \eta_n \|_{L^\infty(\Omega)} \leq \| \eta_1 \|_{L^\infty(B_a)} \cdots \| \eta_n \|_{L^\infty(B_a)} \sup_{x \in \Omega} \| \nabla G(x, \cdot) \|_{L^2(B_a)} \| \nabla u_0 \|_{L^2(B_a)} \| T \|^{n-1},
\]
where \( T \) is defined by (17).

Proof. Using the definition of the operator norm, we obtain the estimate
\[
\| T(\eta f) \|_{L^2(\Omega)} \leq \| T \| \| \eta f \|_{L^2(\Omega)}
\]
\[
\leq \| T \| \| \eta \|_{L^\infty(B_a)} \| f \|_{L^2(B_a)},
\]
where \( f \) is an arbitrary vector field. We also see from definition (15) of the operator \( S \) that
\[
\| S(\eta f) \|_{L^\infty(\Omega)} \leq \| \eta \|_{L^\infty(B_a)} \sup_{x \in \Omega} \| \nabla G(x, \cdot) \|_{L^2(B_a)} \| f \|_{L^2(B_a)},
\]
and thus
\[
\| S(\eta_1(T(\cdots(\eta_{n-1}T(\eta_n \nabla u_0))) \cdots)) \|_{L^\infty(\Omega)} \leq \| \eta_1 \|_{L^\infty(B_a)} \sup_{x \in \Omega} \| \nabla G(x, \cdot) \|_{L^2(B_a)} \| T(\cdots(\eta_{n-1}T(\eta_n \nabla u_0))) \|_{L^2(\Omega)}.
\]
Applying (23) iteratively, we find that
\[
\| T(\cdots(\eta_{n-1}T(\eta_n \nabla u_0))) \|_{L^2(\Omega)} \leq \| T \|^{n-1} \| \eta_1 \|_{L^\infty(B_a)} \cdots \| \eta_n \|_{L^\infty(B_a)} \| \nabla u_0 \|_{L^2(B_a)},
\]
which combines with the above to obtain the statement of the lemma. \( \Box \)

We now obtain a bound on the norm of the operator \( T \).

Lemma 2.2. The operator \( T : [L^2(\Omega)]^d \rightarrow [L^2(\Omega)]^d \) defined by (15) and (17) is bounded and its norm obeys the estimate
\[
\| T \| \leq 1.
\]

Proof. Let \( f \in C_0^\infty(\Omega) \). Then
\[
Sf(x) = \int_\Omega \nabla_x G(x, y) f(y) \, dy
\]
\[
= -\int_\Omega G(x, y) \nabla_y f(y) \, dy,
\]
where we have integrated by parts. Since \( G \) is Green’s function for the boundary value problem for the Laplacian with homogeneous Robin boundary conditions, we have
\[
Sf = \phi,
\]
where \( \phi \) is the unique solution to
\[
-\Delta \phi = \nabla \cdot f \quad \text{in} \quad \Omega
\]
\[
\phi + \zeta \sigma_0 \frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \partial \Omega.
\]
It follows from (17) that
\[
Tf = \nabla \phi,
\]
from which we compute

$$\|Tf\|_{L^2(\Omega)}^2 = \int_\Omega \nabla \phi \cdot \nabla \phi \, dx$$

(33)

$$= \int_\Omega \nabla \cdot f \, \phi \, dx + \int_\partial \Omega \frac{\partial \phi}{\partial v} \phi \, dx$$

(34)

$$= - \int_\Omega f \cdot \nabla \phi \, dx - \frac{1}{z\sigma_0} \int_\partial \Omega \phi^2 \, dx.$$  

(35)

Hence,

$$\|Tf\|_{L^2(\Omega)}^2 + \frac{1}{z\sigma_0} \int_\partial \Omega \phi^2 \, dx = - \int_\Omega f \cdot \nabla \phi \, dx.$$  

(36)

Since both terms on the left-hand side are non-negative and \(Tf = \nabla \phi\),

$$\|Tf\|_{L^2(\Omega)}^2 \leq - \int_\Omega f \cdot \nabla \phi \, dx$$

(37)

$$\leq \|f\|_{L^2(\Omega)} \|Tf\|_{L^2(\Omega)}.$$  

(38)

We thus have

$$\|Tf\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$  

(39)

We now use a standard density argument and the fact that \(C^\infty_0(\Omega)\) is dense in \(L^2(\Omega)\) to obtain the statement of the lemma.

We can extend \(K_n\) to be a map on all of \(L^\infty(B_a \times \cdots \times B_a)\) by replacing the product \(\eta(y_1) \cdots \eta(y_n)\) in (13) by a function \(f(y_1, \ldots, y_n) \in L^\infty\). We can thereby obtain an estimate on the norm of \(K_n\).

**Lemma 2.3.** The operator \(K_n : L^\infty(B_a \times \cdots \times B_a) \rightarrow L^\infty(\partial \Omega)\) defined by (13) is bounded and

$$\|K_n\| \leq v,$$

where

$$v = \sup_{x \in \partial \Omega} \|\nabla G(x, \cdot)\|_{L^2(B_a)} \|\nabla u_0\|_{L^2(B_a)}.$$  

(40)

**Proof.** Using lemmas 2.1 and 2.2, we have

$$\|K_n\| \leq \sup_{x \in \partial \Omega} \|\nabla G(x, \cdot)\|_{L^2(B_a)} \|\nabla u_0\|_{L^2(B_a)} \|T\|^{n-1}$$

(41)

$$\leq \sup_{x \in \partial \Omega} \|\nabla G(x, \cdot)\|_{L^2(B_a)} \|\nabla u_0\|_{L^2(B_a)},$$

(42)

from which the result follows at once.

As an application of lemma 2.3, we obtain a sufficient condition for the convergence of the Born series (10).

**Proposition 2.1** (Convergence of the Born series for the Calderon problem). If the smallness condition \(\|\eta\|_{L^\infty(B_a)} < 1\) holds, then the Born series (10) converges in the \(L^\infty(\partial \Omega)\) norm.
Proof. We majorize the sum
\[ \sum_n \|K_n \eta \otimes \cdots \otimes \eta\|_{L^\infty(\partial \Omega)} \]
by a geometric series
\[ \sum_n \|K_n \eta \otimes \cdots \otimes \eta\|_{L^\infty(\partial \Omega)} \leq \sum_n \|K_n\| \|\eta\|_{L^\infty(\mathcal{B}_\eta)}^n \]
which converges if \( \|\eta\|_{L^\infty(\mathcal{B}_\eta)} < 1 \).

Remark 2.1. The above result asserts that the Born series converges under a smallness condition on the norm of \( \eta \). However, we expect that this bound is not sharp when the support of \( \eta \) is small. That is, for small volume perturbations, it should be possible to weaken the convergence condition by making use of the smallness of \( \mathcal{B}_\eta \) in the estimates.

3. Inverse Born series

The inverse problem is to determine the coefficient \( \eta \) everywhere within \( \Omega \) from measurements of the data \( \phi \) on \( \partial \Omega \). The function \( \phi \) depends implicitly upon the position of the source. For example, in the case of the dipole source (3), if we fix the point \( x_1 \in \partial \Omega \) and vary \( x_2 \in \partial \Omega \), then \( \phi \) will depend upon both \( x_2 \) and the point \( x \) at which we measure the field on \( \partial \Omega \). We will indicate this dependence explicitly and, accordingly, will assume that \( \phi \in L^\infty(\partial \Omega \times \partial \Omega) \).

Following [15, 16], we express \( \eta \) as a series in tensor powers of \( \phi \) of the form
\[ \eta = K_1 \phi + K_2 \phi \otimes \phi + K_3 \phi \otimes \phi \otimes \phi + \cdots , \]
where the \( K_m \) are the operators that are to be determined. To proceed, we substitute expression (10) for \( \phi \) into (46) and equate terms with the same tensor power of \( \eta \). We thus obtain the relations
\[ K_1 = K_1^+, \]
\[ K_2 = -K_1 K_2 K_1 \otimes K_1, \]
\[ K_3 = -(K_2 K_1 \otimes K_2 + K_2 K_2 \otimes K_1 + K_1 K_3) K_1 \otimes K_1 \otimes K_1, \]
\[ K_m = -\left( \sum_{i=1}^{m-1} K_m \sum_{i_1+i_2+\cdots+i_m} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) K_1 \otimes \cdots \otimes K_1. \]

We will refer to (46) with the operators defined in this manner as the inverse Born series. Here \( K_1^+ \) is a regularized pseudoinverse of the operator \( K_1 \). The regularized solution to the linearized inverse problem is given by \( \eta = K_1^+ \phi \).

In [11, 16] we showed that if the operators \( K_m \) defined by (13) obey the generic condition \( \|K_m\| \leq \mu^{n-1} \), then the corresponding inverse Born series converges if
\[ \|K_1\|, \|K_1 \phi\|_{L^\infty(\mathcal{B}_\eta)} < 1/(\mu + \nu). \]

For the Calderon problem, it follows from lemma 2.3 that \( \mu = 1 \). The analysis of [16] then shows that if conditions (51) are obeyed (with \( \mu = 1 \)), then the estimate
\[ \|\tilde{\eta} - \sum_{n=1}^N K_n \phi \otimes \cdots \otimes \phi\|_{L^\infty} \leq C \frac{(1 + \nu)\|K_1 \phi\|_{L^\infty}^{N+1}}{1 - (1 + \nu)\|K_1 \phi\|_{L^\infty}} \]
for the series limit \( \tilde{\eta} \) holds, where \( C = C(\nu, \|K_1\|) \) does not depend on \( N \) nor on \( \phi \).
Next, as shown in [16], we note that when the inverse Born series converges, its limit is stable under perturbations in the measured data. Suppose that \( \|K_1\| < 1/(1 + \nu) \), and \( \phi_1 \) and \( \phi_2 \) are data for which \( M\|K_1\| < 1/(1 + \nu) \), where \( M = \max(\|\phi_1\|, \|\phi_2\|) \). Then if \( \eta_1 \) and \( \eta_2 \) are the corresponding limits of the inverse Born series, the following stability estimate holds,
\[
\|\eta_1 - \eta_2\|_{L^\infty(B_1)} < C\|\phi_1 - \phi_2\|_{L^\infty(\Omega \times \Omega)},
\]
where \( C = C(\nu, \|K_1\|, M) \) is a constant which is otherwise independent of \( \phi_1 \) and \( \phi_2 \).

The limit of the inverse Born series will not, in general, coincide with the true coefficient \( \eta \).

The approximation error may be characterized as follows. Suppose that \( \|K_1\| < 1/(1 + \nu) \) and \( \|\eta\|_{L^\infty(B_1)} < 1/(1 + \nu) \). Let \( M = \max(\|\phi\|, \|K_1\|, \|\eta\|_{L^\infty(B_1)}) \) and assume that \( M < 1/(1 + \nu) \).

Note that there is no ambiguity in the definition of \( M \) since \( \|K_1\|, \|\eta\|_{L^\infty(B_1)} \). Due to the regularization needed to calculate \( K_1 \), then the norm of the difference between the limit of the inverse series \( \eta \) and the true coefficient \( \eta \) obeys the estimate
\[
\|\eta - \eta\|_{L^\infty(B_1)} \leq C\|I - K_1\|\|\eta\|_{L^\infty(B_1)},
\]
where \( C = C(\nu, \|K_1\|, M) \) is independent of \( N \) and \( \phi \) [16].

**Remark 3.1.** In the above, we emphasize that \( K_1 \) is the regularized pseudoinverse of \( K_1 \), not the true inverse. As a consequence in (53) we obtain Lipschitz stability for the recovery of the *low-frequency* part of \( \eta \). Similarly, the error estimate in (54) applies only at low frequencies. As the regularization of \( K_1 \) is weakened (that is, the regularization parameter \( \beta \) becomes small), the constant \( C \) blows up and the inverse Born series diverges. Thus, there is no contradiction to global logarithmic stability for the recovery of \( \eta \) at all frequencies. Nevertheless, if \( \eta \) is known *a priori* to belong to the subspace on which it is possible to invert \( K_1 \), then the limit of the series \( \eta \) coincides precisely with \( \eta \). Thus, we obtain an explicit characterization of what may be reconstructed from the inverse Born series.

It is straightforward to compute the constant \( v \) given by (40) for special geometries. To make contact with the numerical results in section 4, we consider the case of a disk of radius \( R \), for which Green’s function is given by the formula
\[
G(x, x') = -\frac{1}{2\pi} \ln \left( \frac{\|x - x'\|}{R} \right) - \frac{1}{2\pi} \sum_{n=1}^\infty \frac{1}{n} \left( \frac{rr'}{R^2} \right) \frac{R - n\sigma_0}{R + n\sigma_0} \cos(n(\theta - \theta')), \tag{55}
\]
where in polar coordinates \( x = (r, \theta) \) and \( x' = (r', \theta') \). Using this result and the definition of \( v \) in lemma 2.3, we obtain
\[
v = \frac{(\sigma_0)^2}{\pi} \sum_{n=1}^\infty \frac{n}{(R + n\sigma_0)^2} \left( \frac{a}{R} \right)^{2n}. \tag{56}
\]
Note that \( v = 0 \) for the case of Dirichlet boundary conditions, where \( \sigma = 0 \). For Neumann boundary conditions, where \( \sigma \to \infty \), the above series may be summed with the result
\[
v = \frac{1}{\pi} \ln \frac{R}{R - a}, \tag{57}
\]
when \( a < R \).

We will refer to the quantity \( R = 1/(1 + \nu) \) as the radius of convergence of the inverse Born series. In figure 1, we have plotted \( R \) as a function of \( a/R \) for various values of the parameter \( \xi = \sigma_0/R \).

4. Numerical reconstructions

We have performed numerical studies to test the inverse Born series reconstruction method in two dimensions. We note that there is no fundamental reason why the method could not be implemented in three dimensions.
4.1. Finite element method

We implement the solution to (1) obeying the boundary condition (2) using a Galerkin finite element method (FEM) defined on the triple \([N, T, U]\). Here \(T = \{\tau_i\}, i = 1, \ldots, P\), are the set of disjoint elements of the FEM mesh with union \(\bar{\Omega} = \bigcup_{i=1}^{P} \tau_i\), which approximates the domain \(\Omega\). The nodes of the mesh are \(N = \{N_j\}\) and the interpolating shape functions are \(U = \{u_j(x)\}, j = 1, \ldots, N\). We represent

\[ U \simeq U^h = \sum_{j=1}^{N} U_j u_j(x) \tag{58} \]

and define the \(N \times N\) system matrices \(A, B\) and the source vector \(f\) by

\[ A_{jj}' = \int_{\Omega} \sigma \nabla u_j \cdot \nabla u_j' \, dx, \quad B_{jj}' = \int_{\partial \Omega} u_j u_j' \, dx, \quad f_j = \int_{\partial \Omega} u_j g \, dx. \tag{59} \]

We also define the vector of coefficients \(U\) and introduce the system of linear equations

\[ \left[ A + \frac{1}{z} B \right] U = f, \tag{60} \]

which lead to the solution to (1).

4.2. Forward operators

Inversion of the Born series calls for the construction of the operators \(K_n\).

\textbf{Algorithm 1} Construction of forward operators

\begin{verbatim}
Function \( \phi = K(\ell, [\eta_1, \ldots, \eta_\ell]) \)
Solve \( [A + 1/zB]U_0 = f \)
for all orders \( \ell' = 1, \ldots, \ell \) do
    Solve \( [A + 1/zB]U_{\ell'} = H(\eta_{\ell'})U_{\ell'-1} \)
end for
return measurement \( M U_{\ell} \)
\end{verbatim}
Here $M$ is the measurement trace operator taking the potential on the boundary $\partial \Omega_1$, and $H(\eta)$ is the discrete implementation of the operator $\nabla \cdot \eta \nabla$ which has matrix elements

$$H_{ij} = \int_{\text{supp}(u_j, u_j')} \eta \nabla u_j \cdot \nabla u_j' \, dx. \quad (61)$$

**Remark 4.1.** The above algorithm computes $K_n$ from compositions of the $S$ and $T$ operators, each of which is found by solving a partial differential equation. Note that only the right-hand side of the equation changes at each step of this procedure.

### 4.3. Inverse series operators

The operators $K_n$ in the inverse Born series are constructed recursively according to the following algorithm.

**Algorithm 2 Construction of inverse series operators**

```plaintext
Function $\eta = K(j, [\phi_1, \ldots, \phi_j])$
If $j = 1$ return $\eta = K_1^+ \phi_1$
else
$\Sigma_1 = 0$;
for all $m = 1, \ldots, j - 1$ do
$\Sigma_2 = 0$;
for all $[i_1, \ldots, i_m] = \text{composition of } j \text{ of size } m$ do
$\phi_{\text{temp}} = [K_{i_1}(K_1^+ \phi_1, \ldots, K_1^+ \phi_{i_1}), \ldots, K_{i_m}(K_1^+ \phi_{j-i_m+1}, \ldots, K_1^+ \phi_j)]$
$\Sigma_2 = \Sigma_2 + K(m, \phi_{\text{temp}})$
end for
$\Sigma_1 = \Sigma_1 + \Sigma_2$
end for
return $\eta = \Sigma_1$
```

**Remark 4.2.** Note that a composition of integer $j$ is an ordered set of integers $i_1, \ldots, i_m$ such that $i_1 + \cdots + i_m = j$.

### 4.4. Linearized inversion

The solution to the linearized inverse problem is given in terms of the operator $K_1^\dagger$ defined in (47), which is constructed in the discrete setting and inverted explicitly with a first-order Tikhonov regularization scheme according to

$$K_1^\dagger = (K_1^T K_1 + \beta L)^{-1} K_1^T, \quad (62)$$

where $L$ is the discrete Laplacian on the reconstruction grid and $\beta$ is the regularization parameter. In certain geometries, including those with planar boundaries, it is possible to construct the operator $K_1^\dagger$ analytically [14].

### 4.5. Iterative Gauss–Newton solver

We have implemented an iterative Gauss–Newton solver for comparison with the inverse Born series. To do so, we relinearize at each step and use a line search to update the solution where we introduce
Algorithm 3 Gauss–Newton solver

Set $k = 0$ and initialize $\sigma_0$.

for all iterations $k = 1, \ldots, n$ do

update: Solve $h = (K_1(\sigma_{k-1})^TK_1(\sigma_{k-1}) + \beta L)^{-1}K_1(\sigma_{k-1})^T(g - F(\sigma_{k-1}))$

line search: set $\sigma_k = \sigma_{k-1} + \text{argmin}_t \Phi(\sigma_{k-1} + th)$

end for

\[
\Phi(\sigma) = \|f - F(\sigma)\|^2_2 + \beta \|L\sigma\|^2_2,
\]

which is the penalized error norm between the measured and computed data, and $F(\sigma)$ is the solution to the forward problem.

4.6. Results

Simulated reconstructions of a circular chest phantom were performed. The chest was taken to be a disk of radius $R = 40$ with $\sigma_0 = 1$, the parameter $z\sigma_0 = 1$, and 32 uniformly spaced electrodes were employed. A dipole source of the form (3) was employed. The delta functions in (3) were smoothed and taken to be of the form

\[
g(x) = \exp\left[-\frac{(x - x_1)^2}{2h^2}\right] - \exp\left[-\frac{(x - x_2)^2}{2h^2}\right], \quad x_1, x_2 \in \partial\Omega,
\]

where $h$ is a constant. In the reconstructions reported below, $h = 6$. A circular mesh with quadratic shape functions, $N = 21,697$ nodes and $P = 10,752$ triangular elements was used for the finite element forward solver. A circular mesh with linear shape functions, $N = 2857$ nodes and $P = 5544$ triangular elements was used for the finite element Gauss–Newton inverse solver. Both meshes are shown in figure 2. The reconstruction grid consisted of $64 \times 64$ linear pixels on which the images are displayed.

We now present a series of numerical simulations in which reconstructions carried out using the inverse Born series are compared to Gauss–Newton reconstructions. In all cases, the inverse series is calculated to fourth order and the Gauss–Newton iterations are run to
convergence with an error of $4 \times 10^{-4}$. We note that the first iteration of the Gauss–Newton reconstruction is not equivalent to the first term of the inverse Born series, because of the use of a line search in the former method. We further note that the inverse series does not guarantee positivity of the reconstructed coefficient. Thus, negative values of $\sigma$ are not shown in the reconstructed images (which does not noticeably affect the results).

In figure 3 we show a low-contrast reconstruction where $\sigma_{\text{lungs}} = 0.5$ and $\sigma_{\text{heart}} = 2$. These values of $\sigma$ are typical in electrical impedance tomography [10]. It can be seen that the series appears to converge rapidly and the results are similar to the Gauss–Newton reconstruction. Next, in figure 4, we present a reconstruction at intermediate contrast with $\sigma_{\text{lungs}} = 0.2$ and $\sigma_{\text{heart}} = 2$. In this case, the series converges more slowly, and at fourth order the artifacts
present in the linear reconstruction are largely removed. Once again, the Gauss–Newton method produces essentially similar results. Finally, we show a high-contrast reconstruction in figure 5 with $\sigma_{\text{lungs}} = 0.05$ and $\sigma_{\text{heart}} = 2$. Although there is some improvement at fourth order compared to the linear reconstruction, it is evident that the series has not converged. Likewise, the Gauss–Newton method, which has converged at 12 iterations, is similarly ineffective, as shown in figure 6.

We note that the condition for the convergence of the inverse Born series

\[
\|K_i\|, \|K_1\phi\|_{L^\infty(B_{k_j})} < R
\]

(65)
is partially met for the low- and medium-contrast reconstructions. It can be seen that for the case of medium contrast, $\|K_1\phi\|_{L^{\infty}(B)} \approx 0.5$ and $R < 1$. However, $\|K_1\| > 1$ and thus condition (65) is not strictly satisfied. It has been noted, in the case of optical tomography, that the condition on $\|K_1\phi\|$ is more useful than the corresponding condition on $\|K_1\|$ [17], and that the associated estimates are conservative. Evidently, it would be of interest to more fully understand this point.

We have tested the stability of the reconstructions in the presence of noise. In figures 7–9 we present results corresponding to intermediate contrast in the presence of additive Gaussian noise. That is, for each realization of the noise, the data $\phi$ becomes $\phi(1 + nx)$, where $x$ is a
**Figure 6.** High-contrast Gauss–Newton reconstruction of $\sigma$ run to convergence, with $\sigma_{\text{lungs}} = 0.05$, $\sigma_{\text{heart}} = 2$ and $\beta = 2 \times 10^{-3}$.

**Figure 7.** Reconstructions of $\sigma$ at medium contrast in the presence of 0.01% noise with $\sigma_{\text{lungs}} = 0.2$, $\sigma_{\text{heart}} = 2$ and $\beta = 2 \times 10^{-3}$.

Gaussian random variable with zero mean and unit variance, and $n = 0.001\%$, 0.01\%, 0.1\%. It can be seen that the reconstructions are relatively stable up to a noise level of 0.1\%.
5. Discussion

In summary, we have investigated the inverse Born series for the Calderon problem. We have analyzed the convergence of the series and have conducted numerical simulations to illustrate the use of the method.

We conclude with several remarks. (i) Numerical evidence suggests that the estimates we have obtained on the convergence of the inverse Born series are conservative. A similar
situation also occurs in optical tomography [17]. (ii) It is important to note that in the stability result (53) and the error estimate (54), the hypothesis of $L^\infty$ regularity for the coefficient $\eta$ and the scattering data $\phi$ is due to the smallness assumptions (51). In contrast, global uniqueness of solutions to the Calderon problem in two dimensions with $L^\infty$ regularity was a tour de force of mathematical analysis [2] and is an open problem in three dimensions [20]. (iii) Although the numerical studies described in this work were carried out for two-dimensional systems, there is no fundamental obstruction to implementing the inverse Born series method in higher dimensions. We plan to report three-dimensional reconstructions for both the Calderon problem and optical tomography in future work.

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