Physics 523, Quantum Field Theory II

Homework 4

Due Wednesday, 4th February 2004

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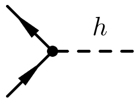
The Anomalous Magnetic Moments of e^- and μ^-

We are to investigate the possible contributions of scalar loops to the QED anomalous magnetic moments of the electron and muon. First we will consider contributions from a Higgs particle, h. We casually note that because the interaction Hamiltonian is given by,

$$H_{\text{int}} = \int d^x \; \frac{\lambda}{\sqrt{2}} h\overline{\psi}\psi,$$

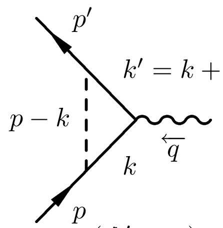
our vertex rule is



$$= -i\tfrac{\lambda}{\sqrt{2}}.$$

Therefore, we may now compute the amplitude for the following interaction.

$$i\mathscr{M} =$$



$$= \int \frac{d^4k}{(2\pi)^4}\overline{u}(p')\frac{-i\lambda}{\sqrt{2}}\frac{i}{((p-k)^2 - m_h^2 + i\epsilon)}\frac{e^-}{}\frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)}(-ie\gamma^\mu)\frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)}\frac{-i\lambda}{\sqrt{2}}u(p),$$

$$\boxed{\therefore i\mathscr{M} = \frac{e\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\right]u(p)}{(k^2 - m^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)((p-k)^2 - m_h^2 + i\epsilon)}.} \tag{a.2}$$

Let us now simplify the denominator using Feynman parametrization. Using the same procedure as before, we see that we may reduce the denominator to the form,

$$\frac{1}{(k^2 - m_e^2 + i\epsilon)(k'^2 - m_e^2 + i\epsilon)((p-k)^2 - m_h^2 + i\epsilon)},$$

$$= \int dxdydz\delta^{(3)}(x+y+z-1)\frac{2}{[xk^2 + yk^2 + zk^2 + 2yqk + yq^2 + zp^2 - 2zpk - xm^2 - ym^2 - zm_h^2 + (x+y+z)i\epsilon]^3},$$

$$= \int dxdydz\delta^{(3)}(x+y+z-1)\frac{2}{[k^2 + 2k(yq - zp) + yq^2 + zp^2 - (1-z)m^2 - zm_h^2 + i\epsilon]^3},$$

Introducing the terms,

$$\ell \equiv k + yq - zp \qquad \text{and} \qquad \Delta = -xyq^2 + (1-z)^2m^2 + zm_h^2,$$

we see that the denominator becomes,

$$\boxed{\int dxdydz\delta^{(3)}(x+y+z-1)\frac{2}{[\ell^2 - \Delta + i\epsilon]^3}} \tag{a.3}$$

We are now ready to simplify the numerator of the integrand using the parameters ℓ for equation (a.2) above. There are arguably more elegant ways to go about this calculation, but we will simplify by brute force. We will use, without repeated demonstration, several identities that were shown in homework 2. Specifically, we will expand the integrand with the knowledge that all terms linear in ℓ will integrate to zero and so may be ignored. Furthermore, we are only interested in terms that do not involve a γ^μ so in the below tabulation of results from the Dirac algebra, we will simply write $\not{q}\gamma^\mu \to -2p^\mu$ with knowledge that $\not{q}\gamma^\mu = 2m\gamma^\mu - 2p^\mu$ because we are uninterested in terms proportional to γ^μ.

We will begin our simplification with a full expansion of the numerator as follows:

$$\begin{aligned}
\mathscr{N} &= \overline{u}(p')\left[(\not{k}' + m)\gamma^\mu(\not{k}+m)\right],\\
&= \overline{u}(p')\left[\not{k}'\gamma^\mu\not{k} + m\not{k}'\gamma^\mu + m\gamma^\mu\not{k} + m^2\gamma^\mu\right]u(p),\\
&\to \overline{u}(p')\Big[\underbrace{\not{\ell}\gamma^\mu\not{\ell}}_{\mathbf{i}} - y(1-y)\underbrace{\not{q}\gamma^\mu\not{q}}_{\mathbf{ii}} + z(1-y)\underbrace{\not{q}\gamma^\mu\not{p}}_{\mathbf{iii}} - zy\underbrace{\not{p}\gamma^\mu\not{q}}_{\mathbf{iv}} + z^2\underbrace{\not{p}\gamma^\mu\not{p}}_{\mathbf{v}} + m^2\gamma^\mu\\
&\qquad + m(1-y)\underbrace{\not{q}\gamma^\mu}_{\mathbf{vi}} + mz\underbrace{\not{p}\gamma^\mu}_{\mathbf{vii}} - my\underbrace{\gamma^\mu\not{q}}_{\mathbf{viii}} + mz\underbrace{\gamma^\mu\not{p}}_{\mathbf{ix}}\Big]u(p).
\end{aligned}$$

Using Dirac algebra and our results from homework 2, we see that

$$\begin{aligned}
&(\mathbf{i}) \to 0, && (\mathbf{ii}) \to 0, && (\mathbf{iii}) \to -2mp^\mu,\\
&(\mathbf{iv}) \to 2mp^\mu, && (\mathbf{v}) \to 2mp^\mu, && (\mathbf{vi}) \to -2p^\mu,\\
&(\mathbf{vii}) \to 2p^\mu, && (\mathbf{viii}) \to 2p'^\mu, && (\mathbf{xi}) \to 0.
\end{aligned}$$

Using this result (which ignores all terms linear in ℓ and γ^μ), we see that

$$\begin{aligned}
\mathscr{N} &\to \overline{u}(p')\Big[-2mz(1-y)p^\mu - 2mzyp'^\mu + 2mz^2p^\mu - 2m(1-y)p^\mu + 2mzp^\mu - 2myp'^\mu\Big]u(p),\\
&= \overline{u}(p')\Big[mp'^\mu(-2zy-2y) + mp^\mu(2zy+2y+2z^2-2)\Big]u(p),\\
&= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(-2zy-2y) + mp^\mu(2z^2-2)\Big]u(p),\\
&= \overline{u}(p')\Big[m(p'^\mu - p^\mu)(-2zy-2y) + mp^\mu(2z^2-2) + mp'^\mu(z^2-1) - mp'^\mu(z^2-1)\Big]u(p),\\
&= \overline{u}(p')\Big[(p'^\mu + p^\mu)m(z^2-1) + (p'^\mu - p^\mu)m(1-z^2-2zy-2y)\Big]u(p),
\end{aligned}$$

$$\boxed{\therefore \mathscr{N} \to \overline{u}(p')\Big[(p'^\mu + p^\mu)m(z^2-1) + (p'^\mu - p^\mu)m(y-x)(z-1)\Big]u(p).} \tag{a.4}$$

We notice almost trivially that this satisfies the Ward identity because the term proportional to $q^\mu = (p'^\mu - p^\mu)$ is odd under the interchange of $x \leftrightarrow y$ while the integral is symmetric under $x \leftrightarrow y$. Therefore the term proportional q^μ will vanish when integrated.

Recall that our goal is to discover this diagram's contribution to the anomalous magnetic moment, the $F_2(q^2)$ term. We recall that we have defined the corrected vertex function Γ^μ in terms of the functions F_1 and F_2 as

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Because the term proportional to $(p'^\mu + p^\mu)$ is multiplied on the outside by $\overline{u}(p')$ and $u(p)$, we may use the Gordon identity to express it in terms of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ and γ^μ. Because we are generally ignoring all terms proportional to γ^μ, we may substitute

$$m(z^2-1)(p'^\mu + p) \to 2m^2(1-z^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m}.$$

Because $F_2(q^2)$ is the term proportional to the $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ term, we see that this implies that

$$F_2(q^2) = \int dxdydz\delta^{(3)}(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{i\lambda^2}{2}\frac{2m^2(1-z^2)2}{[\ell^2-\Delta+i\epsilon]^3}.$$

We may simplify this integral substantially by recalling our work in homework 2 when we computed general integrals of this form. Taking the limit of $q \to 0$, we see that

$$\begin{aligned}
F_2(q^2) &= \int dxdydz\delta^{(3)}(x+y+z-1)\int \frac{d^4\ell}{(2\pi)^4}\frac{i\lambda^2}{2}\frac{2m^2(1-z^2)2}{[\ell^2-\Delta+i\epsilon]^3},\\
&= \int dxdydz\delta^{(3)}(x+y+z-1)\left[\frac{i\lambda^2}{2}\frac{-i}{(4\pi)^2}\frac{4m^2(1-z^2)}{2}\frac{1}{\Delta}\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2}\int dxdydz\delta^{(3)}(x+y+z-1)\frac{(1-z^2)}{zm_h^2+(1-z)^2m_e^2},\\
&= \frac{\lambda^2 m_e^2}{16\pi^2}\int_0^1 dz\frac{(1-z)(1-z^2)}{zm_h^2+(1-z)^2m_e^2},\\
&\approx \frac{\lambda^2 m_e^2}{16\pi^2}\left[\int_0^1 dz\frac{1}{zm_h^2+(1-z)^2m_e^2}-\frac{1}{m_h^2}\int_0^1 dz 1+z-z^2\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\int_0^1 dz\frac{1}{z+(1-z)^2\frac{m_e^2}{m_h^2}}-\frac{7}{6}\right]. \qquad \text{(a.8)}
\end{aligned}$$

Now, let us simplify this formula in the limit where the Higgs mass is very much larger than the electron.

$$\begin{aligned}
F_2(q^2) &\approx \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\frac{1}{1-\frac{m_e^2}{m_h^2}}\int_{\frac{m_e^2}{m_h^2}}^1 du\frac{1}{u}-\frac{7}{6}\right],\\
&= \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\frac{1}{1-\frac{m_e^2}{m_h^2}}\left(\ln(1)-\ln\left(\frac{m_e^2}{m_h^2}\right)\right)-\frac{7}{6}\right],
\end{aligned}$$

$$\boxed{\therefore F_2(q^2)\approx \frac{\lambda^2 m_e^2}{16\pi^2 m_h^2}\left[\ln\left(\frac{m_h^2}{m_e^2}\right)-\frac{7}{6}\right].} \qquad \text{(b.1)}$$

Let us try to compute this contribution for real experimental numbers. We can take a more or less 'good' estimate of the Higgs vacuum expectation value as $v = 246\text{GeV}$. We know that the coupling constant λ may be written in terms of the experimental mass of the electron as $\lambda_e = \frac{m_e}{v}\sqrt{2} \approx 2.94 \times 10^{-6}$. If we take a rather hopeful estimate for the Higgs mass, we can assume it is near its lower experimental bound at $m_h \approx 114\text{GeV}$. Using these numbers, we calculate an anomalous magnetic moment contribution of

$$\boxed{\delta_{\text{higgs}} a_e \approx 2.58 \times 10^{-23}.} \qquad \text{(b.2)}$$

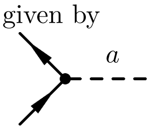
For the muon, we get a coupling to the Higgs of $\lambda_\mu = \frac{m_\mu}{v}\sqrt{2} \approx 6.03 \times 10^{-4}$. Using the same approximate Higgs mass of 114GeV, we see that the anomalous magnetic moment of the muon is altered by

$$\boxed{\delta_{\text{higgs}} a_\mu \approx 2.51 \times 10^{-14}.} \qquad \text{(b.3)}$$

Let us now consider the contribution given for an interaction with an axion particle given by the interaction Hamiltonian

$$H = \int d^x \frac{i\lambda}{\sqrt{2}} a\overline{\psi}\gamma^5\psi.$$

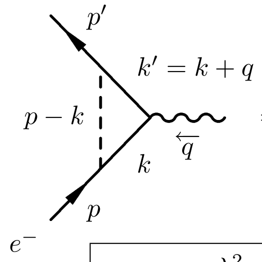
We see immediately that our vertex rule is given by



$$= \tfrac{\lambda}{\sqrt{2}}\gamma^5.$$

Let us now write out the amplitude for the axion's contribution to the vertex function. We see that

$i\mathscr{M} =$



$$= \int \frac{d^4k}{(2\pi)^4} \overline{u}(p') \frac{\lambda}{\sqrt{2}} \gamma^5 \frac{i}{((p-k)^2 - m_a^2 + i\epsilon)} \frac{i(\not{k}' + m)}{(k'^2 - m^2 + i\epsilon)} (-ie\gamma^\mu) \frac{i(\not{k} + m)}{(k^2 - m^2 + i\epsilon)} \gamma^5 \frac{\lambda}{\sqrt{2}} u(p),$$

$$\boxed{\therefore i\mathscr{M} = \frac{e\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-\overline{u}(p') \left[\gamma^5 (\not{k}' + m)\gamma^\mu (\not{k} + m)\gamma^5\right] u(p)}{(k^2 - m^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)((p-k)^2 - m_a^2 + i\epsilon)}.} \tag{c.1}$$

We can simplify the numerator and demoninator as before. Notice that the only change in the denominator algebra is that $\Delta = -xyq^2 + (1-z)^2 m_e^2 - z m_a^2$. In the numerator, we can commute the γ^5 through each of the terms to get a minus sign relative to the 'slash' terms. When we also take into account the overall minus which multiplies the numerator, we arrive at

$$i\mathscr{M} = \frac{e\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-\overline{u}(p') \left[(\not{k}' - m)\gamma^\mu (\not{k} - m)\right] u(p)}{(k^2 - m^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)((p-k)^2 - m_a^2 + i\epsilon)}.$$

This is of course very similar to the equation derived in parts (a). Recall when we expanded all of the terms for the Higgs, we had some of the 'm' terms that came from the Dirac algebra and some explicit the equation as above. Taking these differences into account, we can use our work from part (a) to arrive at a simplified numerator.

$$\begin{aligned}
\mathscr{N} \to &-\overline{u}(p')\Big[-2mz(1-y)p^\mu - 2mzyp'^\mu + 2mz^2 p^\mu + 2m(1-y)p^\mu - 2mzp^\mu + 2myp'^\mu\Big]u(p),\\
&= -\overline{u}(p')\Big[mp^\mu(-2z(1-y) + 2z^2 + 2 - 2y - 2z) + mp'^\mu(-2zy + 2y)\Big]u(p),\\
&= -\overline{u}(p')\Big[m(p'^\mu - p^\mu)(2y - 2zy)m + mp^\mu(-4z + 2z^2 + 2)\Big]u(p),\\
&= -\overline{u}(p')\Big[m(p'^\mu - p^\mu)(2y - 2zy)m + mp^\mu(-4z + 2z^2 + 2) + mp'^\mu(1-z)^2 - mp'^\mu(1-z)^2\Big]u(p),\\
&= -\overline{u}(p')\Big[(p'^\mu + p^\mu)(1-z)^2 m + (p'^\mu - p^\mu)(2y - 2zy - (1-z)^2)m\Big]u(p).
\end{aligned}$$

Again, using the Gordong identity, we may write the contribution to $F_2(q^2)$ as

$$\begin{aligned}
F_2(q^2) &= \int dxdydz\delta^{(3)}(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{i\lambda^2}{2} \frac{2m^2(1-z)^2 2}{[\ell^2 - \Delta + i\epsilon]^3},\\
&= \int dxdydz\delta^{(3)}(x+y+z-1) \left[\frac{i\lambda^2}{2} \frac{-i}{(4\pi)^2} \frac{4m^2(1-z)^2}{2} \frac{1}{\Delta}\right],
\end{aligned}$$

$$\boxed{\therefore F_2(q^2) = \frac{\lambda^2 m_e^2}{16\pi^2} \int_0^1 dz \frac{(1-z)^3}{zm_a^2 + (1-z)^2 m_e^2}.} \tag{c.2}$$

Now, this integral cannot be so easily takn in the limit of a heavy axion. In fact, experimental evidence strongly limits the mass of the axion to be very, very light. The most restrictive data, from Supernova 1987a, restricts $m_a \lesssim 10^{-5}$eV. In the limit where the axion is very, very much lighter than the electron, we see that

$$\begin{aligned}
F_2(q^2) &= \frac{\lambda^2 m_e^2}{16\pi^2} \int_0^1 dz \frac{(1-z)^3}{zm_a^2 + (1-z)^2 m_e^2},\\
&\approx \frac{\lambda^2}{16\pi^2} \int_0^1 dz \frac{(1-z)^3}{(1-z)^2} = \frac{\lambda^2}{32\pi^2},
\end{aligned}$$

$$\boxed{\therefore \delta_{\text{axion}} a_e \approx \delta_{\text{axion}} a_\mu \approx \frac{\lambda^2}{32\pi^2}.} \tag{c.3}$$