Problem 5.1

Let us consider a closed loop carrying a current I. Using the fact that

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\ell' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3},$$

we are to show that the magnetic induction at any point $\mathbf{x}$ is given by

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega,$$

where Ω is the solid angle about $\mathbf{x}$ subtended by the loop.

The required identity is equivalent to showing that

$$\int_{\partial S} d\ell' \times \frac{\mathbf{x}-\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla\Omega = \nabla\int_S d\Omega.$$

This is clear from the fact that the coefficients are the same and $\mathbf{B} = \int d\mathbf{B}$. Throughout our proof, we will rely on many of the vector identities listed in the front cover Jackson's text although we will not reference these explicitly for sake of convenience on the author's part.

Now, before we begin, we notice that we can rewrite

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|},$$

by simple evaluation. We will prove the required identity in by checking that each of the components of the vector fields are the same. Specifically, let $\hat{x}_i$ be a basis vector in the Cartesian coordinate system. It is obvious that the i^{th} component of $\mathbf{B}$ is simply $\hat{x}_i \cdot \mathbf{B}$. Therefore, let us evaluate the i^{th} component of the left-hand-side of the desired identity.

$$\begin{aligned}
\hat{x}_i \cdot \int_{\partial S} d\ell' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} &= \int_{\partial S} \hat{x}_i \cdot d\ell' \times \left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right),\\
&= \int_{\partial S} d\ell' \cdot \left[\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{x}_i\right],\\
&= \int_S d\mathbf{a}' \cdot \left\{\nabla' \times \left[\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{x}_i\right]\right\},\\
&= \int_S d\mathbf{a}' \cdot \left\{\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\underbrace{(\nabla'\cdot\hat{x}_i)}_{=0} - \hat{x}_i\underbrace{\left(\nabla'\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)}_{=\nabla'^2\frac{1}{|\mathbf{x}-\mathbf{x}'|}=0} + (\hat{x}_i\cdot\nabla')\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) - \underbrace{\left(\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\cdot\nabla'\right)\hat{x}_i}_{=0}\right\},\\
&= \int_S d\mathbf{a}' \cdot \frac{\partial}{\partial x_i'}\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned}$$

Now, notice that $\partial_{x_i} = -\partial_{x_i}$ and so we can bring this outside of the integral, picking up an extra minus sign,

$$\hat{x}_i \cdot \int_{\partial S} d\ell' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\frac{\partial}{\partial x_i}\int_S d\mathbf{a}' \cdot \nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

During a discussion of a similar problem in section 1.6, Jackson points out that

$$d\mathbf{a}' \cdot \nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -d\Omega.$$

Therefore, we see that that

$$\hat{x}_i \cdot \int_{\partial S} d\ell' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{\partial}{\partial x_i}\int_S d\Omega = \hat{x}_i \cdot \nabla\Omega.$$

Because $\hat{x}_i$ was an arbitrary Cartesian coordinate, we see that the more general vector equality is true, as desired. In particular, this implies directly that

$$\boxed{\therefore \mathbf{B} = \int d\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{\partial S} d\ell' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{\mu_0 I}{4\pi}\nabla\Omega.}$$

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