

That was more than expected. The charge distribution acts similarly when  $d \rightarrow 0$ ; namely, only the  $\ell = 0$  term in the sum survives so that

$$\lim_{d \rightarrow 0} \sigma(\theta) = -\frac{3Q}{4\pi} \frac{1}{3} \frac{1}{b^2} P_0(\cos \theta) = -\frac{Q}{4\pi b^2}.$$

This was also expected, as it is just a uniform charge distribution on the sphere.

**Problem 3.17**

Let us consider the Dirichlet Green's function for the unbounded space between two grounded planes at  $z = 0, L$ .

a) We are to show that a form of the Green's function in cylindrical coordinates is

$$G(x, x') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi\rho_-}{L}\right) K_m\left(\frac{n\pi\rho_+}{L}\right).$$

Our work will largely follow that of Jackson's section 3.11. We start by simply restating the requirement that  $G(x, x')$  be a Green's function; namely, that it satisfy,

$$\nabla^2 G(x, x') = -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z').$$

Recall that we have many rather convenient ways of expressing a  $\delta$ -function. In particular, recall that

$$\begin{aligned} \delta(\phi - \phi') &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{-im(\phi-\phi')}, \\ \delta(z - z') &= \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right). \end{aligned}$$

Inserting these into the definition of  $\nabla^2 G(x, x')$  in cylindrical coordinates above, we see that

$$\nabla^2 G(x, x') = -\frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) \frac{\delta(\rho - \rho')}{\rho}.$$

Now, recall that  $G(x, x')$ , like any suitable function (in the  $l_2$  topology), can be expanded in terms of orthogonal functions. In particular, we can in complete generality write

$$G(x, x') = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{A}_{nm} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right).$$

Now, in general, the 'coefficients'  $\mathcal{A}_{nm}$  will be functions of the coordinates  $x'$  and  $\rho$ . If we act on this expansion for  $G(x, x')$  with the Laplacian in cylindrical coordinates, we see that

$$\begin{aligned} \nabla^2 G(x, x') &= \nabla^2 \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{A}_{nm} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right), \\ &= \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{A}_{nm} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right), \\ &= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} - \frac{n^2 \pi^2}{L^2} \right] \mathcal{A}_{nm} \right\} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right). \end{aligned}$$

But recall that

$$\nabla^2 G(x, x') = -\frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left\{ e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right) \frac{\delta(\rho - \rho')}{\rho} \right\} e^{im\phi} \sin\left(\frac{n\pi z}{L}\right).$$

Now, orthogonal function expansions are unique in a suitable sense (that is, they can disagree on at most a set of measure zero if the domain of definition is a nice). Therefore, we see that

$$\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} - \frac{n^2 \pi^2}{L^2} \right] \mathcal{A}_{nm} = -\frac{4}{L} e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right) \frac{\delta(\rho - \rho')}{\rho}.$$

Because the differential operator acting on  $\mathcal{A}_{nm}$  does not act on  $\phi'$  or  $z'$ , we can divide both sides by  $e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)$ :

$$\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} - \frac{n^2 \pi^2}{L^2} \right] \left( \frac{\mathcal{A}_{nm}}{e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)} \right) = -\frac{4}{L} \frac{\delta(\rho - \rho')}{\rho}.$$

Setting  $x \equiv \frac{n\pi\rho}{L}$ , it is clear that the expression in parenthesis is a solution to the inhomogeneous modified Bessel equation. We know that the most general solution will be a linear combination of  $I_m\left(\frac{n\pi\rho}{L}\right)$  and  $K_m\left(\frac{n\pi\rho}{L}\right)$ . We know from the discussion in Jackson's section 3.11 that  $I_m\left(\frac{n\pi\rho}{L}\right)$  is regular precisely when  $\rho < \rho'$  and  $K_m\left(\frac{n\pi\rho}{L}\right)$  is regular precisely when  $\rho' < \rho$ . Therefore, because the Green's function must be regular over the entire domain (and symmetric in  $\rho, \rho'$ ), the only possibility is if

$$\left( \frac{\mathcal{A}_{nm}}{e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right)} \right) = \eta I_m\left(\frac{n\pi\rho_-}{L}\right) K_m\left(\frac{n\pi\rho_+}{L}\right),$$

where the constant  $\eta$  will be set by the discontinuity criterion for the derivative of the Green's function at  $\rho = \rho'$ . This computation is extremely easy because we already know the Wronskian of  $I_m$  and  $K_m$  from our work through Jackson. Specifically,

$$W[I_m(x), K_m(x)] = -\frac{1}{x},$$

and therefore, we see that

$$\eta = \frac{4}{L}.$$

We have finished our challenge; all that is now required is to put it all back together.

Our work immediately above has shown us that

$$\mathcal{A}_{nm} = \frac{4}{L} I_m\left(\frac{n\pi\rho_-}{L}\right) K_m\left(\frac{n\pi\rho_+}{L}\right) e^{-im\phi'} \sin\left(\frac{n\pi z'}{L}\right).$$

This leads us immediately to conclude,

$$\therefore G(x, x') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi\rho_-}{L}\right) K_m\left(\frac{n\pi\rho_+}{L}\right).$$

b) Similar to above, we are to show that the Green's function can also be written in the form,

$$G(x, x') = 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_-) \sinh[k(L-z_+)]}{\sinh(kL)}.$$

Our work and method will follow that above. First we recall that the Laplacian acting on a Green's function  $G(x, x')$  satisfies

$$\nabla^2 G(x, x') = -4\pi \frac{\delta(\rho - \rho')}{\rho} \delta(z - z') \delta(\phi - \phi').$$

This time, we will make use of the following two convenient ways of writing  $d$ -functions.

$$\begin{aligned} \delta(\phi - \phi') &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}, \\ \frac{\delta(\rho - \rho')}{\rho} &= \int_0^{\infty} k J_m(k\rho) J_m(k\rho') dk. \end{aligned}$$

Using these, we can write

$$\begin{aligned} \nabla^2 G(x, x') &= -2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} k e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \delta(z - z') dk, \\ &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} \left\{ -2k e^{-im\phi'} J_m(k\rho') \delta(z - z') \right\} e^{im\phi} J_m(k\rho) dk. \end{aligned}$$

But we can always (with caveats, as always) expand  $G(x, x')$  in terms of these orthogonal functions and some ‘coefficients’  $\mathcal{A}_{mk}$ ,

$$G(x, x) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathcal{A}_{mk} e^{im\phi} J_m(k\rho) dk.$$

Acting on this expression with the Laplacian in cylindrical coordinates, we see

$$\begin{aligned} \nabla^2 G(x, x') &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] (\mathcal{A}_{mk} e^{im\phi} J_m(k\rho)) dk, \\ &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} \left\{ \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} - \frac{m^2}{\rho^2} \right] \mathcal{A}_{mk} J_m(k\rho) \right\} e^{im\phi} dk, \\ &= \sum_{m=-\infty}^{\infty} \int_0^{\infty} \left\{ \left[ \frac{\partial^2}{\partial z^2} - k^2 \right] \mathcal{A}_{mk} \right\} e^{im\phi} J_m(k\rho) dk. \end{aligned}$$

In the last line, we used the fact that  $J_m(k\rho)$ , being a Bessel function, solves Bessel’s equation.

As before, we see that the two series must agree for each term. Therefore,

$$\left[ \frac{\partial^2}{\partial z^2} - k^2 \right] \mathcal{A}_{mk} = -2k e^{-im\phi'} J_m(k\rho') \delta(z - z').$$

Notice that the differential operator acting on  $\mathcal{A}_{mk}$  does not act on  $\phi'$  or  $\rho'$  and therefore, we may rearrange the expression to see that

$$\left[ \frac{\partial^2}{\partial z^2} - k^2 \right] \left( \frac{\mathcal{A}_{mk}}{e^{-im\phi'} J_m(k\rho')} \right) = -2k \delta(z - z').$$

Equations like the one above are solved by freshman in their dreams. Because we need the potential to vanish on the plates, the function in parenthesis will be composed of hyperbolic sines (instead of hyperbolic cosines) and by the symmetry of  $z, z'$ , we see that

$$\left( \frac{\mathcal{A}_{mk}}{e^{-im\phi'} J_m(k\rho')} \right) = \eta \sinh(k_-) \sinh(k(L - z_+)),$$

where  $\eta$  is some constant that we will need to determine using the discontinuity of its derivative at  $z = z'$ . We compute this quite directly,

$$\begin{aligned} -2k &= \frac{d}{dz} \left( \eta \sinh(k_-) \sinh(k(L - z_+)) \right) \Big|_{z_{\downarrow}} - \frac{d}{dz} \left( \eta \sinh(k_-) \sinh(k(L - z_+)) \right) \Big|_{z_{\uparrow}}, \\ &= -k\eta \sinh(kz) \cosh(k(L - z)) - k\eta \cosh(kz) \sinh(k(L - z)), \\ &= -k\eta \sinh(kL), \end{aligned}$$

$$\therefore \eta = \frac{2}{\sinh(kL)}.$$

Putting this work back into our expressions above, we see that we have shown

$$\mathcal{A}_{mk} = 2e^{-im\phi'} J_m(k\rho') \frac{\sinh(kz_-) \sinh(k(L - z_+))}{\sinh(kL)},$$

and this allows us to conclude that

$$\boxed{\therefore G(x, x') = 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk e^{im(\phi - \phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_-) \sinh[k(L - z_+)]}{\sinh(kL)}}.$$

ὅπερ ἔδει δειξάναι