Physics 523, General Relativity
Homework 7
Due Wednesday, 6th December 2006

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Problem 1

Consider a gyroscope moving in circular orbit of radius R about a static, spherically-symmetric planet of mass m.

a. We are to derive the equations of motion for the gyroscopic spin vector as a function of azimuthal angle and show that the spin precesses about the direction normal to the orbital plane.

This calculation will be far from elegant, and will probably not give rise to much insight. Nevertheless, we start by recalling the Lagrangian describing a particle's worldline(in the $\theta = \frac{\pi}{2}$ plane) in a static, isotropic spacetime,

$$\mathscr{L} = -g_{ab}u^a u^b = f(r)(u^t)^2 - \frac{1}{f(r)}(u^r)^2 - r^2(u^\varphi)^2, \tag{1.1}$$

where $u^a \equiv \frac{dx^a}{d\tau}$ for some affine parameter τ. Because our analysis will be limited to circular geodesics, we will not have much use for the u^r coordinate; however, its equation of motion will be necessary to relate the various integrals of motion. First observe that u^φ is non-dynamical in the Lagrangian and so it gives us our first integral of motion,

$$J \equiv r^2 u^\varphi. \tag{1.2}$$

For circular geodesics, u^a will of course only have 0 and φ components; u^t is also non-dynamical, and so we are free to set u^t by the normalization of the affine parameter τ:

$$u^2 = -g_{ab}u^a u^b = f(R)(u^t)^2 - \frac{J^2}{R^2} \equiv 1, \qquad \Longrightarrow \qquad u^t = \sqrt{\frac{1}{f(R)}\left(1 + \frac{J^2}{R^2}\right)}. \tag{1.3}$$

Now, it is easy to see that the equation of motion for the r-component is

$$-2\frac{\ddot{r}}{f(r)} + 2\frac{\dot{r}^2}{f^2(r)}f'(r) = -\frac{\dot{r}^2}{f^2(r)} - 2\frac{J^2}{r^3} + f'(r)(u^t)^2. \tag{1.4}$$

Because we are looking for solutions where both $\dot{r}$ and $\ddot{r}$ vanish—and $r = R$—we see at once that this implies the relation

$$\begin{aligned}
J^2 &= \frac{1}{2}f'(R)(u^t)^2 R^3 = \frac{m}{R^2}\frac{R^4}{(R-2m)}\left(1 + \frac{J^2}{R^2}\right), \\
&= \frac{mR^2}{R-2m}\frac{1}{1 - \frac{m}{R-2m}}, \\
&= \frac{mR^2}{R-3m}.
\end{aligned} \tag{1.5}$$

Above, we made use of the definition of the Schwarzschild metric's $f(r) = 1 - \frac{2m}{r}$. We have now completely specified the circular geodesic of radius R in which we are interested.

The direction of a gyroscope's spin is therefore simply a vector S^a which satisfies the orthogonality condition $u^a S^b g_{ab} = 0$ along the geodesic. Recall that two parallelly-transported vectors have the property that the gradient of their scalar product vanishes. This immediately allows us to write down the equation for the evolution of the components of S^a along τ,

$$\frac{dS_a}{d\tau} = \Gamma^b_{ac}S_b u^c, \tag{1.6}$$

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which, upon using the Christoffel symbols for the Schwarzschild metric[1], becomes

$$\frac{dS_t}{d\tau} = \Gamma^r_{tt}S_r u^t = \frac{1}{2}f(R)f'(R)S_r u^t = \frac{1}{2}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)}S_r; \tag{1.7}$$

$$\frac{dS_r}{d\tau} = \Gamma^t_{rt}S_t u^t + \Gamma^\varphi_{r\varphi}S_\varphi u^\varphi = \frac{f'(R)}{2f(R)}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)}S_t + \frac{J}{R^3}S_\varphi; \tag{1.8}$$

$$\frac{dS_\theta}{d\tau} = \Gamma^\varphi_{\theta\varphi}S_\varphi u^\theta + \Gamma^\theta_{\theta r}S_\theta u^r + \Gamma^r_{\theta\theta}S_r u^\theta = 0; \tag{1.9}$$

$$\frac{dS_\varphi}{d\tau} = \Gamma^\theta_{\varphi\varphi}S_\theta u^\varphi + \Gamma^r_{\varphi\varphi}S_r u^\varphi = -\frac{J}{R}f(R)S_r. \tag{1.10}$$

This almost completes our analysis. Indeed, notice that the above system of equations implies that the θ-component of the gyroscope's spin is fixed. All the motion of S^a as it is transported along τ is confined to the plane normal to $\hat{\theta}$. Therefore, we may conclude that the gyroscope will precess about the axis normal to its orbital plane.

The finicky reader may object that the system of equations (1.6-9) are over-specified. To be thorough we should eliminate redundancy. The first of the relations among these expressions comes from the orthogonality condition on the spin vector $S_a u^a = 0$. In components this reads

$$S_t u^t + S_\varphi u^\varphi = 0 \qquad \Longrightarrow \qquad S_t\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)} = -\frac{J}{R^2}S_\varphi. \tag{1.11}$$

Also, it is more physically interesting to compute evolution relative to the angle φ as observed by a stationary observer on the planet. Replacing S_t in favour of S_φ and making us of the fact $\frac{d\tau}{d\varphi} = \frac{R^2}{J}$,

$$\frac{dS_t}{d\varphi} = \frac{R^2}{2J}\sqrt{\frac{1}{f(R)}\left(1+\frac{J^2}{R^2}\right)}S_r;$$

$$\frac{dS_r}{d\varphi} = \left(\frac{1}{R} - \frac{f'(R)}{2f(R)}\right)S_\varphi;$$

$$\frac{dS_\varphi}{d\varphi} = -Rf(R)S_r;$$

$$\frac{dS_\theta}{d\varphi} = 0.$$

The last redundancy to take care of comes from the geodesic equation for $S^a S^b g_{ab}$—namely, that this scalar is preserved. Let us choose to normalize $S^a S^b g_{ab} = +1$ so that

$$1 = -\frac{1}{f(R)}S_t^2 + f(R)S_r^2 + \frac{1}{R^2}S_\theta^2 + \frac{1}{R^2}S_\varphi^2,$$

$$= \frac{1}{R^2}S_\varphi^2\left(1 - \frac{J^2}{R^2}\frac{1}{\left(1+\frac{J^2}{R^2}\right)}\right) + f(R)S_r^2 + \frac{1}{R^2}S_\theta^2,$$

$$= \frac{S_\varphi^2}{(R^2+J^2)} + f(R)S_r^2 + \frac{1}{R^2}S_\theta^2.$$

Bearing in mind that S_θ is a constant of motion, me may therefore write

$$S_\varphi^2 = f(R)\left(R^2 + J^2\right)\left(\frac{1}{f(R)} - \frac{1}{f(R)R^2}S_\theta^2 - S_r^2\right) \qquad \text{or} \qquad S_r^2 = \frac{1}{f(R)\left(R^2+J^2\right)}\left((R^2+J^2) - \frac{(R^2+J^2)}{R^2}S_\theta^2 - S_\varphi^2\right). \tag{1.12}$$

The two substantive equations of motion are clearly $\frac{dS_r}{d\varphi}$ and $\frac{dS_\varphi}{d\varphi}$. Squaring the equations derived above, and using the normalization condition to reexpress unlike components,

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[1]And specializing to the obvious coordinate choice $\theta \mapsto \frac{\pi}{2}$ everywhere it is encountered.

we find

$$\left(\frac{dS_r}{d\varphi}\right)^2 = \left(\frac{1}{R} - \frac{f'(R)}{2f(R)}\right)^2 f(R)\left(R^2 + J^2\right)\left(\frac{1}{f(R)} - \frac{1}{f(R)R^2}S_\theta^2 - S_r^2\right), \tag{1.13}$$

$$\left(\frac{dS_\varphi}{d\varphi}\right)^2 = R^2 f(R)^2 \frac{1}{f(R)\left(R^2 + J^2\right)}\left(\left(R^2 + J^2\right) - \frac{\left(R^2 + J^2\right)}{R^2}S_\theta^2 - S_\varphi^2\right). \tag{1.14}$$

Despite how horrendous these equations look at first glance, the structure present is very simple. Notice that any function $g(\varphi) \equiv \alpha\cos(\beta\varphi)$ (or $g(\varphi) = \alpha\sin(\beta\varphi)$) satisfies the differential equation

$$\left(\frac{d}{d\varphi}\alpha\cos(\beta\varphi)\right)^2 = \left(\frac{dg}{d\varphi}\right)^2 = \beta^2\left(\alpha^2 - g^2(\varphi)\right). \tag{1.15}$$

The initial conditions will determine the coefficients β, α, but the general result is now complete[2].

b. If the gyroscope studied in part (a) is observed to have its spin entirely in the orbital plane, then how much precession is observed? What is the precession observed in the case of a satellite in low-earth orbit?

When we finished part (a), we had done everything necessary to determine the precession of a gyroscope in circular orbit given suitable boundary conditions. In the case of a gyroscope with spin lying in its orbital plane, $S_\theta = 0$. This greatly simplifies our algebra. Let us proceed to simplify the expressions (1.13) and (1.14).

Using the expression for the angular momentum J (1.5) derived above, expanding, and collecting terms, we find

$$\begin{aligned}\left(\frac{dS_r}{d\varphi}\right)^2 &= \frac{(2f(R) - Rf'(R))^2}{4R^2 f(R)}\left(R^2 + J^2\right)\left(\frac{1}{f(R)} - S_r^2\right),\\ &= \frac{(2f(R) - Rf'(R))^2}{4R(R - 2m)}\frac{R^2(R - 2m)}{(R - 3m)}\left(\frac{1}{f(R)} - S_r^2\right),\\ &= \frac{(R - 2m - m)^2}{R}\frac{1}{(R - 3m)}\left(\frac{1}{f(R)} - S_r^2\right),\\ &= \frac{R - 3m}{R}\left(\frac{1}{f(R)} - S_r^2\right).\end{aligned}$$

As described in part (a), a solution to this differential equation is of the form $\alpha\cos(\beta\varphi)$. If we define φ so that S_r is maximum when $\varphi = 0$, we have

$$\therefore S_r(\varphi) = \sqrt{\frac{R}{R - 2m}}\cos\left(\sqrt{\frac{R - 3m}{R}}\varphi\right). \tag{1.16}$$

<div align="right">ὅπερ ἔδει ποιῆσαι</div>

We can follow the same analysis, or simply differentiate this to obtain S_φ. Either way, one finds that

$$\therefore S_\varphi(\varphi) = -R\sqrt{\frac{R - 2m}{R - 3m}}\sin\left(\sqrt{\frac{R - 3m}{R}}\varphi\right). \tag{1.17}$$

<div align="right">ὅπερ ἔδει ποιῆσαι</div>

Using the fact that S_a is a unit covector, we know that the angle between $S^a(0)$ and $S^a(2\pi n)$ after n orbits will be given by

$$\cos(\vartheta) = f^{-1}(R)f(R)\cos\left(\sqrt{\frac{R - 3m}{R}}2\pi n\right), \quad\text{or,}\quad \vartheta = 2\pi n\sqrt{\frac{R - 3m}{R}}, \tag{1.18}$$

which is well-approximated by

$$\delta\vartheta \simeq \frac{3m}{R}\pi n. \tag{1.19}$$

For a low-earth orbit satellite in circular motion, we therefore expect the gyroscopic precession to be on the order of 1.66×10^{-9} degrees per orbit.

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[2]If you had hoped to see us simplify these expressions enormously, please read our solution to part (b) below.

Problem 2

Consider a 4+1-dimensional AdS spacetime described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \qquad \text{where} \qquad f(r) = 1 + r^2 - \frac{\mu}{r^2}. \tag{2.1}$$

We are to determine the radial coordinate of the black hole horizon, calculate the proper time of a massive object to free-fall from the surface of the black hole to the singularity at $r = 0$, and determine the radius and period of the null-circular orbit.

The horizon radius is that for which $f(r)$ vanishes. A child's experience with the quadratic formula is sufficient to see that there is exactly one real root of $f(r)$ and this corresponds to a horizon radius of

$$\therefore r_h = \sqrt{\frac{1}{2}\left(\sqrt{4\mu^2 + 1} - 1\right)}. \tag{2.2}$$

$$\text{ὅπερ ἔδει ποιῆσαι}$$

To calculate the proper time for free-fall from the horizon, we need to quickly derive the equations for motion in only the r-direction. Because we'll need the angular dependence later, we'll start a bit more generally. First, look at the Lagrangian for the particle's motion (its worldline),

$$\mathscr{L} = -g_{ab}u^a u^b = f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\varphi}^2. \tag{2.3}$$

Now, as always, a '·' indicates differentiation with respect to an affine parameter, say τ, along the worldline of the particle. We will eventually impose the normalization condition (think Lagrange multipliers)

$$\kappa \equiv -g_{ab}u^a u^b, \tag{2.4}$$

where $\kappa = 1$ for time-like worldlines and $\kappa = 0$ for null. The first thing that should be apparent form the Lagrangian is that there are two non-interacting degrees of freedom, $\dot{t}$ and $\dot{\varphi}$, giving rise to two integrals of motion[3]

$$E \equiv f(r)\dot{t}, \qquad \text{and} \qquad J \equiv r^2\dot{\varphi}. \tag{2.5}$$

Now, in the case of purely radial motion of a massive object, $J = 0$ and $\kappa = 1$; so we are left with only

$$1 = \frac{1}{f(r)}E^2 - \frac{1}{f(r)}\dot{r}^2, \qquad \Longrightarrow \dot{r}^2 = E^2 - f(r). \tag{2.6}$$

Notice that this means that E must be chosen so that $\dot{r}^2 = 0 = E^2 - f(R)$ for some R. In the problem under consideration, we want to find the motion of an object dropped *from rest* at $R = r_h$—and r_h is defined to be such that $f(r_h) = 0$. Therefore, $E^2 = 0$ for our present problem, and

$$\frac{dr}{d\tau} = \sqrt{-f(r)}; \tag{2.7}$$

which is easy enough to formally invert:

$$\tau = \int_0^{r_h} \frac{dr}{\sqrt{-f(r)}}. \tag{2.8}$$

Our computer algebra software had no difficulty evaluating this, showing that

$$\therefore \tau = \frac{\pi}{4} - \frac{1}{2}\text{arccot}\left(2\sqrt{\mu}\right). \tag{2.9}$$

$$\text{ὅπερ ἔδει ποιῆσαι}$$

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[3]We could have framed this discussion in terms of Killing fields, but we'll stick to Euler while we can.

Lastly, we are asked to find the radius at which light can orbit circularly, and determine the coordinate time of this orbit's period. To do this, we need only to re-instate J into our expression for $\dot{r}^2$ and set $\kappa \to 0$ for null geodesics. Written suggestively, this gives

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}f(r)\frac{J^2}{r^2} = \frac{1}{2}E^2. \tag{2.10}$$

Reminiscent of effective potentials, we are inspired to consider an analogue problem in 1+1-dimensions governed by the effective potential

$$V_{eff} = \frac{J^2}{2r^2}\left(1 + r^2 - \frac{\mu}{r^2}\right). \tag{2.11}$$

This effective potential has only one turning point, at

$$-\frac{J^2}{r^3} + 2\frac{\mu J^2}{r^5} = 0 \qquad \Longrightarrow \qquad r = \sqrt{2\mu}. \tag{2.12}$$
$$\overset{\text{''}}{o}\pi\epsilon\rho\ \overset{\text{'}}{\epsilon}\delta\epsilon\iota\ \pi o\iota\widehat{\eta}\sigma\alpha\iota$$

This is the radius at which there are circular, null geodesics—as evidenced by the fact that $\dot{r} = 0$ at this radius. Inserting $r = \sqrt{2\mu}$ into (2.10),

$$E^2 = f(\sqrt{2\mu})\frac{J^2}{2\mu} = J^2\left(1 + \frac{1}{4\mu}\right) \qquad \Longrightarrow \qquad \frac{J^2}{E^2} = \frac{4\mu}{4\mu + 1}. \tag{2.13}$$

This is needed for us to compute the coordinate-time orbit period. Recall from our definitions of E and J that

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{J}{r^2}\frac{f(r)}{E}, \tag{2.14}$$

—which when combined with the above implies

$$\frac{d\varphi}{dt} = \frac{1}{2}\sqrt{\frac{4\mu + 1}{\mu}}. \tag{2.15}$$

This is trivially integrated. We find that the coordinate time of one orbit is

$$\therefore t_p = 4\pi\sqrt{\frac{\mu}{4\mu + 1}}. \tag{2.16}$$
$$\overset{\text{''}}{o}\pi\epsilon\rho\ \overset{\text{'}}{\epsilon}\delta\epsilon\iota\ \pi o\iota\widehat{\eta}\sigma\alpha\iota$$

Problem 3

Consider a clock in circular orbit at radius $R = 10m$ about a spherically symmetric star.

a. We are to determine the proper time of the $R = 10m$ orbit.

We can draw heavily on our work above. Using the notation and conventions of problem one, we see that

$$\tau_p = \int d\tau = \int \frac{d\tau}{d\varphi}d\varphi = \frac{R^2}{J}\int d\varphi = 2\pi\frac{R^2}{J}.$$

Using our equation (1.5) for J at a given R, we find

$$\therefore \tau_p = 2\pi\frac{R\sqrt{R - 3m}}{\sqrt{m}}. \tag{3.a.1}$$
$$\overset{\text{''}}{o}\pi\epsilon\rho\ \overset{\text{'}}{\epsilon}\delta\epsilon\iota\ \pi o\iota\widehat{\eta}\sigma\alpha\iota$$

For the particular question at hand, $r = 10m$, we find the period to be

$$\tau_p = 20\sqrt{7}\pi m. \tag{3.a.2}$$

b. If once each orbit the clock transmits a signal to a distant observer, what time interval does this observer observe?

The time coordinate t is precisely the time observed by a distant observer in Schwarzschild geometry. Therefore, we simply modify the calculation above as follows.

$$t_p = \int dt = \int \frac{dt}{d\tau} \frac{d\tau}{d\varphi} d\varphi = 2\pi \frac{R^2}{J} \sqrt{\frac{1}{f(R)} \left(1 + \frac{J^2}{R^2}\right)} = 2\pi \frac{R^{3/2}}{\sqrt{m}}. \tag{3.b.1}$$

$$ \ddot{o}\pi\epsilon\rho \ \dot{\epsilon}\delta\epsilon\iota \ \pi o\iota\widehat{\eta}\sigma\alpha\iota $$

We point out that this agrees identically with Kepler's third law.
For $R = 10m$ we find

$$t_p = 20\sqrt{10}\pi m. \tag{3.b.2}$$

c. If another observer is stationed in stationary orbit at $R = 10m$, what time do their clocks report as the orbit period?

The proper time of a shuttle on a fixed distance from the origin is given by

$$\Delta\tau^2 = \left(1 - \frac{2m}{R}\right) \Delta t^2 + 0; \tag{3.c.1}$$

$$\therefore \tau_p = \frac{2\pi R^{3/2}}{\sqrt{m}} \sqrt{1 - \frac{2m}{R}} = 40\sqrt{2}\pi. \tag{3.c.2}$$

d. We are to redux the calculation of part (b), this time for the case of an orbit at $R = 6m$ where $m = 14M_\odot$ and explain why this bound is interesting.

It is not very challenging to simply put real numbers into our calculation above; we find

$$t_p = 2\pi \frac{R^{3/2}}{\sqrt{m}} = 2 \times 10^{-8} \text{ s}. \tag{3.d.1}$$

The reason why this is the minimum for fluctuations to be observed from x-ray sources is that $R = 6m$ is the minimum radius at which there is a stable circular orbit.

e. If forty years go by according to the watch of a distant observer, how long has passed on a spaceship orbiting at $R = 6m$ for $m = 14M_\odot$.

The one thing that both observers will agree on is that during the interval in question the orbiting observer made 6.6×10^{16} orbits; this was of course calculated using the result of part (b) above. Using part (a), we learn that the person living inside the orbiting spaceship observed 28 years pass to complete these 6.6×10^{16} orbits.