Budapest Semesters in Mathematics

ADVANCED COMBINATORICS HANDOUTS

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INTRODUCTION

This handout is based on the Advanced Combinatorics course I have taught through many years at Budapest Semesters. It contains a basic material covered in each semester (sections 1-5) and two further parts (fruit salad and advanced menu) from which I have selected material according to the taste and appetite of the students.

The objective of the course is to introduce hypergraphs together with widely applicable proof methods of combinatorial mathematics.

The material is self contained, apart from basic facts of linear algebra. However, at least one elementary course - for example counting methods, graph theory, discrete structures - is strongly recommended.

Each section is supplemented with a set of easy questions (self-test) to check the understanding of the material and with an exercise set.

At certain places the presentation of the material follows closely the following sources:
Babai, Frankl: Linear Algebra Methods in Combinatorics.
Graham, Rothchild, Spencer: Ramsey Theory.
Spencer: Ten Lectures on the Probability Method.
Van Lint, Wilson: A course in Combinatorics.

Acknowledgement. First of all I express my thanks to the active students of this course through the years from 1988 to the present time. They understood my English, read my handwritten notes, found misprints, caught errors, invented new proofs, some of them even found new results that eventually deserve to be published (or already have been published). The support of the organizers of the Budapest Semesters (in America and in Budapest) is also acknowledged together with the help of my home institute, SZTAKI (Computer and Automation Research Institute of the Hungarian Academy of Sciences). Thanks to Péter Juhász for implementing the figures.

CONTENT

4. COUNTING AND PROBABILITY

4.1. Proofs by counting. (Antichains, intersecting hypergraphs, 3-chromatic uniform hypergraphs.)

4.2. Probability method. (The Erdős lower bound on $R(n)$, Lower bound for $W(k)$, Tournaments, Large transitive subtournaments, Existence versus construction, Paradoxical tournaments, Hamiltonian paths.)

4.3. Local Lemma. (Erdős-Lovász theorem, Improved lower bound on $W(k)$, Even cycles in regular digraphs.)

4.4. Jokes (Triangle is 2-chromatic, Spencer's injections.)

5. LINEAR ALGEBRA METHODS

5.1. The dimension bound. (Oddtown, Fisher inequality, A cubic lower bound for $R(n)$, Two-distance sets, Cross-intersecting hypergraphs.)

5.2. Homogeneous linear equations. (Partition of K_n, discrepancy.)

5.3. Eigenvalues. (Regular graphs of girth five.)

6. FRUIT SALAD

Distinct representatives, symmetric chain decomposition, properties of hypergraphs with n vertices and $n, n+1, n+2$ *edges, critical not 2-colorable hypergraphs, sunflower theorem, sum-free subsets of numbers.*

7. ADVANCED MENU

Baranyai theorem, normal hypergraphs and perfect graphs, constructive super polynomial lower bound for Ramsey numbers, Bursuk conjecture.

1 BASIC NOTIONS AND EXAMPLES

1.1 DEFINITIONS

Hypergraph: $\mathcal{H} = (V, \mathcal{E})$ where V is a set and $\mathcal{E}$ is a collection of subsets of V. The set V is called the *vertex set* and $\mathcal{E}$ is called the *edge set* of the hypergraph $\mathcal{H}$. The word "collection" is used to allow selection of the same subset of V more than once, if this happens we have *multiedges.*

Simple hypergraph: no multiedges.

What can be empty? To be precise, there are some details to discuss. Do we allow that $V = \emptyset$? Frank Harary has a paper with the pun title: "Is the null-graph a pointless concept?" Do we allow $\mathcal{E} = \emptyset$? What if $V, \mathcal{E}$ are nonempty but $e = \emptyset$ for some $e \in \mathcal{E}$? We shall assume that V and $\mathcal{E}$ are nonempty but allow empty edges for technical reasons.

Singletons, isolated vertices. An edge $e \in \mathcal{E}$ is called *singleton* if $|e| = 1$. A vertex is *isolated* if no edge contains it. We allow singleton edges and isolated vertices (unless stated otherwise).

Finite or infinite? In most cases we shall work with finite hypergraphs. However, sometimes it is natural to consider hypergraphs with $|V| = \infty$, for example $V = R^d$, but usually we shall still assume $|\mathcal{E}| < \infty$.

Degree of $v \in V$: The number of edges containing vertex v (in a hypergraph $\mathcal{H} = (V, \mathcal{E})$). Notation for the degree: $d(v)$ or $d_{\mathcal{H}}(v)$ if not clear from the context.

Maximum and Minimum degree: $\Delta(\mathcal{H})$ denotes the largest and $\delta(\mathcal{H})$ denotes the smallest among the degrees of the vertices of $\mathcal{H}$.

Regular hypergraph: $d(x) = d(y)$ for all $x, y \in V$.

t-regular hypergraph: regular hypergraph with common degree t.

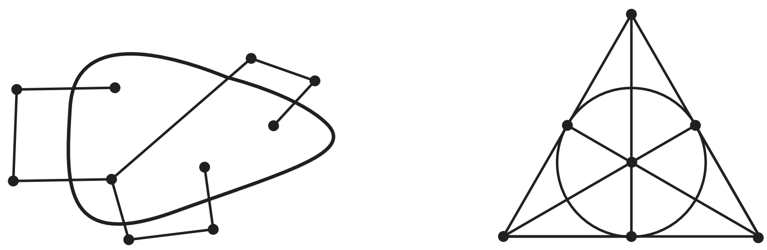
Uniform hypergraph: $|e| = |f|$ for all $e, f \in \mathcal{E}$.

t-uniform hypergraph: uniform hypergraph with edges of t vertices.

Graphs can be defined as 2-uniform hypergraphs, *simple graphs* are the simple 2-uniform hypergraphs. (A singleton edge is similar to the notion of the loop in Graph theory but a loop contributes 2 and a singleton contributes 1 to the degree of a vertex.) There are some terms used only for graphs. The most important is the adjacency: two vertices x, y of a graph are called *adjacent* if $\{x, y\}$ is an edge of the graph. The *complementary graph (or complement)* of a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the graph $\overline{\mathcal{G}} = (V, \overline{\mathcal{E}})$ where $\overline{\mathcal{E}}$ is the set of pairs of V not in $\mathcal{E}$. Complements of simple uniform hypergraphs can be defined in a similar way.

Isomorphic hypergraphs: Two hypergraphs are *isomorphic* if there exists an edge preserving bijection between their vertex sets.

How to draw a hypergraph? One possibility to represent a hypergraph is to draw it. In case of graphs this is very usual. It is more complicated to draw a hypergraph.



Figures 1.1. and 1.2. Drawings of hypergraphs

On Figure 1.1 the "potato" represents an edge with four vertices, the other nine edges have two vertices. Figure 1.2 is a drawing of the so called Fano plane, a famous 3-regular, 3-uniform hypergraph on seven vertices. In that case the edges are represented with lines with one exception, the circle also represents an edge.

The incidence matrix. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ can be represented by its incidence matrix, defined as follows. Assume that $V = \{v_1, \dots v_n\}$ and $\mathcal{E} = \{e_1, \dots e_m\}$. The incidence matrix of $\mathcal{H}$ is the $m \times n$ matrix $[a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in e_i \\ 0 & \text{if } v_j \notin e_i \end{cases}$$

Remember that rows correspond to edges and columns correspond to vertices in the incidence matrix of the hypergraph. (Of course, this is merely a convention.)

Hypergraphs are rather general structures but there is a theorem valid for all hypergraphs.

Theorem 1.1 *For any hypergraph $\mathcal{H} = (V, \mathcal{E})$*

$$\sum_{x \in V} d(x) = \sum_{e \in \mathcal{E}} |e|$$

Proof. Interpret the statement for the incidence matrix of $\mathcal{H}$. □

Duality. Assume that $\mathcal{H}$ is a hypergraph with incidence matrix M. Let M^T be the transpose of M ($M^T[a_{ij}] := M[a_{ji}]$). The hypergraph $\mathcal{H}^*$ with incidence matrix M^T is called the *dual of* $\mathcal{H}$. The definition obviously shows that $(\mathcal{H}^*)^* = \mathcal{H}$. There is a bijection f from the vertices of $\mathcal{H}$ to the edges of $\mathcal{H}^*$ and a bijection g from the edges of $\mathcal{E}$ to the vertices of $\mathcal{H}^*$. There is a dictionary of duality to translate concepts and theorems. Some lines from this dictionary might look like this:

$\mathcal{H}$	$\mathcal{H}^*$
$x \in e$	$f(x) \ni g(e)$
$d(x)$	$\|f(x)\|$
$\|e\|$	$d(g(e))$
t-regular	t-uniform
t-uniform	t-regular

Intersection graph (alias line graph) of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\mathcal{E} = \{e_1, \ldots, e_m\}$ is a graph with vertices $x_1, \ldots, x_m$ so that $\{x_i x_j\}$ is an edge of the graph if and only if $e_i \cap e_j \neq \emptyset$. The intersection graph reflects the intersections only between pairs of edges of a hypergraph, the finer structure of the hypergraph is lost.

Theorem 1.2 *Every simple graph is the intersection graph of some hypergraph.*

Proof. Exercise 1.1 □

1.2 EXAMPLES OF HYPERGRAPHS

Paths and cycles. Assume that $k \geq 2$ is an integer. A *path* in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a sequence $x_1, e_1, x_2, e_2, \ldots, x_{k-1}, e_{k-1}, x_k$ where the x_i-s are *distinct* vertices and the

e_i-s are *distinct* edges and $x_i, x_{i+1} \in e_i$ for $1 \le i \le k-1$. For convenience, a vertex is considered also as a path (this corresponds to the case $k = 1$). The *length* of the path is $k-1$, the number of edges in it.

The definition of a *cycle* is very similar, it is more symmetric: add e_k to the end of a sequence forming a path, requiring $x_1, x_k \in e_k$. More precisely, for $k \ge 2$, a *cycle* is a sequence $x_1, e_1, x_2, e_2, \dots, x_{k-1}, e_{k-1}, x_k, e_k$ where the x_i-s are *distinct* vertices and the e_i-s are *distinct* edges and $x_i, x_{i+1} \in e_i$ for $1 \le i \le k-1$ and $x_1, x_k \in e_k$. The *length of the cycle* is k, the number of vertices (or the number of edges). An important special case is the cycle of length two (the first object on Figure 1.3).

Paths and cycles can form rather complicated structures in hypergraphs. In the special case of graphs, paths and cycles are determined by just giving the sequence of their vertices. For graphs, paths (and cycles) with k vertices are isomorphic, a rather standard notation is to use P_k for paths and C_k for cycles (with k vertices). Figure 1.3 gives the drawing of some paths and cycles.

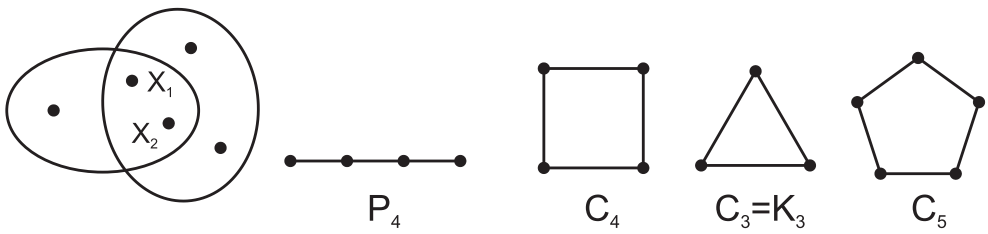


Figure 1.3. Paths and Cycles

The complete uniform hypergraph K_n^r is the hypergraph with n vertices and with all distinct r-element subsets as edges. Therefore K_n^r has $\binom{n}{r}$ edges. We always assume $1 \le r \le n$ and (as usual) abbreviate K_n^2 as K_n.

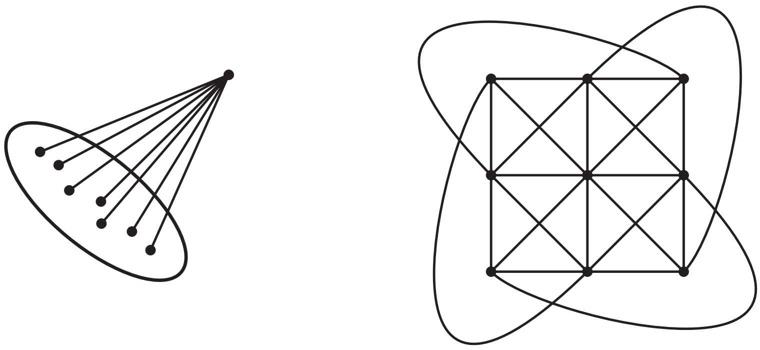
A k-partite hypergraph is a hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that V is partitioned into k non-empty sets, called *partite classes* and every edge $e \in \mathcal{E}$ intersects each partite class in at most one vertex. Notice that every hypergraph is k-partite for some k since the vertex set can be partitioned into one element partite classes. The 2-partite graphs are called *bipartite graphs.*

Intersecting hypergraphs are hypergraphs in which any pair of edges have a non-empty intersection. For example, the Fano plane (Figure 1.2) is an intersecting hypergraph.

Steiner systems. Let t, k, n be integers satisfying $2 \leq t \leq k < n$. A Steiner system $S(t, k, n)$ is a k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with n vertices such that for each t-element set $T \subset V$ there is exacly one edge $e \in \mathcal{E}$ satisfying $T \subseteq e$. The complete graph K_n is the same as a $S(2, 2, n)$ Steiner system. The Steiner systems $S(2, 3, n)$ are called *Steiner triple systems* . The Fano plane is an example of a Steiner triple system on 7 vertices. A trivial example of a Steiner system with $t = 3$ is K_4^3, the smallest interesting one is $S(3, 4, 8)$ (Exercise 1.4).

Linear spaces. These are hypergraphs in which each pair of distinct vertices is contained in precisely one edge. To exclude trivial cases, it is always assumed that there are no empty or singleton edges.

The rest of the section gives examples and results about linear spaces. An obvious example is a hypergraph with only one edge which contains all vertices, this is called a *trivial linear space* . Another example is the so called *near pencil* with one 'big' edge which contains all but one vertices (the other edges have two vertices, see Figure 1.4). Another straightforward example is the complete graph K_n. A more complicated one, the affine plane of order 3 is shown in Figure 1.5. This example (like the Fano plane) is a special case of Steiner triple systems $S(2, 3, n)$. Observe that uniform linear spaces are precisely the Steiner systems $S(2, k, n)$.



Figures 1.4. and 1.5. The near pencil and the affine plane of order 3

Drawings of linear spaces. One can try to represent a linear space by drawing its vertices as points of the plane so that edges are precisely the lines determined by these points. If there are at most seven vertices then such a drawing is possible apart from one exception, the Fano plane. Such a drawing is more clear if only the lines having at least three points are shown. Figure 1.6 gives the drawings of linear spaces with at most five vertices according to this convention.

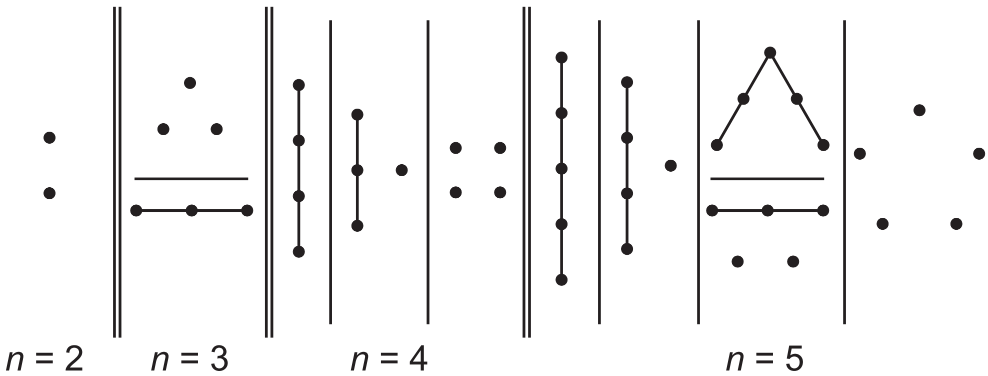


Figure 1.6. Drawings of linear spaces up to five vertices

After observing the examples, it seems that (apart from the trivial one) a linear space must have at least as many edges as vertices.

Theorem 1.3 *If a non-trivial linear space has n vertices and m edges then $m \geq n$ (de Bruijn - Erdős).*

Proof. (Tricky proof, due to Conway.) Let $\mathcal{H} = (V, \mathcal{E})$ be a linear space, $|V| = n$, $|\mathcal{E}| = m$ and suppose $1 < m \leq n$. Select $v \in V$, $e \in \mathcal{E}$ such that $v \notin e$. Observe that this implies $d(v) \geq |e|$ which (together with $m \leq n$) gives

$$\frac{1}{n(m-d(v))} \geq \frac{1}{m(n-|e|)}$$

Adding these inequalities for all pairs $v \notin e$ one gets

$$1 = \sum_{v \in V} \sum_{e \not\ni v} \frac{1}{n(m-d(v))} \geq \sum_{e \in \mathcal{E}} \sum_{v \notin e} \frac{1}{m(n-|e|)} = 1$$

because the inner sums are never empty ($1 < m$) and the terms in them do not depend on e and v (and have $m - d(v)$ resp. $n - |e|$ terms). Therefore all inequalities must be equalities, in particular $m = n$. □

The next result shows that intersecting linear spaces are rather restrictive.

Theorem 1.4 *Assume $\mathcal{H} = (V, \mathcal{E})$ is an intersecting linear space not isomorphic to a trivial one or to a near pencil. Then (for some integer $k \geq 2$)*

$$|\mathcal{E}| = |V| = k^2 + k + 1, \mathcal{H} \text{ is } (k+1)\text{-uniform and } (k+1)\text{-regular} . \quad (1)$$

Proof. (Outline.) If the union of at most two edges cover the vertex set then $\mathcal{H}$ is trivial or a near pencil. Otherwise for $e, f \in \mathcal{E}$ select $p \in V \setminus (e \cup f)$. For $x \in e$, let $g(x)$ be a vertex of f such that $x, p, g(x)$ are in an edge of $\mathcal{H}$. Then g is a bijection therefore $|e| = |f| := k + 1$. This shows that $\mathcal{H}$ is $(k+1)$-uniform. Then $(k+1)$-regularity, $|V| = k^2 + k + 1$ and $|\mathcal{E}| = k^2 + k + 1$ follows easily. □

Intersecting linear spaces with property (1) are called *finite planes of order k* . Their existence is known only if k is a prime power. The following important result of Bruck and Ryser rules out the existence of finite planes for certain values of k: if a finite plane of order k exists for $k \equiv 1$ or $k \equiv 2 \pmod 4$ then $k = x^2 + y^2$ has integer solution. Apart from these results, only the nonexistence of a finite plane of order 10 is known. The finite planes of order at most eight are unique but this is not true for order nine.

Self-test 1.

[T1.1] Show that all $0-1$ matrices are incidence matrices of hypergraphs. What about zero rows or columns?

[T1.2] Find the smallest k for which the Fano plane is a k-partite hypergraph. Answer the same question for the affine plane of order 3.

[T1.3] Is it obvious that the dual of a finite plane is a finite plane?

[T1.4] Is it obvious that the dual of a finite plane is isomorphic to itself?

[T1.5] Formulate Theorem 1.3 in dual form!

[T1.6] What is the intersection graph of the Fano plane?

[T1.7] What is the intersection graph of the affine plane of order 3 ?

[T1.8] Formulate Theorem 1.1 for graphs!

[T1.9] Find how to get the finite plane of order 3 from the affine plane of order 3 (Figure 1.5).

Exercise set 1.

[1.1] Prove by induction that every simple graph is the intersection graph of some hypergraph.

[1.2] Prove the statement in [1.1] by using the notion of duality!

[1.3] Work out the details of the proof of Theorem 1.4

[1.4] Construct $S(3,4,8)$ starting from the Fano plane. Explain why does it work!

[1.5] The Fano plane has cyclic representation: shift the set $\{1,2,4\}$ (by adding 1 to its elements) six times, using arithmetic $\pmod 7$. Find similar representation for the finite plane of order 3 and order 4. What property of $q+1$ positive integers ensures that they give a cyclic representation of a finite plane of order q?

[1.6] Show that if a Steiner triple system $S(2,3,n)$ exists then $n \equiv 1$ or $n \equiv 3 \pmod 6$.

[1.7] Find two divisibility conditions for the existence of $S(2,k,n)$.

[1.8] Find t divisibility conditions for the existence of $S(t,k,n)$.

[1.9] Prove or give counterexample for the following two statements. Regular linear spaces are uniform. Uniform linear spaces are regular.

[1.10] Give a catalogue of linear spaces with six vertices. (Follow the convention of Figure 1.6.)

2 CHROMATIC NUMBER AND GIRTH

2.1 CHROMATIC NUMBER

A proper coloring of a hypergraph is a coloring of the vertices so that there are no monochromatic edges, i.e. each edge of the hypergraph receives at least two distinct colors. Observe that it is impossible to find a proper vertex coloring if the hypergraph has a singleton edge. *Therefore in this section we shall exclude singleton edges.* Of course, instead of real colors, we shall use positive integers most of the time. However, it is also traditional to use red and blue for 2-colorings. It is clear that one needs at least two colors for a proper coloring of a hypergraph. The hypergraphs which have proper 2-colorings are called 2-*colorable* hypergraphs. These hypergraphs are very important and we shall see some of their properties later. The 2-colorable graphs are called *bipartite graphs* , they have a well-known characterization (see Excercise 2.1). However, no characterization is known for 2-colorable hypergraphs.
The chromatic number $\chi(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the minimum number of colors needed for a proper vertex coloring of $\mathcal{H}$.
Greedy coloring algorithm. The simplest way to find a proper coloring of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is to order V in an arbitrary way and color the vertices in this order by assigning the smallest positive integer to the current vertex so that it does not create a (completely) monochromatic edge.

Theorem 2.1 *Every hypergraph* $\mathcal{H} = (V, \mathcal{E})$ *satisfies*

$$\chi(\mathcal{H}) \leq \Delta(\mathcal{H}) + 1$$

Proof. Exercise 2.2 □

The girth of a hypergraph $\mathcal{H} = (V, \mathcal{E})$, $g(\mathcal{H})$, is the minimum among all cycle lengths taken over all possible cycles of $\mathcal{H}$. If $\mathcal{H}$ has no cycles, we define $g(\mathcal{H}) = \infty$. Observe that $g(\mathcal{H}) \geq 2$ and $g(\mathcal{H}) = 2$ if and only if $\mathcal{E}$ has two edges intersecting in at least two vertices.

We show later in this section that there are hypergraphs with large girth and large chromatic number. The existence of such hypergraphs have been proved first by P.Erdős and A.Hajnal with the probability method, the first explicit construction was given by L.Lovász. First we look at the case of graphs.

2.2 GRAPHS FROM THE HALL OF FAME

The easiest example of a graph of chromatic number n is the complete graph K_n, it has girth 3. Can we find an n-chromatic graph G with girth at least 4? This means that G has no K_3 subgraph (triangle). Some famous constructions are shown below.

Zykov graphs. Set $Z_1 = K_1$ and construct recursively Z_{i+1} by taking i disjoint copies of Z_i and taking a disjoint vertex set A_i of $|V(Z_i)|^i$ elements. Each vertex of A_i is made adjacent to the vertices of the vector $[x_1, x_2, \ldots, x_i]$ where x_j is a vertex of the j-th copy of Z_i. Distinct elements of A_i are made adjacent to distinct vectors.

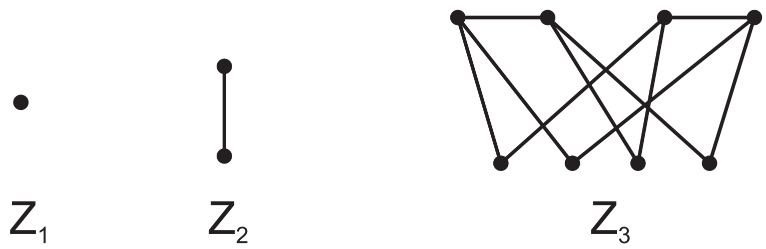


Figure 2.1. Zykov graphs.

Theorem 2.2 *The Zykov graph Z_i has no triangle and its chromatic number is i.*

Proof. First we prove by induction that Z_i has no triangle. This is obvious for Z_2, assume that true for Z_i. Select three vertices from Z_{i+1} and assume they form a triangle T. Then, from the construction, no two vertices of T are in A_{i+1} and no two vertices of T are from different copies of Z_i and if two vertices of T are in the same copy of Z_i then the third is not from A_{i+1}. Therefore the only possibility is that all three vertices of T are from the same copy of Z_i which contradicts the inductive hypothesis.

Next we prove that $\chi(Z_i) = i$, it is clear for $i = 2$. If true for some i, then a proper coloring of Z_{i+1} with $i + 1$ colors can be obtained by taking the same proper i-coloring on all copies of Z_i and coloring A with a new color. But perhaps there is a tricky proper coloring of Z_{i+1} with i colors...

Assume there is such a tricky proper coloring of Z_{i+1} with colors $1, 2, \ldots, i$. Since $\chi(Z_i) = i$ (by induction), the tricky coloring must use all colors in each copy of Z_i. In particular, there is a vector $[x_1, x_2, \ldots, x_i]$ such that for all j, x_j is a vertex of the j-th copy of Z_i and x_j is colored with color j. There exists a vertex $a \in A_i$ adjacent with the vertices of this vector! (Contradiction). $\square$

Mycielski graphs.

This construction is more economic than the Zykov construction. Set $M_1 = K_1$, $M_2 = K_2$. To define M_{i+1}, let G_i be a copy of M_i. For each vertex v of G_i define a twin vertex v^* which is adjacent to the same set of vertices as v in G_i. (The set of twins are distinct and disjoint from the vertex set of G_i.) Finally, add a new vertex w_i adjacent to all twin vertices.

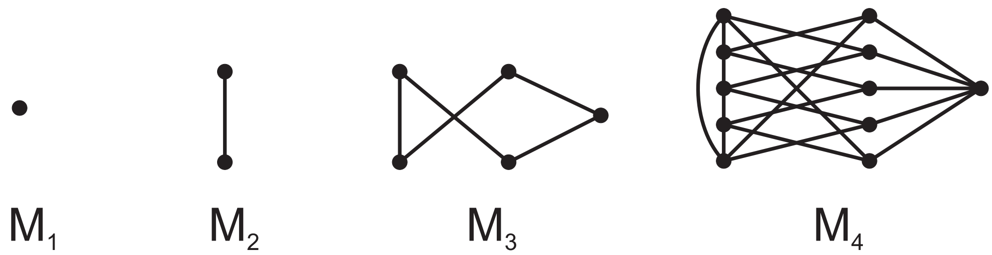


Figure 2.2. Mycielski graphs.

Theorem 2.3 *The Mycielski graph M_i has no triangles and $\chi(M_i) = i$.*

Proof. Exercise 2.8 $\square$

Tutte graphs.

The Tutte graph T_i will be a graph of chromatic number i with girth 6, so T_i is not only without triangles but without cycles of length four and five. Like the previous constructions, we start with $T_1 = K_1$ and proceed recursively. However, the number of copies of T_i needed to construct T_{i+1} will be enormous...

Assume that T_i is already constructed and has n_i vertices. Take $m_i = \binom{(n_i-1)i+1}{n_i}$ disjoint copies of T_i, the reason will be explained soon. These copies will be glued together along the hyperedges of a complete n_i-uniform hypergraph $\mathcal{H}_i = K^{n_i}_{(n_i-1)i+1}$ whose vertices are disjoint from all copies of T_i. The gluing is done as follows. Due to the definitions, we can make a one-to-one onto correspondence between the copies of T_i and the hyperedges of $\mathcal{H}_i$. Assume that the hyperedge e_j corresponds to the j-th copy of T_i ($j = 1, 2, \ldots, m_i$). For every fixed j, define an injection f_j from the vertices of e_j to the vertices of the j-th copy of T_i. The new graph edges (in addition to the edges inside the copies of T_i) are the pairs $\{x, f_j(x)\}$ where x runs over the vertices of e_j and j runs over $\{1, 2, \ldots, m_i\}$.

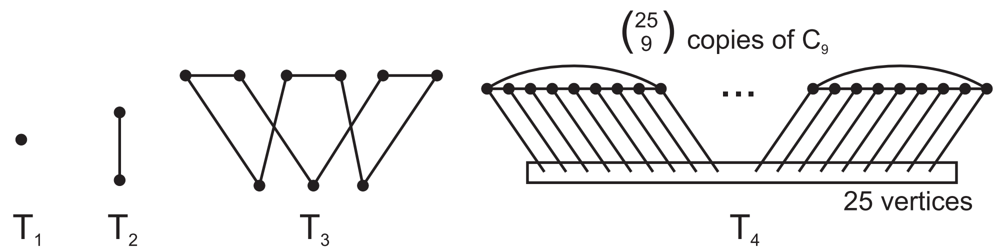


Figure 2.3. Tutte graphs.

Theorem 2.4 *The Tutte graph T_i is of girth 6 and $\chi(T_i) = i$.*

Proof. Since the vertices of the gluing hypergraph $\mathcal{H}_i$ can be colored with a new color and all copies of T_i can be colored the same way by i colors, $\chi(T_i) \le i$ comes easily by induction. The problem is to show $\chi(T_i) \ge i$. This is also done by induction, of course there is no difficulty to launch it. Assume that we already know this for some i. Assume indirectly that T_{i+1} has a proper coloring with i colors. Then, by the inductive hypothesis, all copies of T_i in T_{i+1} are colored using all the i colors.

The vertices of the gluing hypergraph $\mathcal{H}_i$ are also colored and by the pigeonhole principle, there exists a hyperedge, say e_j, whose vertices are all colored with the same color, say color 1! (Check this carefully to enjoy the beauty of Tutte's idea.) Now we have a contradiction because the j-th copy of T_i has a vertex y colored with color 1 so both vertices of the edge $\{f_j^{-1}(y), y\}$ are colored with color 1.

There is still a nice part ahead: imagine that you walk on a cycle of T_{i+1}. Look around carefully while you complete your tour on the cycle. It can certainly happen that your tour is completely inside of a copy of T_i. Then, by induction you are convinced that you visited at least six vertices. Watch your step when you leave a copy, you are in the gluing hypergraph. Where to go? You are forced to go into a new copy of T_i from where you can not escape immediately. Therefore you visit at least two vertices in at least two copies and at least two vertices of the gluing hypergraph, a total of at least six vertices.□

The Shift graph (of Erdős and Hajnal).

The new feature of this graph is that it is defined without recursion. The vertex set of the shift graph SH_n is the set of pairs (i, j), $1 \le i < j \le n$ so it has $\binom{n}{2}$ vertices. What about the edges? The vertices $(p, q), (r, s)$ are adjacent if and only if $q = r$ or $p = s$.

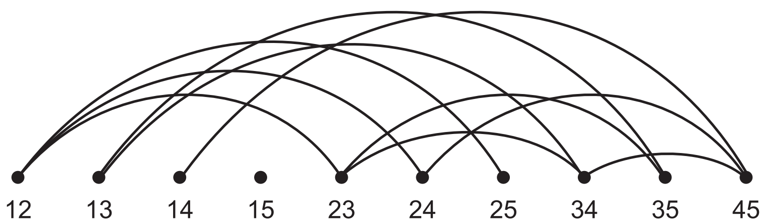


Figure 2.4. The Shift graph SH_5.

Theorem 2.5 *The Shift graph SH_n has no triangles and $\chi(SH_n) = \lceil log_2(n) \rceil$.*

Proof. The triangle free property is left as an (easy) exercise. To see that $\chi(SH_n) \le \lceil log_2(n) \rceil$, color the vertex (p, q) with color 1 if $1 \le p \le \frac{n}{2} < q \le n$. The vertices of SH_n are partitioned into A_1, A_2 as follows.

$$A_1 = \{(i, j) : 1 \le i < j \le \frac{n}{2}\},\ A_2 = \{(i, j) : \frac{n}{2} < i < j \le n\}$$

Observe that $x \in A_1$ and $y \in A_2$ are not adjacent and A_1, A_2 define shift graphs, isomorphic to $SH_{\lfloor \frac{n}{2} \rfloor}$ and to $SH_{\lceil \frac{n}{2} \rceil}$, respectively. Continuing this process, a proper coloring of SH_n is obtained with $\lceil log_2(n) \rceil$ colors.

To prove the reverse inequality, $\chi(SH_n) \geq \lceil log_2(n) \rceil$, assume that the vertices of SH_n are properly colored with t colors. For each $i < n$, let S_i denote the set of vertices $\{(i,j) : j = i+1, i+2, \ldots, n\}$. We claim that the sets S_k and S_l can not be colored with the same color set if $k \neq l$. Indeed, without loss of generality, $k < l$ and the color of (k,l) is red. Then red is a color used on S_k but red can not be used on S_l because all vertices of S_l are adjacent to the vertex (k,l). Now the claim is proved and it gives that $n-1$, the number of sets S_i is not larger than the number of nonempty subsets of t colors. It follows that $n - 1 \leq 2^t - 1$ i.e. $n \leq 2^t$ which implies $\lceil log_2(n) \rceil \leq t$. $\square$

Kneser graphs. In 1955 Kneser posed the following problem. For fixed positive integers n, k, consider the hypergraph K^n_{2n+k}, i.e. all n element subsets of a ground set of $2n + k$ elements. Kneser was interested in partitioning the hyperedges into as few classes as possible so that hyperedges in the same partition class are pairwise intersecting and conjectured that this minimum is $k + 2$. One can reformulate this as a problem to determine the chromatic number (of a very symmetric) graph, the Kneser graph, $KN(n,k)$ whose vertices correspond to the edges of K^n_{2n+k}, two vertices being adjacent if and only if the corresponding hyperedges are disjoint. Then the conjecture is that the chromatic number of this graph is $k+2$. It is easy to see that $k+2$ is an upper bound (Exercise 2.10), the problem was whether one can do better. Kneser's conjecture was proved by Lovász in 1977 and Bárány simplified it very soon thereafter (but Lovász's proof gives more). The main surprise (of both proofs) is the method applied.

Theorem 2.6 *(Lovász) It is impossible to partition the n-element subsets of a $2n + k$ -element ground set into $k + 1$ classes so that each class has pairwise intersecting sets.*

Proof. Bárány's short proof uses two ingredients (the first was used in Lovász's proof as well):
Borsuk theorem (1933). Assume that surface of the unit sphere, $S^k = \{\mathbf{x} \in R^{k+1}, ||\mathbf{x}|| = 1\}$ is covered by the union of $k+1$ open sets. Then some of these sets has two antipodal points.

Gale theorem (1956). For positive integers k, n it is possible to place $2n+k$ points on S^k so that each open hemisphere contains at least n of them. (An open hemisphere $H(\mathbf{a})$ with center $\mathbf{a} \in S^k$ is the set $\{\mathbf{x} \in S^k\}$ for which the inner product $\mathbf{xa}$ is positive.)

This is the way to put together the two ingredients. Assume that the n-element subsets of a $2n+k$-element are partitioned into $k+1$ classes so that each class contains pairwise intersecting sets. Place the vertices of the ground set on S^k as described by Gale theorem. Define the sets $U_1, U_2, \ldots, U_k, U_{k+1}$ as follows: let U_i be the set of points $\mathbf{x} \in S^k$ for which the open hemisphere $H(\mathbf{x})$ contains an n-element set from the i-th partition class. By Gale theorem, the union of the sets U_i cover S^k. The sets U_i are all open sets, Borsuk theorem says that in some of them, say in U_j, there are two antipodal points. These points are centers of disjoint open hemispheres, by definition both contain an n-element set from the j-th partition class - contradiction. □

Josh Green, a former BSM student simplified the proof even further by eliminating Gale's theorem from the proof! His modification (appeared in 2002 in the American Mathematical Monthly) is as follows. Instead of placing the vertices of the ground set on S^k by Gale's theorem, he just placed them on S^{k+1} as points in general position! General position here means no $k+2$ points are on a great k-sphere, $S^k \subset S^{k+1}$. The sets $U_1, U_2, \ldots, U_k, U_{k+1}$ are defined similarly: U_i is the set of points $\mathbf{x} \in S^{k+1}$ for which the open hemisphere $H(\mathbf{x})$ contains an n-element set from the i-th partition class. The $k+2$ sets

$$U_1, \ldots, U_{k+1}, F = S^{k+1} \setminus \bigcup_{i=1}^{k+1} U_i$$

cover S^{k+1}. Borsuk's theorem remains true if open and closed sets are mixed in the cover (one of the sets, F, is closed). Thus some of these sets contain an antipodal pair $\mathbf{a}, -\mathbf{a}$. This pair can not be in F because the great circle separating the hemispheres $H(\mathbf{a}), H(-\mathbf{a})$ contains at most k+1 points so one of $H(\mathbf{a}), H(-\mathbf{a})$ contain at least n points so $\mathbf{a}$ or $-\mathbf{a}$ would be in some U_i. Thus - as in the original proof - we have a pair of antipodal points in some U_i and this contradiction finishes the proof.

2.3 HOW TO GLUE HYPERGRAPHS TO GET GRAPHS?

Recall that Tutte graphs have arbitrary large chromatic number and girth six. No simple construction is known if graphs of girth seven are required with large chromatic number, although their existence was proved by Erdős in 1961 using his probability method. The first construction of such graphs was found by Lovász in 1968. Interestingly, his construction is based on hypergraphs. The following construction is somewhat simpler (?!). It is also based on hypergraphs.

The Nesetril-Rödl construction.

Assume that three positive integers, p, k, n are given, $k, n \geq 2$. Our objective is to construct a hypergraph $\mathcal{H}$ with the following properties: 1. $\mathcal{H}$ has no cycles of length at most p (the girth of $\mathcal{H}$ is at least $p+1$) . 2. $\mathcal{H}$ is k-uniform . 3. $\chi(\mathcal{H}) \geq n$. A hypergraph $\mathcal{H}$ with properties 1,2,3 is called a $[p, k, n]$-*hypergraph.* Observe that we have already seen $[3, 2, n]$-hypergraphs, i.e. graphs without triangles and chromatic number at least n (Zykov graphs, Mycielski graphs) and $[5, 2, n]$-hypergraphs, i.e. graphs with girth six and chromatic number at least n.

The construction of $[p, k, n]$-hypergraphs is recursive. The first step is to construct a $[1, k, n]$-hypergraph. Note that in this case we have no restrictions about cycles because, by the definition of cycles, every cycle has length at least two. We define (in terms of k and n) a number s as follows.

$$s := (k-1)(n-1) + 1.$$

What is the significance of s? We plan to construct an s-partite $[p, k, n]$-hypergraph. Why do we make our life more difficult by imposing further restrictions? The answer is the same as in many other cases in mathematics: to prove a stronger statement is easier. Now we are ready to start at $p = 1$.

Base step: the $[1, k, n]$-*hypergraph.* The complete hypergraph $\mathcal{H} = K_s^k$ is an s-partite $[1, k, n]$-hypergraph. The proof of this statement is immediate from the pigeonhole principle. The only nontrivial statement is that the chromatic number of $\mathcal{H}$ is at least n. Assume that there is a proper coloring of the vertices of $\mathcal{H}$ with $n-1$ colors. From the definition of s there are k vertices of $\mathcal{H}$ colored with the same color i.e. the edge of $\mathcal{H}$

determined by those k vertices is monochromatic. This contradicts the definition of a proper coloring.

Recursive step. Assume that $[p-1, K, n]$-hypergraphs are already constructed. Unfortunately, there is no misprint here! Although we wish to construct an s-partite $[p, k, n]$-hypergraph, we shall use $[p-1, K, n]$-hypergraphs with values of K much larger than k. (This means that graphs ($k = 2$) can not be constructed without using hypergraphs already constructed.) Notice also that s depends on k therefore the $[p-1, K, n]$-hypergraphs we shall use are S-partite where S is not the same as s. It might be a consolation than n never changes.

In fact, we shall build hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_{s+1}$, each of them will be s-partite and k-uniform. The last one, $\mathcal{H}_{s+1}$ will be the final aim, the s-partite $[p, k, n]$-hypergraph.

The definition of $\mathcal{H}_1$*.* It will be refreshing to see that $\mathcal{H}_1$ is not defined recursively. Moreover, $\mathcal{H}_1$ will have pairwise disjoint edges. After these promises, look at the definition.

The vertex set V of $\mathcal{H}_1$ is partitioned into sets $X_1, X_2, \ldots, X_s$ where each set X_i has the same number of vertices, namely $\binom{s-1}{k-1}$. This allows to define pairwise disjoint k-element edges so that for all choices of k partite sets (there are $\binom{s}{k}$ choices) there is exactly one edge which has vertices from the chosen partite classes (one vertex from each). Figure 2.5 illustrates $\mathcal{H}_1$ for $k = 3, n = 3$ in that case $s = (3-1)(3-1) + 1 = 5$, $|X_i| = \binom{4}{2} = 6$.

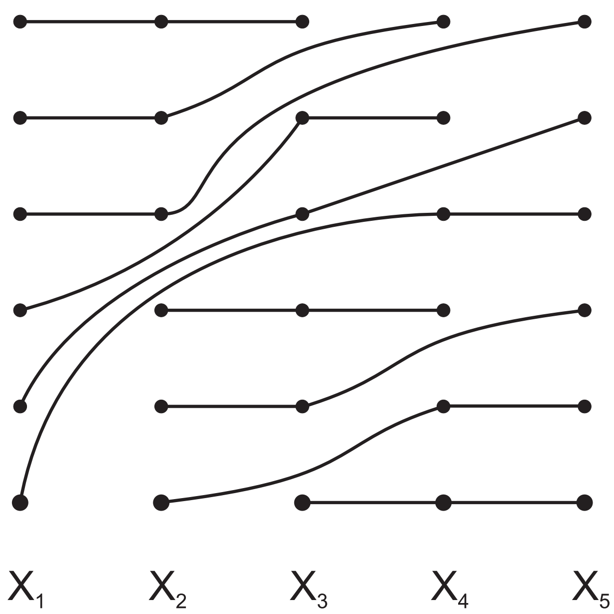


Figure 2.5. $\mathcal{H}_1$

The definition of $\mathcal{H}_{j+1}$ $(1 \le j \le s)$. Assume that $\mathcal{H}_j$ is already constructed for some j, $1 \le j \le s$ with partite classes $X_1, \ldots, X_s$.

$$\text{Set } K = |X_j|.$$

Let $\mathcal{F}_j$ be a $[p-1, K, n]$-hypergraph with vertex set V_j and with edge set $\mathcal{E}_j$. The set V_j is called the gluing base and the edges in $\mathcal{E}_j$ are called the gluing edges because they serve to keep together many copies of $\mathcal{H}_j$. The definition of $\mathcal{H}_{j+1}$ is as follows. Take as many copies of $\mathcal{H}_j$ as the number of gluing edges and, for each gluing edge, identify the j-th partite class of a copy with the gluing edge. For distinct gluing edges distinct copies of $\mathcal{H}_j$ are used. Apart from their j-th classes, all partite classes of the copies are kept disjoint. The result is an s-partite hypergraph whose t-th partite class is the union of the t-th partite classes of the copies used, except for $t = j$ when the t-th partite class is V_j. This finishes the (recursive) definition of $\mathcal{H}_j$ which gives $\mathcal{H}_{s+1}$ in the last step.

Theorem 2.7 *The hypergraph* $\mathcal{H}_{s+1}$ *is an* s*-partite* $[p, k, n]$*-hypergraph.*

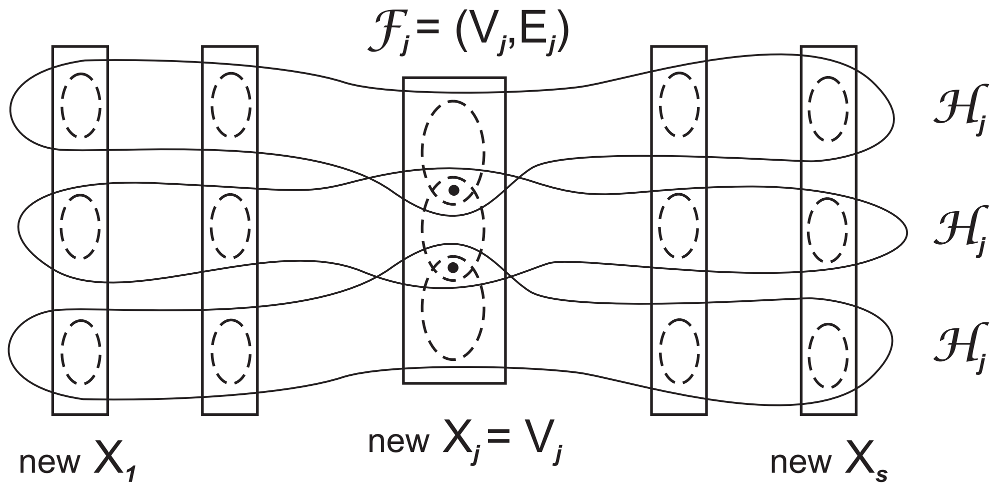


Figure 2.6. To get $\mathcal{H}_{j+1}$ from copies of $\mathcal{H}_j$

Proof. The problem is to understand the construction, after that the proof is not difficult. Only two statements need a proof. The first is that there are no cycles up to length p. This comes from an argument similar to the one used for Tutte graphs to show that their smallest cycle is of length at least six. Of course, it is a bit more difficult to travel through a cycle of a hypergraph than through a cycle of a graph... Prove first for $\mathcal{H}_1$ (trivial) then use induction to prove it for $\mathcal{H}_{s+1}$. (Exercise 2.11 asks to elaborate the details.)

The second statement is that it is impossible to color properly the vertices of $\mathcal{H}_{s+1}$ with $n-1$ colors. Assume there is such a coloring C. Then C colors the vertices of X_s which is the gluing base of $\mathcal{F}_s$ which is at least n-chromatic so there exists a gluing edge in it monochromatic under C. This gluing edge serves as X_s, the last partite class in a copy of $\mathcal{H}_s$. Therefore we can select a copy of $\mathcal{H}_s$ whose last partite class is monochromatic. Repeat the argument, C colors the $(s-1)$-th partite class, X_{s-1} of the selected copy of $\mathcal{H}_s$ which is the gluing base of $\mathcal{F}_{s-1}$... Going back like this, we find a copy of $\mathcal{H}_1$ in which all partite classes are monochromatic under C. And now the choice of s (and the contradiction) becomes (hopefully) clear (Exercise 2.12). □

Self-test 2.

[T2.1] What is the chromatic number of K_n?

[T2.2] What is the chromatic number of K_n^r?

[T2.3] Show that the Fano plane and the affine plane of order 3 are 3-chromatic hypergraphs.

[T2.4] Find a 3-chromatic graph with girth g.

[T2.5] Give some hypergraphs (graphs) with equality in Theorem 2.1.

[T2.6] Show that (with suitably chosen parameters) the Kneser graph has no triangle and its chromatic number is arbitrary large.

Exercise set 2.

[2.1] Show that a graph is bipartite if and only if it does not have cycles of odd length.

[2.2] Prove that the Greedy algorithm colors every hypergraph $\mathcal{H} = (V, \mathcal{E})$ properly with at most $\Delta(\mathcal{H}) + 1$ colors!

[2.3] Give an example of a bipartite graph G such that for a certain ordering of its vertices the Greedy algorithm uses 2000 colors for the proper coloring of G.

[2.4] Use the greedy algorithm to prove: if $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph (no singleton edges) such that for all pairs of distinct edges $e, f \in \mathcal{E}$ we have $|e \cap f| \neq 1$ then $\chi(\mathcal{H}) = 2$.

[2.5] Prove that a finite plane of order at least 3 is 2-colorable!

[2.6] Prove that Steiner triple systems are not 2-colorable!

[2.7] Modify the definition of the Zykov graph Z_{n+1} so that instead of n copies of Z_n one copy of Z_i is used for $1 \leq i \leq n$. Show that it works (no triangles, and it has chromatic number $n + 1$).

[2.8] Prove that the Mycielski graph M_n has no triangle and has chromatic number n. Hint: one possible proof uses the following lemma. Assume that G is an n-chromatic graph and consider a proper coloring of G with n colors, $1, 2, \ldots, n$. Then, for each color i, there exists a vertex v colored with color i such that v is adjacent to at least one vertex of color j for all $j \neq i$.

[2.9] Prove that there are no triangles in the Shift graphs.

[2.10] Prove that the chromatic number of the Kneser graph $KN(n, k)$ is at most $k + 2$.

[2.11] Show that $\mathcal{H}_{s+1}$ has no cycles of length $2, 3, \ldots, p$ (hint is in the text).

[2.12] Show that the chromatic number of $\mathcal{H}_{s+1}$ is at least n (outline is in the text).

3 A LOOK AT RAMSEY THEORY

3.1 RAMSEY NUMBERS

Pigeonholes and Parties. According to the Pigeonhole Principle, if $2n-1$ persons are present at a party then either there are n men or n women. The same statement is not necessarily true if only $2n-2$ persons are present. Of course, similar statement is true for any 2-partition of the elements of a ground set of $2n-1$ elements. The work of Ramsey (in 1932) extended the pigeonhole principle by introducing 2-partitions of the *pairs* of a ground set. Let $R(n)$ be the smallest integer m with the following property: if the (unordered) pairs of a m-element ground set are partitioned into two classes in any fashion then there exists n elements such that all of the $\binom{n}{2}$ pairs determined by them are in the same class. It is not obvious that $R(n)$ is well defined but it will be clear soon. The most famous value is $R(3) = 6$, in terms of parties and with a symmetric relation "to know each other" it is usually formulated as follows.

Theorem 3.1 *If six persons are present at a party then either three of them pairwise know each other or three of them pairwise do not know each other. This is not necessarily true for five persons.*

Proof. Select an arbitrary person P. Then there are three other persons P_1, P_2, P_3 such that P either knows all of them or none of them. In the former case, either P, P_i, P_j pairwise know each other or P_1, P_2, P_3 pairwise do not know each other. The latter case is similar.

The graph C_5 (vertices represent persons and edges represent the pairs who know each other) show the second part of the theorem. $\square$

Existence of Ramsey numbers. It is usual to consider partitions as colorings, in case of two colors red and blue are used most frequently. The objects to be colored can be t-element subsets of a set. Then, in terms of hypergraphs, Ramsey's existence theorem can be formulated as follows.

Theorem 3.2 *Assume that n, t are integers satisfying $n \geq t$. There exists an integer m (depending on n and t) with the following property: if the edges of K_m^t are colored with red and blue in any fashion then there exists a monochromatic K_n^t, i.e. n vertices with all the $\binom{n}{t}$ edges determined by them having the same color.*

The smallest m in Theorem 3.2 is called the Ramsey number and denoted by $R_t(n)$. Then, clearly $R_1(n) = 2n - 1$ (as mentioned above discussing the Pigeonhole Principle) and $R_2(n)$ is the same as $R(n)$ introduced above.

Proof. We prove by induction on t. The case $t = 1$ is the pigeonhole principle ($R_1(n) = 2n - 1$).

Consider t fixed and assume that $R_t(k)$ exists for *any* $k \geq t$. We are going to prove that $R_{t+1}(n)$ exists.

The heart of the proof is the definition of the "shadow coloring". If the edges of $\mathcal{H} = K_m^{t+1}$ are 2-colored and a vertex x is removed from $\mathcal{H}$ then $\mathcal{H} - x$ can be considered as a 2-colored K_{m-1}^t, namely: a t-element subset T of $\mathcal{H} - x$ is considered to be colored with the same color as $T \cup x$ in $\mathcal{H}$. We call briefly this coloring as the *shadow coloring of $\mathcal{H} - x$* . Observe that if m is large (depending on k) then the induction hypothesis guarantees a monochromatic K_k^t in the shadow coloring of $\mathcal{H} - x$. In other words, there exists a set K with k vertices such that the color of each $t+1$-element subset containing x in the set $x \cup K$ is the same.

Using that k can be chosen arbitrarily in terms of n, one can select a huge m which allows to find a set S of $2n - 1$ vertices, $S = \{x_1, x_2, \dots x_{2n-1}\}$, so that the color of any $t + 1$-element subset of S depends only on the smallest index of its elements. By the pigeonhole principle, there exist n elements of S determining a monochromatic K_n^{t+1}.□

Upper bound on $R(n)$.

The next theorem gives the explicit bound for m from the proof above (for the case $t = 2$).

Theorem 3.3 $R(n) \leq 4^n$.

Proof. Set $m = 4^n = 2^{2n}$, consider a 2-coloring of the edges of the complete graph K_m. The shadow coloring at any vertex x_1 is a vertex coloring. By the pigeonhole principle, one can find 2^{2n-1} vertices all adjacent to x_1 in the same color. Select x_2 arbitrarily among those vertices and continue... One can select $S = \{x_1, x_2, \ldots, x_{2n-1}\}$ so that the color of any edge $\{x_i, x_j\}$ within S depends only on the minimum of i and j. This gives a monochromatic K_n in S (as in the proof of Theorem 3.2). □

Many colors and non-diagonal Ramsey numbers. There are natural generalizations of the numbers $R_t(n)$. The Ramsey number

$$R_t(n_1, n_2, \ldots, n_r)$$

is the smallest integer m with the following property: if the edges of K_m^t are colored with colors $1, 2, \ldots, r$ in any fashion then, for some i, there exists a $K_{n_i}^t$, monochromatic in color i (i.e. all the $\binom{n_i}{t}$ edges within some set of n_i vertices are colored with color i).

The number $R_t(n_1, n_2, \ldots, n_r)$ is sometimes called *non-diagonal Ramsey number* (for r colors) emphasizing that the numbers n_i can be different. In the same spirit, if all n_i are equal than the term *diagonal Ramsey number* is used (for example, $R(n) = R_2(n, n)$ is a diagonal Ramsey number). The existence of non-diagonal Ramsey numbers follows immediately from the existence of diagonal ones. The order of magnitude of $R(3, n)$ ($\frac{n^2}{log(n)}$) was clarified in 1995 (Kim) after a a long process of advances. The role of non-diagonal Ramsey numbers can be illustrated by the following basic inequality of Erdős and Szekeres.

Theorem 3.4 $R(p, q) \leq R(p-1, q) + R(p, q-1)$.

Proof. Exercise [3.10] □

Corollary 3.5 $R(p, q) \leq \binom{p+q-2}{p-1}$.

Proof. Exercise [3.11] □

Exact values of Ramsey numbers. It is known that $R(3, 4) = 9$ and $R(4) = 18$ (exercises [3.1],[3.2],[3.3]). Further known values: $R(3, 5) = 14, R(3, 6) = 18, R(3, 7) = 23, R(3, 8) = 28, R(3, 9) = 36, R(4, 5) = 25$.

Paul Erdős thought that it would take enormous effort to determine $R(5)$ but to determine $R(6)$ is impossible.

The only nontrivial exact value known for three colors is $R(3,3,3) = 17$ (see exercise [3.4]). The only known value for hypergraphs is $R_3(4,4) = 13$.

Convex n-gons. The following surprising application of Ramsey numbers is a celebrated theorem of Erdős and Szekeres.

Theorem 3.6 *There exists a function $f(n)$ with the following property: if $f(n)$ points of the plane are in general position (no three on a line) then there exist n of these points forming the vertices of a convex n-*gon.

Proof. We show that $R_4(5,n)$ is a good choice for $f(n)$. Consider a set P of $R_4(5,n)$ points in general position. A four-element subset of P is called convex if the convex hull of the four points is a quadrangle, otherwise it is called concave. This is a partition of all four-element subsets of P therefore, from the definition of the Ramsey number $R_4(5,n)$, either there exist five points of P whose four-element subsets are concave or there exist n points of P whose four-element subsets are convex. However, the former possibility is impossible ([T3.2]) and the latter implies that we have n points which are the vertices of a convex n-gon ([T3.3]). □

3.2 VAN DER WAERDEN NUMBERS

Returning to the Pigeonhole Principle, clearly any 2-coloring of $[2k-1]$ guarantees a monochromatic k-element subset. In Ramsey theory one looks for special monochromatic sets, the first famous example is the case of arithmetic progressions. Van der Waerden proved in 1927 that for every integer $k > 1$ there exists an integer m (depending on k) with the following property: if the numbers in $[m]$ are 2-colored in any fashion then there is a monochromatic k-term A.P. (arithmetic progression). The smallest m with this property is called the Van der Waerden number $W(k)$. Clearly, $W(2) = 3$ because any two numbers form a 2-term A.P. The next paragraph gives an argument showing that $W(3)$ is well-defined.

Claim: $W(3) \leq 325$. Call a 3-term A.P. *almost monochromatic* if its first two terms have the same color. A *base block* is five consecutive numbers. With these definitions the following is obvious:

Proposition 3.7 *A* 2*-colored base block contains an almost monochromatic* 3*-term A.P.*

Consider a 2-coloring C of [325] and partition [325] into 65 base blocks, $B_1, B_2, \ldots B_{65}$, starting with $B_1 = [5]$. Since there are only 2^5 ways to color a base block, the Pigeonhole Principle guarantees the existence of $1 \leq i < j \leq 33$ such that C colors B_i and B_j precisely the same way. Now we can select $l \leq 65$ so that the three base blocks B_i, B_j, B_l form a 3-*term A.P. of blocks* which means that the first elements of the blocks form a 3-term A.P. Applying Proposition 3.7 for B_i, we can find an almost monochromatic 3-term A.P. in B_i. If this is monochromatic, the claim follows. Otherwise, w.l.o.g., C colors it as RRB (red,red,blue). Since B_j is colored precisely like B_i, one can find two 3-term A.P.-s with identical third term such that one of them is colored RR? and the other is colored BB?. This finishes the proof of the claim (see Figure 3.1).

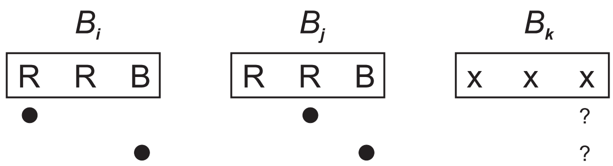


Figure 3.1. RR? and BB?

Of course, 325 is just an upper bound on $W(3)$. It is not very difficult to prove that $W(3) = 9$, but that proof is not suitable to generalize. The next step shows how to prove the existence of $W(4)$ from the existence of $W(3, r)$, which is the smallest m with the property: if $[m]$ is r-colored in any fashion then there exists a monochromatic 3-term A.P. It is strange that a theorem about 2-colorings can not be proved without proving it for r-colorings... (This interesting phenomena occurred earlier. The construction of $[p, 2, n]$-hypergraphs (graphs!) relied on constructions of $[p-1, K, n]$-hypergraphs.)

$W(4) < \infty$ **if** $W(3, r) < \infty$. A base block of length $t = 2W(3) - 1$ ensures that Proposition 3.7 can be generalized: a 2-colored base block contains an almost monochromatic 4-term A.P. (first three terms of the same color). But now we need *three* identically colored base blocks forming a 3-term A.P. of blocks! Since a base block of length t can be 2-colored in 2^t ways we need $W(3, 2^t)$ base blocks to ensure that. To extend it with a fourth base block is just a modest doubling and then we have the possibility to find two 4-term A.P.-s colored as RRR? and BBB? such that their fourth terms are identical. In fact, $t = 13$ works here, (using $W(3) = 9$ instead of the upper bound 325 from the proof) we have

$$W(4) \leq t(2W(3, 2^t) - 1) < 13W(3, 2^{13}).$$

Now the existence of $W(4)$ follows from the existence of $W(3, r)$ where $r = 2^{13}$. One can guess that this is not very small, especially not from our proof which will be roughly

$$2^{2^{\cdot^{\cdot^{2}}}}$$

where the hight of the tower is r. Concerning the existence of $W(3, r)$ only the first step of the iteration process is shown.

$W(3, 3) < \infty$. Blocks of length 7 are the 'small base blocks', one can find almost monochromatic 3-term A.P.-s in them. Blocks of length $2 \times 7 \times 3^7$ are the 'medium base blocks', one can find in them the pattern shown in Figure 3.2. The final step is to find two identically colored medium blocks and extend them to a 3-term A.P. of medium blocks (see Figure 3.2). The argument gives

$$W(3, 3) < 2(2 \times 7 \times 3^7)(3^{2 \times 7 \times 3^7})$$

and Exercise [3.14] asks to work out the details.

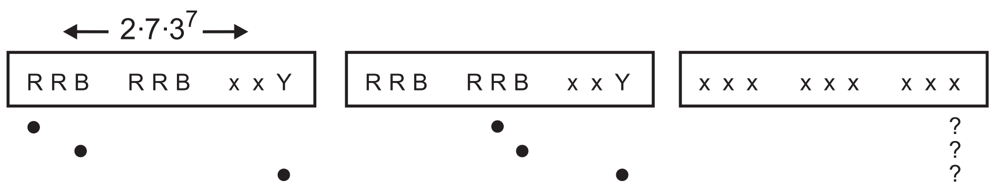


Figure 3.2. Focusing three copies of medium blocks.

Hopefully the presented argument is convincing enough to accept

Theorem 3.8 *(Van der Waerden) For every k, r there exists m (depending on k, r) such that any r-coloring of $[m]$ contains a monochromatic k-term arithmetic progression.*

The proof outlined above gives a bound for $W(k,2)$ which grows as the Ackermann function. In 1989 Shelah found a better bound, $W(k,2) \le wow(k+1)$, where "wow" is the iterated tower function

$$wow(2) = 2, wow(3) = 2^2, wow(4) = 2^{2^{2^2}} = 65536$$

and $wow(5)$ is a tower of 2's with height 65536 - wow! For comparison, note that $W(4) = 35$ and $W(3,3) = 27$. It is also known that $W(5) = 178$. The best lower bound of $W(k,2)$ is exponential (see Section 4).

3.3 TIC-TAC-TOE AND HALES-JEWETT THEOREM

The cube and its lines. The *cube* C_t^n is defined as the set of vectors of length n using t symbols. These vectors are simply called the *points* of the cube. Clearly, C_t^n has t^n points. For example, C_2^n with symbols $0,1$ is the usual n-dimensional cube.

We shall define *lines* of the cube C_t^n as follows. Assume that the symbol set is ordered, for example consider the symbols $1, 2, \ldots, t$ with natural order. Each line of C_t^n will contain t points, $P_1, P_2, \ldots, P_t$. Certain coordinates (at least one) are called moving coordinates. The symbol in P_i at a moving coordinate must be i (the i-th symbol). The coordinates which are not moving, are called the constant coordinates. At any constant coordinate all points P_j $(1 \le j \le t)$ must have the same symbol (but that common symbol may vary at distinct constant coordinates).

At this point it is useful to check that C_t^n has $\sum_{i=1}^{n} \binom{n}{i} t^{n-i}$ lines ([T3.5]). We note that lines are defined as a subset of 'geometric' lines. An even wider definition of lines would be to require only that, at each coordinate, the symbols are all equal or all distinct. Such a definition is used in the game "SET", played with cards representing C_3^4.

The cubes C_3^2 and C_2^3 together with their lines are shown on Figure 3.3.

Theorem 3.9 *(Hales-Jewett) For arbitrary positive integers t, r there exists a positive integer n (depending on t, r) with the following property: for every r-coloring of the points of the cube C_t^n there exists a monochromatic line (i.e. a line whose points are colored with the same color).*

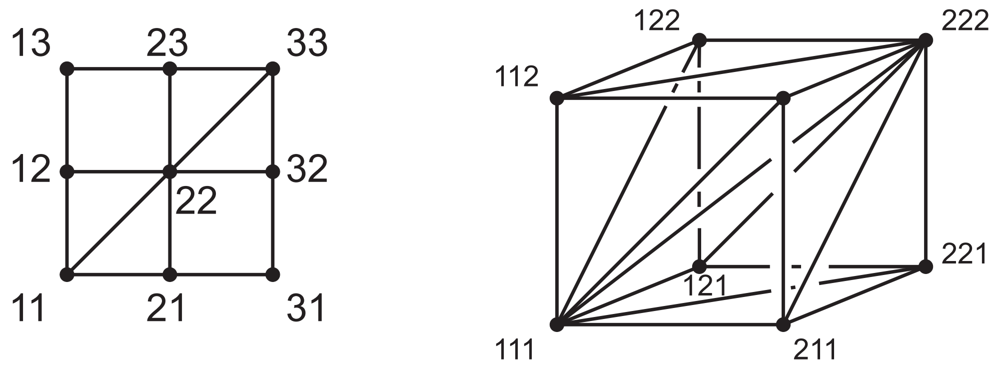


Figure 3.3. The cubes C_3^2 and C_2^3 and their lines.

Corollary 3.10 *Van der Waerden theorem*

Proof. Exercise [3.15] □

Tic-tac-toe. In the r-player Tic-tac-toe game the players color the points of C_t^n (players use distinct colors) and their objective is to create a monochromatic line (using some definition of a line). Since the lines defined above are the most restrictive, Theorem 3.9 implies that there is no draw if r persons play with fixed board size t in sufficiently high dimenson n. In case of two persons, the first player always wins (again, if n is large enough).

Self-test 3.

[T3.1] Show that $R(n) > (n-1)^2$.

[T3.2] Assume that five points of the plane are in general position (no three on a line). Show that four of them are the vertices of a convex quadrangle.

[T3.3] Assume that n points of the plane are in general position and all subsets of four points determine a convex quadrangle. Prove that the n points are the vertices of a convex n-gon.

[T3.4] Consider the k-uniform hypergraph $\mathcal{A}(k,n)$ whose vertex set is $[n]$ and whose edges are the k-term A.P.-s. Formulate Van der Waerden's theorem as a statement about the chromatic number of this hypergraph.

[T3.5] Show that C_t^n has $\sum_{i=1}^{n} \binom{n}{i} t^{n-i}$ lines.

Exercise set 3.

[3.1] Prove that $R(3,4) \leq 9$.

[3.2] Prove that $R(3,4) \geq 9$.

[3.3] Prove that $R(4,4) \leq 18$. (The reverse inequality comes from the following famous coloring of the edges of K_{17}: the vertex set is [17] and the edge (i,j) is red if $i - j$ is a square $\pmod{17}$ otherwise it is blue.)

[3.4] Prove that $R(3,3,3) \leq 17$.

[3.5] Show that $R(3,3,\ldots,3) \leq 3r!$ (assuming r arguments). Extra part: improve the bound to $\lfloor er! \rfloor + 1$ (this is the current record set eighty years ago).

[3.6] Show that $R(3,3,\ldots,3) > 2^r$ (assuming r arguments). Extra: Ideas for improvement?

[3.7] Use the result of exercise [3.5] to prove the following: if $n \geq 3r!$ then no matter how [n] is partitioned into r classes, there is a solution of the equation $x + y = z$ with all three numbers from the same class ($x = y$ is permitted in the solution).

[3.8] Show that if the edges of a countably infinite complete graph are colored red or blue in any fashion then there is a infinite monochromatic complete subgraph.

[3.9] (Kiran Kedlaya, former BSM student) Show that $R(n) \leq R_3(6,n)$.

[3.10] Prove Theorem 3.4!

[3.11] Prove the corollary of Theorem 3.4!

[3.12] Prove that $R_3(n,n)$ is also a good choice for $f(n)$ in Theorem 3.6 . Hint: color triples according to the parity of the number of points inside the triangle spanned by them.

[3.13] Assume (a real life situation) that 'knowing each other' is not necessarily symmetric. Prove that at a party with nine persons there are either three persons among which nobody knows any other or there are three persons A, B, C such that A knows B, B knows C and A knows C (a transitive triple). Show that the same statement is not necessarily true for a party of eight persons.

[3.14] Show that

$$W(3,3) < 2(2 \times 7 \times 3^7)(3^{2\times 7\times 3^7})$$

[3.15] Show that Hales-Jewett theorem implies Van der Waerden theorem.

[3.16] Show that

$$R_{t+1}(n_1, n_2) \leq R_t(R_{t+1}(n_1 - 1, n_2), R_{t+1}(n_1, n_2 - 1)) + 1$$

4 COUNTING AND PROBABILITY

4.1 PROOFS BY COUNTING

In this section some results from extremal hypergraph theory are given with proofs which are based on counting. The extremal problems are of the following type: what is the maximum or minimum number of edges in hypergraphs with a certain property?

Antichains. An *antichain* is a hypergraph $\mathcal{H} = (V, \mathcal{E})$ in which no edge contains any other edge (a *chain* is a hypergraph in which for each pair of edges, one contains the other.)

Theorem 4.1 *(Sperner) Assume that* $V = [n]$ *and* $\mathcal{H} = (V, \mathcal{E})$ *is an antichain. Then*

$$|\mathcal{E}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. (Lubell)We say that a permutation $x_1 x_2 \dots x_n$ of $V = [n]$ *extends* the edge $e \in \mathcal{E}$ if $e = \{x_1, x_2, \dots, x_t\}$ for some t. Observe that

$$|e|!(n - |e|)! \quad \text{permutations extend} \quad e \in \mathcal{E}$$

and no permutation extends distinct edges. Therefore

$$\sum_{e \in \mathcal{E}} |e|!(n - |e|)! \leq n!$$

which is equivalent with

$$\sum_{e \in \mathcal{E}} \binom{n}{|e|}^{-1} \leq 1$$

Each term is at least $\binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}$ because $\binom{n}{k}$ is maximum for $k = \lfloor \frac{n}{2} \rfloor$. This gives

$$|\mathcal{E}| \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} \leq 1$$

and the proof is finished. □

Intersecting hypergraphs. What is the maximum number of edges of an intersecting hypergraph $\mathcal{H} = (V, \mathcal{E})$ on vertex set $V = [n]$? We have to assume that $\mathcal{H}$ is simple, otherwise the question is foolish. With this assumption, the answer is not difficult, guess it before you look at exercise [4.2]. What happens if we ask the same question for simple t-uniform hypergraphs? If $n \leq 2t - 1$ then the answer is immediate ([T4.2]). For larger values of n, the answer is provided by

Theorem 4.2 *(Erdős-Ko-Rado) Assume that $\mathcal{H} = (V, \mathcal{E})$ is a simple intersecting t-uniform hypergraph on vertex set $V = [n]$ and $n \geq 2t$. Then $|\mathcal{E}| \leq \binom{n-1}{t-1}$.*

Proof. (Katona)We say that a *cyclic* permutation of $[n]$ *extends an edge $e \in \mathcal{E}$* if the vertices of e appear in consecutive positions of the cyclic permutation. There are $t!(n-t)!$ cyclic permutations extending e. On the other hand, a cyclic permutation can be the extension of at most t edges. This is not so obvious, exercise [4.3] asks for the proof. The number of cyclic permutations of $[n]$ is $(n-1)!$ therefore

$$|\mathcal{E}|t!(n-t)! \leq t(n-1)!$$

which, like a miracle, gives the statement to be proved. □

3-chromatic uniform hypergraphs. What is the minimum number of edges in a t-uniform 3-chromatic hypergraph? If $t = 2$, i.e. we have a graph, the answer is 3 because a graph with just two edges is clearly 2-chromatic and the triangle is 3-chromatic. If $t = 3$ then the answer is 7, the Fano plane gives an example of a 3-uniform 3-chromatic hypergraph, but it is not so easy to prove that all 3-uniform hypergraphs with six edges are 2-chromatic. For the next step, when $t = 4$, the answer is not known. It is rather surprising that one needs at least 2^{t-1} edges as the following famous theorem with equally famous proof claims.

Theorem 4.3 *(Erdős - Selfridge) A t-uniform hypergraph with less than 2^{t-1} edges is 2-chromatic.*

Proof. Assume that $\mathcal{H} = (V, \mathcal{E})$ is a t-uniform hypergraph with vertex set $V = [n]$. For each edge $e \in \mathcal{E}$ there are $2 \times 2^{n-t} = 2^{n-t+1}$ distinct 2-colorings of V which makes e monochromatic. This implies that at most $|\mathcal{E}|2^{n-t+1}$ 2-colorings of V are monochromatic on *some* edges of $\mathcal{H}$. However, from the assumption $|\mathcal{E}| < 2^{t-1}$, that number is smaller than 2^n, the number of 2-colorings of V. This means that there exists a 2-coloring of V under which no edge of $\mathcal{H}$ is monochromatic! □

4.2 PROBABILITY METHOD

Counting versus probability. First we repeat the proof of Theorem 4.3 in terms of the probability method. Consider the probability space of all possible (2^n) 2-colorings of $[n]$ where each coloring has the same ($\frac{1}{2^n}$) probability. The probability of a monochromatic edge is $\frac{1}{2^{t-1}}$ therefore the probability of the event that some edge is monochromatic is not larger than $\frac{|\mathcal{E}|}{2^{t-1}}$ which is smaller than 1 from the assumption of Theorem 4.3. Thus the complementary event has positive probability which finishes the proof.

In all subsequent proofs such a simple probability space is used. It has only one parameter N, the number of elements of the probability space, each element has probability $\frac{1}{N}$. The probability of subsets (events) of the probability space is the cardinality of the subset divided by N. We shall use the obvious fact that *the probability of union of events is at most the sum of the probabilities of the events.* (Equality if and only if the events are pairwise disjoint.) In many applications the probability space is colorings of vertices or edges of a graph or hypergraph. We shall prove highly nontrivial theorems from a trivial property of the model, namely that only the empty set has probability zero.

The Erdős lower bound on $R(n)$. The best lower bound of $R(n)$ we have seen so far is quadratic in n (Self-test [T3.1]). A cubic one is to be constructed in the next section which can be generalized to a construction providing a lower bound which grows faster than any polynomial of n (with fixed degree). However, no construction is known which reaches c^n (with some constant $c > 1$). Still, the probability method gives such a bound easily.

Theorem 4.4 *(Erdős) $m < R(n)$ if $\binom{m}{n} < 2^{\binom{n}{2}-1}$.*

Proof. The probability space is the 2-colorings of the edges of K_m. This space has $N = 2^{\binom{m}{2}}$ elements. A subset of n vertices in K_m spans a monochromatic complete subgraph with probability $p = \frac{1}{2^{\binom{n}{2}-1}}$. Therefore the probability of having a monochromatic K_n is at most $p\binom{m}{n} < 1$ from the assumption of the theorem. This means that the probability of having no monochromatic K_n is positive. Therefore there exists a 2-coloring of the edges of K_m such that there is no monochromatic K_n. This clearly implies that m is a lower bound for $R(n)$. $\square$

The lower bound m for $R(n)$ was stated in a strange way in Theorem 4.4 to show the idea of the proof clearly. To get an actual bound, one has to calculate the largest m (in terms of n) satisfying the condition

$$\binom{m}{n} < 2^{\binom{n}{2}-1}.$$

Such calculations often require upper bound on binomial coefficients. The better the estimate, the better the bound. If we start with the trivial upper bound

$$\binom{m}{n} < m^n$$

then our lower bound is

$$m = 2^{\frac{n}{2}} \times 2^{-\frac{1}{n}} \times 2^{-\frac{1}{2}}$$

([T4.3]). If Sterling's formula is used for the asymptotic of $\binom{m}{n}$ then

$$m = 2^{\frac{n}{2}} \times \frac{n}{e\sqrt{2}}$$

is obtained so there is no earthshaking difference between the trivial and the best estimate of m.

Corollary 4.5 *(Erdős) For $n \geq 3$, $2^{\frac{n}{2}} < R(n)$.*

Lower bound for $W(k)$.

Theorem 4.6 $W(k) \geq 2^{\frac{k}{2}}$.

Proof. Outline:

$$\frac{\binom{n}{2}}{2^{k-1}} < 1.$$

Tournaments. A tournament T_n is an oriented K_n, which means that each edge of the complete graph K_n gets a one-way orientation. The edge from vertex v to vertex w is denoted by (v, w). We can interpret a tournament T_n as vertices represent players and (v, w) means that v beats w in the game.

There are two non-isomorphic T_3-s, one is the cyclic triangle and the other is the transitive triangle. ([T4.4]: define the isomorphism of tournaments) A tournament is called transitive if the presence of the edges (i, j) and (j, k) imply that (i, k) is also an edge. A tournament T_n is transitive if and only if its vertices can be labeled with $1, 2, \ldots, n$ so that all edges point from smaller label to larger label (exercise [4.4]). This shows that transitive tournaments on the same number of vertices are isomorphic.

Large transitive subtournaments.

Let $f(k)$ be the smallest n such that any T_n contains a transitive T_k. It is obvious that $f(2) = 2$, it is easy that $f(3) = 4$ ([T4.5]). Then $f(4) = 8$ and $f(5) = 14$.

Theorem 4.7 $2^{\frac{k-1}{2}} \leq f(k) \leq 2^{k-1}$.

Proof. The lower bound comes from the probability method. The probability space is the set of all tournaments on $[n]$, there are $N = 2^{\binom{n}{2}}$ elements in the probability space. If

$$\frac{\binom{n}{k} k!}{2^{\binom{k}{2}}} < 1$$

then there exists a T_n which does not contain a transitive subtournament on k vertices. Work out the details! (Exercise [4.5])

The upper bound comes easily by induction (Exercise [4.6]). □

Existence versus construction. Theorem 4.7 shows that there exists a tournament T_{127} which does not contain a transitive T_{15} because the probability of having a transitive T_{15} in every T_{127} is smaller than one. In a "real world" problem, it would be important to know how much smaller? The following situation is from the book of Erdős and Spencer: Probabilistic methods in Combinatorics.

"A computer scientist is told to construct , on a computer, the matrix of a tournament on 127 players that does not contain a transitive subtornament on 15 players. Imagine that if his matrix does contain a transitive subtournament on 15 players, his company (or country or mutual fund) will suffer grave financial consequences. He learns from this book that such a tournament exists. But this is not sufficient - his boss wants a specific matrix stored in the computer. Our scientist calculates that it may cost millions of dollars (or tens of millions of forints) to find such a T by exhaustive methods. However, like many of us, he is on a limited budget. What is he going to do? If his moral character is sufficiently low he would construct T at random, pack his bags to prepare for a hasty departure, and hope for the best. Given our experiences with computer scientists, we surmise that this would be the "real world" solution. If, on the other hand, he is of outstanding moral character his only recourse is to ask for a refund on this book. We would, of course, refuse."

This was written more than twenty years ago so certain statements (definitely the conversion rate of dollars to forints) may not be valid any more. The quoted paragraph continues:

"The "real world" situation is different if it is required that the tournament does not contain a transitive subtournament on 16 players. Then a "random" tournament will have the desired property with probability $\geq .998$. This probability of success is sufficient in many practical situation - our computer scientist can randomize and relax."

Paradoxical tournaments. Who are the best k players in a tournament? This can not be answered easily because, for every k, there are tournaments in which every set of k players is beaten by somebody. Tournaments of this property are called *k-paradoxical* .

Theorem 4.8 *For every positive integer k, there exist k-paradoxical tournaments.*

Proof. At this point, the reader can easily prove it (exercise [4.7]) by interpreting the inequality

$$\binom{n}{k}(1-\frac{1}{2^k})^{n-k} < 1$$

□

Hamiltonian Paths in Tournaments.

A famous theorem of Rédei (1937) says that the players of any tournament can be placed in a line so that each player won the match against her left hand neighbor. More formally, every tournament has a (directed) Hamiltonian path. The simple proof is left as an exercise ([4.8]). (A more difficult result is that the number of Hamiltonian paths is odd in every tournament.) How many Hamiltonian paths can we found in a tournament? If the tournament is transitive then there is only one. On the other hand, there are tournaments with many Hamiltonian paths.

Theorem 4.9 *For every n there is a tournament T_n with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.*

Proof. (Outline.) A given permutation of $V = [n]$ gives a Hamiltonian path with probability $\frac{1}{2^{n-1}}$. This implies that the expected number of H. paths is $\frac{n!}{2^{n-1}}$. Therefore *there exists a* T_n with at least that many H. paths. □

4.3 LOCAL LEMMA

All the proofs we have done by the probability method can be described using the following scheme. There are events $A_1, A_2, \ldots, A_n$ in a probability space and we know that for $A = A_1 \cup A_2 \cup \ldots \cup A_n$, $prob(A) < 1$. Then $prob(\overline{A}) > 0$ which means that the event $\overline{A}$ exists (nonempty). We have also seen that requiring $prob(A) < \epsilon$ will ensure the existence of $\overline{A}$ with probability $1 - \epsilon$.

Another possibility to ensure $prob(\overline{A}) > 0$ is to assume mutual independence of the events A_i and $prob(A_i) < 1$ (for all i). In this case $prob\left(\overline{A_i}\right) > 0$ and

$$prob\left(\overline{A}\right) = prob\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = \prod_{i=1}^{n} \overline{A_i} > 0$$

follows from mutual independence. However, mutual independence is almost never present in applications. The Local Lemma, which is introduced in a paper of Erdős and Lovász, allows a limited number of dependencies among the events A_i. Initially the lemma was applied to a specific problem (to prove Theorem 4.11 below) but later became an important tool applicable in many situations.

Dependency bound. *The set of events* $\{A_1, A_2, \ldots, A_n\}$ *has dependency bound* D if, for each i, A_i is dependent on at most D other events A_j. More precisely, this means that for each i, there is a set $S_i \subseteq [n]$ such that $|S_i| \leq D$, $i \notin S_i$ and A_i is *independent of any boolean combination* of the events $\{A_j : j \notin S_i\}$. (Two events A, B are independent if $prob(A \cap B) = prob(A)prob(B)$.)

Theorem 4.10 *(Local Lemma) Assume that* $A_1, A_2, \ldots, A_n$ *are events with dependency bound* D *and* $prob(A_i) \leq \frac{1}{4D}$ *for* $1 \leq i \leq n$. *Then*

$$prob\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$$

Proof. We shall use occasionally AB for $A \cap B$, pr for $prob$. It is convenient to use the notion of conditional probability. For non-empty B define probability of A with condition B as $pr(A|B) := \frac{pr(AB)}{pr(B)}$. Then A, B are independent if and only if $pr(A|B) = pr(A)$.

Claim: for arbitrary $S \subseteq [n]$ and $i \notin S$

$$P = pr\left(A_i | \bigcap_{j \in S} \overline{A_j}\right) \leq \frac{1}{2D}$$

The claim is proved by induction on $|S|$, for $S = \emptyset$ it is true because $pr(A_i) \leq \frac{1}{4D}$ from the condition of the theorem. From the assumptions, S can be partitioned into S_1 and S_2 so that $|S_1| \leq D$ and A_i is independent of any boolean combination of $\{A_j : j \in S_2, j \neq i\}$. Using the identity $pr(A|BC) = \frac{pr(AB|C)}{pr(B|C)}$ write P as

$$P = \frac{pr\left(A_i \bigcap_{j \in S_1} \overline{A_j} | \bigcap_{j \in S_2} \overline{A_j}\right)}{pr\left(\bigcap_{j \in S_1} \overline{A_j} | \bigcap_{j \in S_2} \overline{A_j}\right)} \tag{2}$$

The numerator in (2) is bounded from above by

$$pr\left(A_i | \bigcap_{j \in S_2} \overline{A_j}\right) = pr(A_i) \leq \frac{1}{4D}$$

and the denominator in (2) is bounded from below by

$$1 - pr\left(\bigcup_{j\in S_1} A_j | \bigcap_{j\in S_2} \overline{A_j}\right) \geq 1 - \sum_{j\in S_1} pr\left(A_j | \bigcap_{j\in S_2} \overline{A_j}\right) \geq 1 - \sum_{j\in S_1} \frac{1}{2D} \geq 1 - \frac{1}{2} = \frac{1}{2}$$

and the claim follows.

The proof is finished because

$$pr\left(\bigcap_{j=1}^{n} \overline{A_j}\right) = \prod_{i=1}^{n} (\overline{A_i} | \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_{i-1}}) \geq \prod_{i=1}^{n} \left(1 - \frac{1}{2D}\right) > 0.$$

□

Erdős - Lovász theorem.

Theorem 4.11 *Assume that $\mathcal{H} = (V, \mathcal{E})$ is a t-uniform hypergraph in which each edge intersects at most 2^{t-3} other edges ($t \geq 3$). Then $\mathcal{H}$ is 2-chromatic.*

Proof. Assume that the vertex set of $\mathcal{H}$ is $[n]$. Consider the probability space of all 2-colorings of V. For each edge $e_i \in \mathcal{E}$, let A_i be the event that e_i is monochromatic. Clearly $prob(A_i) = \frac{1}{2^{t-1}}$ and $D = 2^{t-3}$ is a good dependency bound for the events because A_i is independent from any Boolean combinations of those A_j-s for which e_j does not intersect e_i. Thus the condition for the Local Lemma is satisfied with equality ($\frac{1}{2^{t-1}} = \frac{1}{4D}$). Then the Local Lemma says

$$prob\left(\bigcap_{e_i\in\mathcal{E}} \overline{A_i}\right) > 0$$

i.e. there exists an event (2-coloring of V) under which no edge is monochromatic. □

Improved lower bound on $W(k)$.

Theorem 4.12 $W(k) > \frac{2^k}{8k}$.

Proof. The probability space is all 2-colorings of $[n]$. For each k-term A.P. S, A_S is the event that S is monochromatic. Clearly, $prob(A_S) = \frac{1}{2^{k-1}}$. Since a fixed S can intersect at most nk other S's (Ex.[4.9]), $D = nk$ is a good dependency bound. The condition of the Local Lemma is satisfied with equality if $n = \frac{2^k}{8k}$. □

Even cycles in regular digraphs.

A directed graph is r-regular if the indegree and the outdegree of each vertex is r. It was a long-standing conjecture that for sufficiently large r any r-regular digraph contains an *even* (directed) cycle. A surprising proof was found by Alon.

Theorem 4.13 *For $r \geq 8$ any r-regular digraph has an even cycle.*

Proof. The probability space is the 2-colorings of the vertex set of the given r-regular digraph G. A coloring is 'good' if for each vertex x of G there exists an edge (x, y) such that the color of x and y are different. A coloring is 'bad' at x if there is no vertex y such that the color of x and y are different. Let A_x be the event that a coloring is bad at x. Then $prob(A_x) = \frac{1}{2^r}$. On the other hand, $D = r^2$ is good dependency bound for the set of events $\{A_x : x \in V\}$ (check! Ex.[4.10]). To apply the Local Lemma, the condition $\frac{1}{2^r} \leq \frac{1}{4r^2}$ is needed which is valid since $r \geq 8$. Now the Local Lemma says that there exists a good coloring which immediately gives an even cycle ([T4.6]). □

4.4 JOKES.

The triangle is 2-chromatic!

Theorem 4.14 *Joke 1. The graph K_3 is 2-chromatic.*

Proof. Let $V = [3]$ be the vertex set of the triangle T. The probability space is the 2-colorings of V ($N = 8$ elements). Let A_{ij} be the event that the edge ij is monochromatic. Clearly, $prob(A_{ij}) = \frac{1}{2}$ for all the three choices of index pairs. Any pair of events is independent, for example

$$prob\,(A_{12}A_{13}) = prob\,(A_{12})\,prob\,(A_{13}) = \frac{1}{4}$$

therefore $D = \frac{1}{2}$ is a good dependency bound. The condition for the application of the Local Lemma holds with equality because

$$prob\,(A_{ij}) = \frac{1}{2} = \frac{1}{4D}$$

Therefore there exists a coloring of V so that no edge is monochromatic. $\square$

Spencer's injections.

A function $f : S \to T$ is called *injective* if its domain is S and it is one-to-one.

Theorem 4.15 *Joke 2 (Spencer) Assume that S is a finite set. If* $|T| > \binom{|S|}{2}$ *then there exists an injective function* $f : S \to T$.

Proof. The probability method is used with probability space of all $f : S \to T$ functions. The event A_{xy} is that $f(x) = f(y)$. Clearly

$$prob\,(A_{xy}) = \frac{1}{|T|}$$

for all $x, y \in S$. Since

$$prob\left(\bigcup_{x,y\in S} A_{xy}\right) \leq \sum_{x,y\in S} \frac{1}{|T|} = \frac{\binom{|S|}{2}}{|T|} < 1$$

it follows that

$$prob\left(\bigcap_{x,y\in S} \overline{A_{xy}}\right) > 0$$

so an injective f exists. $\square$

One can try to improve the theorem above by the Local Lemma. Indeed, an order of magnitude better result comes from it.

Theorem 4.16 *Joke 3 (Spencer) Assume that S is a finite set. If* $|T| \geq 8|S|$ *then there exists an injective function* $f : S \to T$.

Proof. Follow the previous proof, but apply the Local Lemma. Clearly, $D = 2(|S| - 2)$ is a good dependency bound for the events A_{xy}. The condition $|T| \geq 8|S|$ makes the Local Lemma applicable. $\square$

Self-test [T4.8] asks for further improvements along this line...

Self-test 4.

[T4.1] Is it true that an antichain is a simple hypergraph?

[T4.2] What is the maximum number of edges of a t-uniform intersecting simple hypergraph on vertex set $[n]$, if $n \leq 2t - 1$?

[T4.3] Check the calculation of the lower bound in Theorem 4.4 if the upper bound m^n is used for $\binom{m}{n}$.

[T4.4] Define isomorphism for tournaments.

[T4.5] Show that any T_4 contains a transitive T_3.

[T4.6] Show that a digraph with a good coloring (defined in the proof of Theorem 4.13 has an even cycle!

[T4.7] Find the mistake in the proof of Joke 1.

[T4.8] Find the strongest theorem generalizing the results of Jokes 2 and 3.

Exercise set 4.

[4.1] Let $\mathcal{H}$ be an antichain with f_k edges of k elements. Prove the so called LYM inequality:

$$\sum_{k=0}^{n} f_k \binom{n}{k}^{-1} \leq 1$$

[4.2] Prove that an intersecting simple hypergraph on vertex set $[n]$ has at most 2^{n-1} edges!

[4.3] Prove that any cyclic permutation extends at most t edges! (See the proof of Theorem 4.2.)

[4.4] Prove that if T_n is transitive then its vertices can be labeled with $1, 2, \ldots, n$ so that all edges point from smaller label to larger label.

[4.5] Work out the details for the lower bound in Theorem 4.7

[4.6] Work out the inductive proof for the upper bound in Theorem 4.7.

[4.7] Prove Theorem 4.8 by interpreting the stated inequality!

[4.8] Prove that every tournament has a directed Hamiltonian path.

[*4.9] Prove that a k-term A.P. in $[n]$ can intersect at most kn other k-term A.P.-s (in $[n]$).

[4.10] Show that $D = r^2$ is a good dependency bound in the proof of Theorem 4.12.

5 LINEAR ALGEBRA METHOD

5.1 THE DIMENSION BOUND

Oddtown. Oddtown is a little town somewhere (probably near to Chicago) whose inhabitants like to form clubs. They insist to have clubs with an odd number of members to settle all matters easily by a majority vote. Their other rule is that each pair of clubs must share an even number of members, because they can prove

Theorem 5.1 *Oddtown (under the stated rules) has no more clubs than the number of inhabitants.*

Proof. Assume that the inhabitants form the vertices and the clubs form the edges of the hypergraph $\mathcal{H} = (V, \mathcal{E})$. Set $V = [n]$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. Consider the linear space of n-component vectors over the two-element field $F_2 = \{0, 1\}$. The edges of $\mathcal{H}$ are viewed as elements of this linear space by looking at e_i as a row vector in the incidence matrix of $\mathcal{H}$. The two rules of Oddtown can be expressed nicely with the standard inner product:

$$\mathbf{e_i e_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Claim: $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}$ are linearly independent.

To see this, assume that

$$\sum_{i=1}^{m} \lambda_i \mathbf{e_i} = \mathbf{0} \qquad (\lambda_\mathbf{i} \in \mathbf{F_2})$$

and take the inner product of both sides by $\mathbf{e_j}$. Using the property of the inner product displayed above, we get $\lambda_j = 0$ and this is valid for any j proving the claim.

Therefore $m \leq$ the dimension of the linear space $= n$. □

The main motive of this cute proof is that the dimension is an upper bound for the number of linearly independent elements in any linear space. One can easily find a similar theorem and proof for "Modprimetown" (exercise [5.1]). A further trick is needed for the proof of the "Modprimepowertown" theorem (exercise [5.2]). Surprisingly, it is unknown whether a similar "Modsixtown" theorem is true or not. But presently there is the strong belief in Modsixtown that they will never have more clubs than the number of inhabitants.

Fisher inequality.

Theorem 5.2 *Assume that $\mathcal{H} = (V, \mathcal{E})$ is a simple hypergraph without empty edges and each pair of distinct edges intersect in precisely λ elements. Then $|\mathcal{E}| \leq |V|$.*

Proof. There are some easy special cases treated separately. For $\lambda = 0$ the edges are pairwise disjoint, so the theorem follows immediately. If there is an edge $e \in \mathcal{E}$ with $|e| = \lambda$ then the theorem follows easily ([T5.1]). Assume these cases are excluded, $|V| = n$, $|\mathcal{E}| = m$.

The edges of $\mathcal{H}$ are considered again as row vectors from the incidence matrix. The intersection property of the edges can be expressed with the inner product as follows (with suitable positive integers γ_i):

$$\mathbf{e_i}\mathbf{e_j} = \begin{cases} \lambda + \gamma_i & \text{if } i = j \\ \lambda & \text{if } i \neq j \end{cases}$$

Claim: The vectors $\mathbf{e_i}$ are linearly independent in the linear space of n-component vectors of real numbers (i.e. in $\mathbf{R^n}$). Assume, on the contrary, that $\sum_{i=1}^{m} \alpha_i \mathbf{e_i} = \mathbf{0}$ and compute the inner product of both sides by $\mathbf{e_j}$. We get that for all j ($j \in [m]$)

$$\lambda\beta + \alpha_j\gamma_j = 0$$

where $\beta = \sum_{i=1}^{m} \alpha_i$. Expressing α_j ($\gamma_j \neq 0$), we get that for all $j \in [m]$

$$\alpha_j = -\frac{\lambda}{\gamma_j}\beta \tag{3}$$

proving the claim if $\beta = 0$. If $\beta \neq 0$ then adding (5) for all $j \in [m]$ we get

$$\beta = -\lambda \left(\sum_{j=1}^{m} \frac{1}{\gamma_j} \right) \beta$$

which is contradiction ($\lambda > 0, \beta \neq 0$). This proves the claim and the theorem follows from the dimension argument. $\square$

Corollary 5.3 *Theorem 1.3 (de Bruijn- Erdős theorem).*
Proof. *Self-test [T5.2]*
A cubic lower bound for $R(n)$.

Proposition 5.4 *(Zs.Nagy)* $R(n) > \binom{n-1}{3}$

Proof. Here is the tricky coloring of the edges of the complete graph $K = K_{\binom{n-1}{3}}$ with red and blue. Associate the vertices of K with the edges (triples) of the complete hypergraph $T = K^3_{n-1}$. Color an edge of K red if the corresponding two triples of T intersect in precisely one vertex. Otherwise color the edge blue.

A red complete subgraph of K corresponds to triples of T pairwise intersecting in precisely one element. From Theorem 5.2, there are at most $n-1$ such triples. A blue complete subgraph of K corresponds to triples of T pairwise intersecting in zero or two elements. Since three is an odd number, we have an oddtown situation, theorem 5.1 shows that there are at most $n-1$ triples again. Therefore in our coloring of the edges of K there is no monochromatic K_n. $\square$

We note here that the above construction can be generalized to give superpolynomial lower bound for $R(n)$. For a fixed prime p define a red-blue coloring by associating vertices to (p^2-1)-element subsets of an m-element set and coloring the edge between two vertices red if the corresponding subsets has intersection size congruent to $-1 \pmod p$. Otherwise the edge is blue. Observe that proposition 5.4 gives this coloring for $p = 2$. However, it is not as easy to show that under this coloring there are no large monochromatic complete subgraphs [not larger than $2\binom{m}{p-1}$].
Two-distance sets. A one-distance set in $\mathbf{R^n}$ is a set of points (vectors) such that any two of them have the same distance. It is easy to see that a one-distance set has at most $n+1$ points and equality is possible (exercise [5.3]). How many points can form a *two-distance set* in $\mathbf{R^n}$, i.e. a set such that the distances between the points take only two values? It is easy to find a two-distance set with $\binom{n}{2}$ points ([T5.3]). The next theorem gives an upper bound not far from this lower bound.

Theorem 5.5 *A two-distance set of* $\mathbf{R^n}$ *has no more than* $\frac{(n+1)(n+4)}{2}$ *points.*

Proof. Assume that $\{\mathbf{a_1}, \ldots, \mathbf{a_m}\}$ is a two-distance set of $\mathbf{R^n}$ with distances δ_1 and δ_2. Using

$$||\mathbf{x}|| = \left(\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{x_k^2}\right)^{\frac{1}{2}}$$

the distance of $\mathbf{x}, \mathbf{y}$ is $||\mathbf{x} - \mathbf{y}||$. Define the polynomials (in n variables) $f_i(\mathbf{x})$ for $i \in [m]$ as

$$f_i(\mathbf{x}) = (||\mathbf{x} - \mathbf{a_i}||^{\mathbf{2}} - \delta_{\mathbf{1}}^{\mathbf{2}})(||\mathbf{x} - \mathbf{a_i}||^{\mathbf{2}} - \delta_{\mathbf{2}}^{\mathbf{2}})$$

Claim: the polynomials f_i are linearly independent over $\mathbf{R}$.

Assume that for real λ_i

$$\sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) = \mathbf{0}$$

and substitute $\mathbf{a_j}$ for $\mathbf{x}$! All but the j-th term vanishes and the j-th term is $\lambda_j(\delta_1\delta_2)^2$ which can be zero only if $\lambda_j = 0$. Since j was arbitrary, the claim is proved.

On the other hand, looking at the definition of f_i, all f_i can be written as a linear combination of polynomials of five types:

$$\left(\sum_{k=1}^{n} x_k^2\right)^2 \quad ; \quad \left(\sum_{k=1}^{n} x_k^2\right) x_j \quad ; \quad x_i x_j \quad ; \quad x_i \quad ; \quad 1$$

thus (adding the number of these polynomials type-to-type) all f_i belong to a linear space of dimension at most

$$1 + n + \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+4)}{2}$$

so their number, m, can not exceed the claimed value. □

Cross-intersecting hypergraphs. Assume that a p-uniform hypergraph $(V, \mathcal{E})$ and a q-uniform hypergraph $(V, \mathcal{F})$ have the same vertex set $V = [n]$ and both hypergraphs have m edges, $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$, $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$. We call the pair $(\mathcal{E}, \mathcal{F})$ *cross-intersecting* if $e_i \cap f_j = \emptyset$ if and only if $i = j$.

Theorem 5.6 *If $(\mathcal{E}, \mathcal{F})$ is cross intersecting then $m \le \binom{p+q}{p}$.*

Proof. We say that an ordering of $[n]$ is compatible with the pair (e_i, f_i) if all elements of e_i precede all elements of f_i in the ordering. There are

$$\binom{n}{p+q} p!q!(n-p-q)! = \frac{n!}{\binom{p+q}{p}}$$

orderings compatible with a pair (e_i, f_i) and the cross-intersecting property implies that each ordering of $[n]$ is compatible with at most one pair (e_i, f_i). Therefore

$$m \frac{n!}{\binom{p+q}{p}} \le n!$$

proving the theorem. □

There are two questions to answer. Why do we look at cross-intersecting hypergraphs? The answer is that they come along in many extremal problems. An interesting application is given in exercise [5.4]. The second question is why do we have this nice proof here, why not in section 4.1 (PROOFS BY COUNTING)? Because there is another proof, by Lovász, which uses the dimension bound and gives a stronger theorem.

Represent the vertices of the cross-intersecting pair $(\mathcal{E}, \mathcal{F})$ by vectors of $\mathbf{R^{p+1}}$ placed in general position, i.e. so that any $p+1$ of them are linearly independent. The vector associated with $v \in V$ is denoted by

$$\mathbf{s}(v) = (s_0(v), s_1(v), \ldots, s_p(v)) \in \mathbf{R^{p+1}}$$

For $i = 1, 2, \ldots, , m$ define the polynomial

$$g_i(\mathbf{x}) = \prod_{v \in f_i} \mathbf{s}(v)\mathbf{x}$$

(product of inner products) which is a homogeneous polynomial of degree q in $p+1$ variables. The dimension of this linear space is $d = \binom{p+q}{p}$ (exercise [5.5]).

The vectors corresponding to the vertices of e_j generate a subspace $\mathbf{A}_j$ of dimension p, let $\mathbf{a}_j$ be a nonzero vector orthogonal to $\mathbf{A}_j$. Since the vectors $\mathbf{s}(v)$ are in general position, $\mathbf{s}(v) \in \mathbf{A}_j$ if and only if $v \in e_j$. This means that $g_i(\mathbf{a}_j) = 0$ if and only if e_j

and f_i have non-empty intersection. Since we have cross-intersecting hypergraphs, this means

$$g_i(\mathbf{a_j}) = \begin{cases} 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

and this condition implies that the polynomials g_i are linearly independent (exercise [5.6]). Therefore their number, m, is at most the dimension d, proving theorem 5.4. In fact, the linear independence of the g_i-s follow from the weaker assumption

$$g_i(\mathbf{a_j}) = \begin{cases} 0 & \text{if } i < j \\ \neq 0 & \text{if } i = j \end{cases}$$

(exercise [5.7]) and it gives a generalization of Theorem 5.4. for so called skew cross-intersecting hypergraphs.

5.2 HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Partitioning into complete bipartite graphs

Assume that we want to partition the edge set of the complete graph K_n into as few complete bipartite graphs as possible. One possibility is to decompose it into $n-1$ stars. One can also try to halve the vertex set, take a complete bipartite graph between these parts and continue. This works also but again gives $n-1$ parts. After some futile attempts to beat these constructions, one feels that perhaps $n-1$ is the best. This is true but it is very surprising that presently nobody can prove it without using linear algebra.

Theorem 5.7 *(Graham-Pollack) If the complete graph K_n is partitioned into m (edge disjoint) complete bipartite graphs then $m \geq n-1$.*

Proof. (Tverberg) Assume that $[n]$ is the vertex set of K_n and we have a partition into m complete bipartite graphs in the form $[A_i, B_i]$, where $i \in [m]$ and the edges of the complete bipartite graphs are between A_i and B_i. Consider n variables x_j ($j \in [n]$) and define $2m$ linear forms as follows:

$$L_i = \sum_{j \in A_i} x_j \quad \text{and} \quad M_i = \sum_{j \in B_i} x_j$$

The condition that the complete bipartite graphs form edge disjoint partition is translated as

$$\sum_{1 \le i < j \le n} x_i x_j = L_1 M_1 + L_2 M_2 + \ldots + L_m M_m \tag{4}$$

Assume indirectly that $m \le n - 2$. Then the system of $m + 1$ homogeneous linear equations with n variables

$$L_i = 0 \text{ for } i = 1, 2, \ldots, m$$

and

$$x_1 + x_2 + \ldots + x_n = 0$$

has a nontrivial solution $a_1, a_2, \ldots, a_n$ because $m + 1 < n$. We get a surprising contradiction:

$$0 < a_1^2 + a_2^2 + \ldots + a_n^2 = (a_1 + a_2 + \ldots + a_n)^2 - 2 \sum_{1 \le i < j \le n} a_i a_j = 0$$

□

Discrepancy of hypergraphs.

The discrepancy of a hypergraph is a measure of the most balanced bipartition of the vertex set. Let Ψ denote a function from V to $\{-1, +1\}$ and let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Then the discrepancy of $\mathcal{H}$ is defined as follows.

$$disc(\mathcal{H}) = min_\Psi max_{e \in \mathcal{E}} |\sum_{x \in e} \Psi(x)|$$

Theorem 5.8 *(Beck-Fiala) Assume $\mathcal{H}$ is a hypergraph with maximum degree t. Then $disc(\mathcal{H}) \le 2t - 1$.*

Proof. Assume $V = [n]$, $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. We allow Ψ to take real values from the interval $[-1, +1]$. A vertex x is called bad for Ψ if $\Psi(x) \in (-1, +1)$. An edge e_j is called bad for Ψ if it has *more than* t bad vertices. We shall define a procedure to get a Ψ with no bad edges and at each step the following condition will be preserved for all bad edges:

$$\sum_{x \in e_j} \Psi(x) = 0 \tag{5}$$

The procedure starts with $\Psi \equiv 0$ which clearly satisfies (5). For the general step, assume that there exist bad edges $e_1, e_2, \ldots, e_r$ for Ψ and let $1, 2, \ldots, s$ be the bad vertices of $\mathcal{H}$ for Ψ.

Claim: $r < s$. (Exercise [5.9])

Consider the unknowns $y_1, y_2, \ldots, y_s$. Using the claim, the system of r homogeneous linear equations

$$\sum_{i \in e_j} y_i = 0$$

has a non-trivial solution $y_1, y_2, \ldots, y_s$. Select λ so that for all $i, 1 \leq i \leq s$ the values $\Psi(i) + \lambda y_i$ are all in $[-1, +1]$ and at least one of them becomes $+1$ or -1. Then a new Ψ can be defined changing the values $\Psi(i)$ to $\Psi(i) + \lambda y_i$ for all $i, 1 \leq i \leq s$. It is left as exercise [5.10] to show that the required λ exists and the new Ψ satisfies condition (5).

The procedure eventually terminates with a Ψ with no bad edges because the set $\{i \in V : \Psi(i) \in \{+1, -1\}\}$ increases at each step. At this point the final Ψ is defined by setting all values Ψ to 1 at the remaining bad vertices. It is easy to see (exercise [5.11]) that this final Ψ satisfies $|\sum_{x \in e_j} \Psi(x)| \leq 2t - 1$ for all j, $1 \leq j \leq m$. □

5.3 EIGENVALUES

Regular graphs of girth five.

Consider an r-regular (simple) graph of girth five. At least how many vertices do we need in such a graph? An arbitrary vertex is adjacent to r further vertices and all of these are adjacent to $r - 1$ further vertices. Could it be possible that we double-counted some vertices so far? The answer is no, thus we have at least $r^2 + 1$ vertices in such a graph. (Check this, [T5.4]). But is it possible to find r-regular graphs with girth five which have only r^2+1 vertices? For $r=2$ the five-cycle, C_5, is a trivial example. For $r=3$ the famous Petersen graph is such an example.

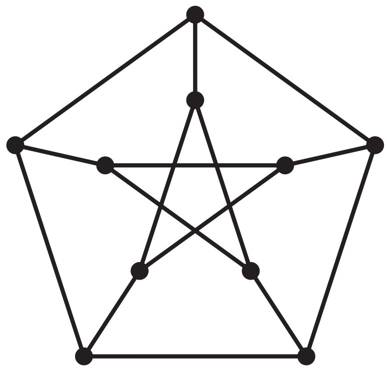


Figure 5.1. The Petersen graph.

Unfortunately such beautiful graphs are rare as the next theorem shows.

Theorem 5.9 *(Hoffman-Singleton) If G is an r-regular graph with girth five and has r^2+1 vertices then $r \in \{2, 3, 7, 57\}$.*

Proof. Set $n = r^2+1$, $V(G) = [n]$ and consider the adjacency matrix $A = [a_{ij}]$ of G defined as the $n \times n$ matrix

$$a_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an edge of } G \\ 0 & \text{if } ij \text{ is not an edge of } G \end{cases}$$

Claim. $A^2 + A - (r-1)I_n = J_n$ where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ all-ones matrix.

To prove the claim, notice that the diagonal entries of A^2 are r so the equation is valid for the diagonal entries (the diagonal of A is zero). Assume $i \neq j$. If (ij) is not an edge of G then there is precisely one vertex of G is adjacent to both i and j (no C_4 in G) therefore the ij entry in A^2 is 1. If (ij) is an edge of G then no vertex of G is adjacent to both i and j (no C_3 in G) therefore the ij entry in A^2 is 0. In both cases the claimed matrix equation is correct.

By the principal axis theorem the symmetric matrix A has an orthogonal basis from eigenvectors. Let $\mathbf{f}$ be the all-ones column vector with n elements. Then $A\mathbf{f} = r\mathbf{f}$ hence $\mathbf{f}$ is an eigenvector with eigenvalue r. Consider an eigenvector $\mathbf{e}$ orthogonal to $\mathbf{f}$, by definition, $A\mathbf{e} = \lambda\mathbf{e}$, $\mathbf{fe} = 0$ (with inner product notation). Multiply the equation of the claim by $\mathbf{e}$ from the right which gives

$$\lambda^2\mathbf{e} + \lambda\mathbf{e} - (r-1)\mathbf{e} = \mathbf{0}$$

implying that $\lambda^2 + \lambda - (r-1) = 0$. Solving this equation we get that the possible eigenvalues are

$$\lambda_{12} = \frac{1}{2}(-1 \pm \sqrt{4r-3}).$$

Let m_i be the multiplicity of λ_i ($i \in [2]$). The sum of the multiplicities of eigenvalues is n, therefore

$$1 + m_1 + m_2 = n = r^2 + 1 \tag{6}$$

and the sum of the eigenvalues is the trace of A which gives

$$r + m_1\lambda_1 + m_2\lambda_2 = 0. \tag{7}$$

Substituting the actual values of λ_i to (7) we get

$$2r - (m_1 + m_2) + (m_1 - m_2)s = 0 \quad \text{where} \quad s = \sqrt{4r-3}.$$

Using (6) this equation changes to

$$2r - r^2 + (m_1 - m_2)s = 0. \tag{8}$$

If s is irrational then $m_1 - m_2 = 0$ implying $r = 2$. Otherwise s is a positive integer and $r = \frac{s^2+3}{4}$. Plug this into (8) to get

$$s^4 - 2s^2 - 16(m_1 - m_2)s - 15 = 0.$$

This equation implies that s is a divisor of 15, its possible values are $1, 3, 5, 15$. Therefore $r = \frac{s^2+3}{4} \in \{1, 3, 7, 57\}$ but $r = 1$ is impossible (girth is ∞). □

Self-test 5.

[T5.1] Prove Fisher inequality if some edge has λ vertices.

[T5.2] How do you get de Bruijn-Erdős theorem from Fisher inequality?

[T5.3] Construct a two-distance set in $\mathbf{R^2}$ with $\binom{n}{2}$ points.

[T5.4] Show that an r-regular graph of girth five has at least $r^2 + 1$ vertices.

Exercise set 5.

[5.1] Assume that p is a prime and $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph in which the edge sizes are not divisible by p but each pair of edges has intersection size divisible by p. Prove that $|\mathcal{E}| \leq |V|$.

[5.2] The same as exercise [5.1] but replace the prime by a power of prime. Hint: show that the edges as row vectors are linearly independent in the linear space of n-component vectors *over the field of rational numbers* .

[5.3] Show that a one-distance set of $\mathbf{R^n}$ has at most $n + 1$ points.

[5.4] A transversal of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a set $T \subseteq V$ such that $T \cap e \neq \emptyset$ for all $e \in \mathcal{E}$. The transversal number of $\mathcal{H}$ is defined as

$$\tau(\mathcal{H}) = min\{|T| : T \quad \text{is a transversal of} \quad \mathcal{H}\}$$

A Hypergraph $\mathcal{H}$ is called p-critical if $\tau(\mathcal{H}) = p$ but $\tau(\mathcal{H} - e) < p$ for each $e \in \mathcal{E}$.

Prove (using Theorem 5.6) that a $(p+1)$-critical t-uniform hypergraph has at most $\binom{p+t}{t}$ edges. Is equality possible (for every p, t)?

[5.5] Prove that the linear space of homogeneous polynomials of degree q in $p+1$ variables has dimension $\binom{p+q}{p}$.

[5.6] Let S be an arbitrary set and $\mathbf{F}$ is a field. Consider the linear space of all functions from S into $\mathbf{F}$. Let $a_i \in S$ and g_i are functions satisfying

$$g_i(a_j) = \begin{cases} 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$$

(Diagonal criterion.) Prove that the functions are linearly independent.

[5.7] The same as [5.6] but from the weaker condition, called triangular criterion:

$$g_i(a_j) = \begin{cases} 0 & \text{if } i < j \\ \neq 0 & \text{if } i = j \end{cases}$$

[5.8] Assume that we want to cover the edge set of the complete graph K_n by the edges of complete bipartite graphs (allowing to cover an edge many times). How many bipartite graphs do we need?

[5.9] Show that $r < s$ at an arbitrary step in the procedure of the proof of Theorem 5.8.

[5.10] Prove that λ can be defined as required in the proof of Theorem 5.8 and condition (*) is preserved for the new Ψ.

[5.11] Show that the final Ψ in the proof of Theorem 5.8 satisfies $|\sum_{x \in e_j} \Psi(x)| \leq 2t - 1$ for all $j, 1 \leq j \leq m$.

6 FRUIT SALAD

6.1 Distinct representatives of sets.

A system of distinct representatives (SDR) of a hypergraph with m edges $e_1, e_2, \ldots, e_m$ is a set of distinct vertices $v_1, v_2, \ldots, v_m$ such that $v_i \in e_i$ for all i.

Theorem 6.1 *A hypergraph has an SDR if and only if $|\cup_{i\in S} e_i| \geq |S|$ for all $S \subseteq [m]$.*

This important result can be formulated for bipartite graphs as follows.

Theorem 6.2 *A bipartite graph $[A, B]$ has a matching from A to B if and only if $|\Gamma(S)| \geq |S|$ for every $S \subseteq A$. (Here $\Gamma(S)$ denotes the set of vertices adjacent to some vertex of S.)*

Outline of a short proof by induction on $|A|$: if strict inequality holds for all nonempty proper subsets S, delete an edge and apply induction; if there is equality for some S then apply induction for the bipartite graphs $[S, \Gamma(S)]$ and $[A \setminus S, B \setminus \Gamma(S)]$. □

The proof above does not provide a fast algorithm to find a matching or a subset $S \subseteq A$ violating the condition. Such a proof can be obtained by using the *alternating path method* .

6.2 Symmetric chain decomposition.

The subsets of $[n]$ can pe partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains. (Apply Theorem 6.2.) This gives another proof for Sperner's theorem (Theorem 4.1 in the handout). A nicer partition is also possible: a chain in $[n]$ is called *symmetric* if it contains sets of cardinality $a, \ldots, n-a$ for some a, $(0 \leq a \leq \lfloor n/2 \rfloor)$.

Theorem 6.3 *The subsets of $[n]$ can be partitioned into symmetric chains.*

Proof. Assume we have a SCD on $[n-1]$ and to each chain C associate C^* by adding the element n to each member of C. Then remove the top element of C^* and place it as the top element of C. □

6.3 A property of n sets on n elements.

A *vertex deletion* of a hypergraph means that we remove a vertex x from the vertex set and each edge e is replaced by the edge $e \setminus x$. In other words, column x is removed from the incidence matrix of the hypergraph.

Theorem 6.4 *Asssume that a simple hypergraph has n vertices and n edges. Then there is a vertex deletion which gives a simple hypergraph.*

Proof. Assume that the edges are $e_1, \ldots, e_n$ and define a graph G with vertex set $[n]$ and with edges ij if and only if the symmetric difference of the edges e_i and e_j is a single vertex x. If Theorem 6.4 is not true then we have n edges in G. However, G can not contain any cycle - prove it! - thus we have a contradiction. □

6.4 A property of $n + 1$ sets on n elements.

Theorem 6.5 *Assume that $e_1, \ldots, e_{n+1}$ are nonempty edges of a hypergraph on vertex set $[n]$. There exists nonempty disjoint subsets I_1, I_2 of $[n + 1]$ such that*

$$\cup_{i \in I_1} e_i = \cup_{i \in I_2} e_i$$

Proof. Consider the edges as $0 - 1$ vectors in R^n. They are linearly dependent. This gives easily the proof. □

6.5 A property of $n + 2$ sets on n elements.

Theorem 6.6 *Assume that $e_1, \ldots, e_{n+2}$ are nonempty edges of a hypergraph on vertex set $[n]$. There exists nonempty disjoint subsets I_1, I_2 of $[n + 2]$ such that*

$$\cup_{i \in I_1} e_i = \cup_{i \in I_2} e_i$$

moreover

$$\cap_{i \in I_1} e_i = \cap_{i \in I_2} e_i$$

Proof. Consider the vectors in R^{2n} whose first n coordinates are determined by e_i and the last n by the complement of e_i. These vectors are in an $n+1$-dimensional subspace of R^{2n} therefore they are linearly dependent. Finish the proof. $\square$

No proofs are known without linear algebra.

6.6 Critical hypergraphs.

Call a hypergraph *critical* if it is not 2-colorable but the edge set within any proper subset of vertices is 2-colorable (like the Fano plane).

Theorem 6.7 *A critical hypergraph has at least as many edges as vertices.*

Proof. Assign real variables x_j to vertices and consider the system of homogeneous linear equations $\sum_{j \in e_i} x_j = 0$. $\square$

The critical hypergraphs have a surprising connection with digraphs without even cycles. Assume that D is a digraph, define the hypergraph $H(D)$ as follows. The vertices of H are the vertices of D and the edges of H are the sets $x \cup \Gamma^+(x)$ where x is a vertex of D and $\Gamma^+(x)$ is the 'outset' of x.

Theorem 6.8 *For a strong digraph D, $H(D)$ is critical if and only if D has no even directed cycles.*

6.7 Sunflower theorem.

A sunflower with s petals is a hypergraph with s edges such that the intersection of any two edges are the same set, called the kernel of the sunflower. Notice that the kernel can be empty...

Theorem 6.9 *An r-uniform hypergraph with more than $r!(s-1)^{r+1}$ edges contains a sunflower with s petals.*

Proof. Induction on r. Take a maximum family of pairwise disjoint edges, $e_1, \ldots, e_k$. If $k \geq s$ we are done. Otherwise

$$|\cup_{i=1}^{k} e_i| \leq r(s-1)$$

and there exists a vertex of degree more than

$$\frac{r!(s-1)^{r+1}}{r(s-1)} = (r-1)!(s-1)^r$$

and we are done by induction. □

6.8 Sum-free subsets of numbers.

Theorem 6.10 *Every set B of n positive integers contain $A \subseteq B$ such that $|A| > n/3$ and A is sum-free (the sum of any two elements of A is not in A).*

Proof. Let $p = 3k+2$ be a prime larger than any number in $B = \{b_1, \ldots, b_n\}$. Set $C = \{k+1, \ldots, 2k+1\}$. Then C is a sum-free subset of the cyclic group Z_p. For each b_i, the set

$$S_{b_i} = \{xb_i : x \in [p-1]\} \pmod{p} = [p-1].$$

Since S_{b_i} covers $[3k+1]$, their union for $b_i \in B$ covers C $n(k+1)$ times. Therefore for some $x \in [p-1]$ at least $\frac{(k+1)n}{3k+1} > n/3$ b_i satisfy $xb_i \in C \pmod{p}$ and these b_i-s form a sum-free subset A in B. □

Remarks. Theorem 6.1 is Hall's theorem (1935). The SCD theorem (Theorem 6.3) is from de Bruijn, Tengbergen and Kruyswijk (1951). Theorem 6.4 is from Bondy (1972), Theorem 6.5 is from Tverberg (1971), Theorem 6.6 is from Lindstrom (1993), Theorems 6.7 and 6.8 are from Seymour (1974). The sunflower theorem (Theorem 6.9) is from Erdős and Rado (1960) - the bound given is not sharp. In fact, it seems to be a very difficult problem to decide whether for s=3 the bound can be improved to c^r. The complete r-partite hypergraph with two vertices in each partite class and with edge multiplicity $r-1$ shows that $2^r(r-1)$ edges do not give a sunflower with three petals. Theorem 10 is from Erdős (1965). It is not known whether $n/3$ is the best bound, Alon and Kleitman constructed a set B of n positive integers with no larger sum-free subset than $12n/29$.

7 ADVANCED MENU

7.1 Factorization of hypergraphs.

Assume that r is a divisor of n, a *factorization* of K_n^r is a partition of the edges into $\binom{n-1}{r-1}$ parts so that each part consists of n/r pairwise disjoint edges. The case $r = 1$ is trivial, the case $r = 2$ is easy: represent the vertices as a regular polygon $(1, 2, \ldots, n-1)$ and its center (n). The edges $1+i, n+i$ for $0 \le i \le n/2 - 1$ define a factor and one can rotate it to get a factorization. The general solution is provided by the following theorem (Baranyai, 1973).

Theorem 7.1 *The complete r-uniform hypergraph K_n^r has a factorization.*

Proof. The first step is the following rounding lemma of Baranyai:

Lemma 7.2 *Let A be a nonnegative real matrix with integral row and column sums. Then the elements of A can be replaced by their floor or ceiling so that all row and column sums retain their values.*

Proof. A non-integral entry (NIE) of A gives another NIE in the same row which gives another NIE in the same column which gives another NIE in the same row, etc. This procedure gives an alternating cycle of NIE-s along which we can replace the elements to get at least one integral entry and keep all row and column sums. Then repeat the procedure. □

Set $M = \binom{n-1}{r-1}$ (the number of factors) and $m = n/r$ (the number of edges in each factor). Define an m-partition of a set X as a partition of X into m parts, such that $\emptyset$ is allowed with multiplicities. Example: $\emptyset, \emptyset, 1$ is a 3-partition of [1].

Lemma 7.3 *For every integer $k \in [0, n]$ there exist m-partitions of $[k]$, $A_1, A_2, \ldots, A_M$ with the following property (*): each subset $S \subseteq [k]$ occurs in exactly $\binom{n-k}{r-|S|}$ partitions.*

Example: $n = 6, r = 2, m = 3, M = 5, k = 2$ then $A_1 = A_2 = A_3 = A_4 = \{\emptyset, 1, 2\}$, $A_5 = \{\emptyset, \emptyset, \{1, 2\}\}$. Then $S = \emptyset$ occurs $\binom{6-2}{2-0} = 6$ times, $S = \{1\}$ and $S = \{2\}$ occurs $\binom{6-2}{2-1} = 4$ times, $S = \{1, 2\}$ occurs $\binom{6-2}{2-2} = 1$ times.

Observe that Lemma 7.3 gives the theorem: for $k = n$ it says that $S \subseteq [n]$ occurs in exactly $\binom{n-n}{r-|S|}$ partitions - this number is zero except for $|S| = r$ when it is one!!

Lemma 7.3 is proved by induction on k. The base is $k = 0$ when all partitions consist of m empty sets. Assume that for some $k < n$ the m-partitions $A_1, A_2, \ldots, A_M$ satisfy (*). Define matrix A as an $M \times 2^k$ matrix where the rows are associated with the partitions the columns are associated with the 2^k subsets of $[k]$. The entry of A associated to row A_i and column S is 0 if $S \notin A_i$, otherwise it is

$$\frac{r - |S|}{n - k}$$

except when $S = \emptyset$: in that case the entry $\frac{r-|S|}{n-k}$ is taken with the multiplicity of S in A_i. In the example above, A is a 5×4 matrix, the columns are associated to $\emptyset, 1, 2, \{1, 2\}$. The first four rows are $[1/2, 1/4, 1/4, 0]$ and the fifth row is $[1, 0, 0, 0]$ because $\emptyset$ has multiplicity two in A_5.

The row sums of A:

$$\sum_{S \in A_i} \frac{r - |S|}{n - k} = \frac{1}{n - k}(mr - \sum_{S \in A_i} |S|) = 1$$

and the column sums of A:

$$\sum_{\{i: S \in A_i\}} \frac{r - |S|}{n - k} = \frac{r - |S|}{n - k}\binom{n - k}{r - |S|} = \binom{n - k - 1}{r - |S| - 1}.$$

Using Lemma 7.2, round A to integral B keeping row and column sums. Since the row sums are all ones, there is precisely one 1 in each row of B, this corresponds to a block $S_i \in A_i$.

Change $A_1, A_2, \ldots, A_M$ by replacing S_i with $S_i \cup \{k + 1\}$ (and keep all other blocks unchanged). This defines $A_1^*, A_2^*, \ldots, A_M^*$ and we show that they satisfy (*) for $k + 1$. Assume $S^* \subseteq [k + 1]$.

Case 1. $k + 1 \in S^*$. Now S^* appeared 0 times in the A_i-s and appears in the A_i^*-s as many times as the column sum of B which is equal to the column sum of A thus

$$\binom{n - k - 1}{r - (|S^*| - 1) - 1} = \binom{n - k - 1}{r - |S^*|}$$

showing property (*) for $k+1$.

Case 2. $k+1 \notin S^*$. Now S^* occurred $\binom{n-k}{r-|S^*|}$ times in the A_i-s (inductive hypothesis) but disappeared as many times as extended by $k+1$ i.e. by column sum of B which is equal to column sum of A thus

$$\binom{n-k}{r-|S^*|} - \binom{n-k-1}{r-|S^*|-1} = \binom{n-k-1}{r-|S^*|}$$

showing property (*) for $k+1$.

This proves Lemma 7.3 and Theorem 7.1. □

7.2 Normal hypergraphs and perfect graphs.

A *partial hypergraph* of $\mathcal{H} = (V, E)$ is a hypergraph $\mathcal{H}' = (V, E')$ where $E' \subseteq E$. Thus partial hypergraphs are defined by removing edges. The *chromatic index* of a hypergraph is the minimum number of colors needed to color the edges so that each color class has pairwise disjoint edges. (Such edge colorings are called proper.)

The chromatic index is denoted by $q(\mathcal{H})$. Clearly, $q(\mathcal{H}) \geq \Delta(\mathcal{H})$ for every hypergraph (Δ is the maximum degree). Example: for the Fano plane $q = 7$ and $\Delta = 3$.

A hypergraph is called *normal* if $q = \Delta$ for every partial hypergraph. The following lemma of Lovász captures an important property of normal hypergraphs.

Lemma 7.4 *Assume that e is an edge of a normal hypergraph $\mathcal{H}$. Form a new hypergraph $\mathcal{H}^+$ by adding a new copy of e. Then $\mathcal{H}^+$ is also a normal hypergraph.*

Proof. We prove by induction on the number of edges of $\mathcal{H}$. The only problem is to prove $q = \Delta$ for $\mathcal{H}^+$. Let e^+ be the new edge. If e contains a vertex of maximum degree in $\mathcal{H}$ then

$$\Delta(\mathcal{H}^+) = \Delta(\mathcal{H}) + 1, q(\mathcal{H}^+) = q(\mathcal{H}) + 1$$

and the proof is done. Otherwise consider a proper coloring of $\mathcal{H}$ with $\Delta(\mathcal{H}) = t$ colors, assume that e is red. Remove all red edges and add e^+. This is a partial hypergraph of $\mathcal{H}$ with maximum degree $t-1$ so there is a proper coloring with $t-1$ colors. Adding back the red edges, we have a proper coloring of $\mathcal{H}^+$ with $\Delta(\mathcal{H}^+) = t$ colors and the lemma is proved.

The next lemma shows that normal hypergraphs behave nicely for another pair of parameters. Let $\nu(\mathcal{H})$ denote the maximum number of pairwise disjoint edges in a hypergraph $\mathcal{H}$ (matching number) and let $\tau(\mathcal{H})$ denote the transversal number, the minimum number of vertices needed to intersect all edges of $\mathcal{H}$ (see Exercise 5.4 in handout). The obvious inequality $\tau \geq \nu$ becomes equality for normal hypergraphs.

Lemma 7.5 *For normal hypergraphs $\tau = \nu$.*

Proof. Assume that $\mathcal{H}$ is normal, set $t = \nu(\mathcal{H})$. We prove by induction on t that $\tau(\mathcal{H}) = t$. Let E_x denote the set of edges of $\mathcal{H}$ incident to vertex x.

Base: $t = 1$. Since $\mathcal{H}$ is intersecting, q equals to the number of edges, which by normality equals to the maximum degree. Thus $\tau(\mathcal{H}) = 1$.

Inductive step. We prove that there exists a vertex x for which $\nu(\mathcal{H} \setminus E_x) < t$, then induction proves the theorem.

Assume indirectly that for each vertex x, $\nu(\mathcal{H} \setminus E_x) = t$. Assume that $\mathcal{H}$ has n vertices $x_1, x_2, \ldots, x_n$. For each x_i let $\mathcal{F}_i$ be a maximum matching of $\mathcal{H} \setminus E_x$. Consider the hypergraph

$$\mathcal{H}_0 = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$$

and note that

(a) $\mathcal{H}_0$ is normal by Lemma 7.4.

(b) $\mathcal{H}_0$ has tn edges.

(c) $\Delta(\mathcal{H}_0) < n$.

In a proper coloring of $\mathcal{H}_0$ each color class has at most t edges thus, by (b), $q(\mathcal{H}_0) \geq n$. However, by (c) $q(\mathcal{H}_0) \neq \Delta(\mathcal{H}_0)$ which contradicts (a) and the proof of Lemma 7.5 is finished □

Intersection graphs of hypergraphs. Every simple graph is the intersection graph of a suitable hypergraph (exercises 1.1 and 1.2 in the handout). For a given (simple) graph G, the dual of G is a suitable hypergraph. This is a special case of the following construction. Assume that on the vertex set of G we have a hypergraph H whose hyperedges span complete subgraphs in G and each edge of G is covered by some hyperedge of H. It is easy to see that the dual of H is a suitable hypergraph, i.e. the intersection graph of H^* is G. The cited special case is when the edge set of G defines H. Another (not very economic) choice of H is to select all complete subgraphs of G as hyperedges. The best choice is to select the minimum number of complete subgraphs needed to cover the edge set of G. However, as we shall see, there is another useful choice for H called the *clique-hypergraph* of G: the edges of H are the complete subgraphs of G maximal for inclusion.

Example. Consider the graph G obtained from a triangle $T = K_3$ by adding three new vertices, each adjacent to a different pair of vertices of T. (G has six vertices and nine edges.) The dual of G has 9 vertices, the dual of all complete subgraphs of G has 19 vertices, the dual of a minimum cover of G has 3 vertices and the dual of the clique-hypergraph of G has 4 vertices.

The advantage of the clique-hypergraph is that its dual has the Helly-property: pairwise intersecting edges have a nonempty intersection.

The perfect graph theorem. Let $\omega(G)$ denote the number of vertices in the largest complete subgraph of G. A graph is called *perfect* if $\chi = \omega$ for all induced subgraphs. The perfect graph theorem (PGT) is the following result of Lovász:

Theorem 7.6 *A graph is perfect if and only if its complement is perfect.*

Proof. Assume that G is perfect, i.e. $\omega = \chi$ for all induced subgraphs. This implies that $\Delta = q$ for all partial hypergraphs of H^*, the dual of the clique hypergraph H of G - in other words, H^* is normal (Helly property is needed!). Lemma 15 gives that $\nu = \tau$ for all partials of H^* which gives that $\omega = \chi$ for all induced subgraphs of the complement of G. □

Remark. Lovász's proof of PGT is from 1972. It was a conjecture of Berge (1960) who also had a stronger conjecture, the perfect graph conjecture (PGC) claiming that a graph is perfect if and only if it does not contain odd cycles of length at least five and their complements (as induced subgraphs!). The PGC is recently proved by Chudnovsky, Robertson, Seymour and Thomas. However, the proof is very long structural analysis and definitely does not make the PGT obsolete. An introduction to the subject is Golumbic: Algorithmic Graph Theory and Perfect graphs (1980).

7.3 Constructive super-polynomial lower bound for Ramsey numbers.

The idea of the cubic lower bound of $R(n)$ (Proposition 5.1) can be generalized as follows. The next theorem is a variant of a result of Ray-Chaudhuri and Wilson (1969, published proof in 1975). The theorem is from Deza,Frankl and Singhi (1983). The simple proof is from Alon, Babai and Suzuki (1991).

Theorem 7.7 *Assume that p is a prime, L is a set of s integers, and we have a hypergraph with n vertices and with m edges $e_1, e_2, \ldots, e_m$ satisfying the following conditions: for each i, $|e_i| \notin L \pmod p$; for each $i, j, i \neq j$, $|e_i \cap e_j| \in L \pmod p$. Then*

$$m \leq \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{s}.$$

Proof. The claimed upper bound is the dimension of polynomials in n variables with degree at most s which are linear in all variables (products of at most s distinct variables form a basis). The proof idea is to define m linearly independent polynomials of that type. Apart from a tiny technical problem, the following definition works:

$$f_i(x) := \prod_{l \in L} (xe_i - l)$$

where e_i is the $0-1$ incidence vector of the edge e_i, xe_i is the inner product $\pmod p$. Linear independence follows by substituting e_j to f_i and applying the diagonal criterion (exercise 5.6). The tiny problem is that to get linear polynomials, the domain of f_i must be restricted to $\{0,1\}^n$. □

Theorem 17 gives a super-polynomial lower bound for the Ramsey number as follows. Let p be a prime and $n > 2p^2$. Associate the vertices of the complete graph K with the

p^2-1-element subsets of an n-element set. Color an edge of K red if the corresponding subsets has intersection size not congruent to $-1 \pmod p$ otherwise color it blue. Apply Theorem 17 with $L=\{0,1,2,\ldots,p-2\}$ for the red color. Then apply Theorem 17 with a prime $q>p^2-1$ and with $L=\{p-1,2p-1,\ldots,p^2-p-1\}$ for the blue color.

7.4 Hypergraphs in geometry: the fall of Borsuk conjecture.

For any prime p define the graph G_p as follows. The vertices of G_p are the $2p-1$-element subsets of a $4p-1$-element set. Two vertices of G are adjacent if and only if the corresponding subsets intersect in precisely $p-1$ elements.

Lemma 7.8 *For the graph G_p the following holds:*

$$\alpha(G_p) < 1.7548^{4p-1}, \chi(G_p) > 1.1397^{4p-1}$$

Proof. Theorem 7.7 is applied with $L=\{0,1,\ldots,p-2\}$. If S is a set of independent vertices in G_p then from Theorem 7.7 we get

$$|S| \le \sum_{i=0}^{p-1} \binom{4p-1}{i} \le 2\binom{4p-1}{p-1} \le 1.7548^{4p-1}$$

(the last step uses Sterling formula). The inequality for the chromatic number follows from $\chi(G_p) \ge \frac{|V(G_p)|}{\alpha(G_p)} \ge \frac{2^{4p-1}}{1.7548^{p-1}}$. □

Borsuk conjectured (in 1933) that every set of R^d with diameter one can be partitioned into $d+1$ sets of diameter less than one. The conjecture was proved for $d\le 3$ and for all d if the set is special (centrally symmetric, have smooth boundary). Let $f(d)$ be the minimum for which every set of diameter one can be partitioned into $f(d)$ sets of diameter smaller than one. Borsuk's conjecture *was* $f(d)\le d+1$. This fails badly as proved by Kahn and Kalai in 1992:

Theorem 7.9 $f(d) > 1.1397^{\sqrt{2d}}$

Proof. Let p be a prime, $d=\binom{4p-1}{2}$. Let H be a set of $4p-1$ elements. The hypergraph $KK(d)$ is defined as follows.

The vertex set of $KK(d)$ is formed by the d pairs of elements of H. Formally

$$V(KK) = \{a_{xy} : x, y \in H\}$$

The edges e_Z of KK are associated with $2p-1$-element subsets Z of H as follows:

$$e_Z = \{a_{xy} : x \in Z, y \notin Z\}$$

thus e_Z is defined by the pairs of H split by Z.

Notice that $KK(d)$ has d vertices, $\binom{4p-1}{2p-1}$ edges and $(2p-1)2p$-uniform. Represent $KK(d)$ in R^d with the rows of its incidence matrix. Observe that the distance of the points representing e_{Z_1} and e_{Z_2} is

$$\sqrt{2((2p-1)2p - |e_{Z_1} \cap e_{Z_2}|)}$$

therefore the maximum distance, i.e. the diameter of $KK(d)$ in R^d is realized if and only if

$$|e_{Z_1} \cap e_{Z_2}| = min\{|e_Z \cap e_{Z'|} : Z, Z' \subseteq H\} = \lambda$$

Claim : $|e_{Z_1} \cap e_{Z_2}| = \lambda$ if and only if $|Z_1 \cap Z_2| = p-1$.

The claim follows from observing that $|e_{Z_1} \cap e_{Z_2}|$ is the number of pairs of H split by both Z_1 and Z_2. Thus, with $r = |Z_1 \cap Z_2|$,

$$\lambda = min\{r(r+1) + (2p-1-r)^2 : 0 \leq r \leq 2p-1\}$$

and it is easy to check that $r = p-1$ minimizes the expression.

Using the claim, the partition of $KK(d)$ in R^d into sets of smaller diameter is equivalent with partitioning the $2p-1$-element sets Z in H into classes so that no class has two sets intersecting in $p-1$ elements. This is at least $\chi(G_p)$ so we get

$$f(d) \geq \chi(G_p) > 1.1397^{4p-1} > 1.1379^{\sqrt{2d}}.$$

□