Math 217 – Final Exam

Winter 2013

Time: 120 mins.

1. Answer all questions in the spaces provided.
2. Remember to show all work.
3. Justify all answers.
4. If you run out of room for an answer, continue on the back of the page.
5. No calculators, notes, or other outside assistance allowed.

Name: ______________________ Section: __________

Question	Points	Score
1	12	
2	12	
3	10	
4	15	
5	12	
6	9	
Total:	70	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (3 points) The $n \times n$ matrix A is *diagonalizable*.

   (b) (3 points) An *eigenvector* of a square matrix A.

   (c) (3 points) A *basis* for a subspace V of $\mathbb{R}^n$.

   (d) (3 points) Let $g : X \to Y$ be a function between sets X and Y. The function g is *one-to-one*.

2. Consider the following matrix A and its row reduced echelon form

$$A = \begin{bmatrix} 3 & 9 & 1 & -2 & 3 & 4 \\ -1 & -3 & -1 & 6 & 3 & 1 \\ 2 & 6 & 1 & -4 & 4 & 1 \\ 0 & 0 & -1 & 8 & 3 & 1 \\ 1 & 3 & 1 & -6 & 0 & 1 \end{bmatrix} \qquad \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(You do not need to check that the row reduction is correct)

(a) (1 point) Find the rank of A.

(b) (2 points) Find a basis for $\mathrm{Col}(A)$.

(c) (3 points) Find a basis for $\mathrm{Nul}(A)$.

(d) (3 points) Find a basis for $(\mathrm{Nul}(A))^\perp$.

(e) (3 points) Given that $A\mathbf{x}_0 = \mathbf{b}$, where

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 9 \\ -3 \\ 6 \\ 0 \\ 3 \end{bmatrix},$$

describe the solution set of the system $A\mathbf{x} = \mathbf{b}$.

3. Consider the matrix $A = \begin{bmatrix} 1 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & -1 \end{bmatrix}$.

   (a) (6 points) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   (b) (4 points) Compute A^{218}.

4. Determine whether the following statements are true or false, and give justification.

   (a) (3 points) Suppose that a matrix A has reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   Then the column space of A has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (b) (3 points) A matrix with real entries has a real eigenvalue.

   (c) (3 points) Let A and B be square matrices of the same size. Then A and B are similar if and only if they are row equivalent.

   (d) (3 points) If an $n \times n$ matrix A has distinct eigenvalues, then $\operatorname{rank}(A) \geq n - 1$.

   (e) (3 points) There exists a unique 4×4 matrix A having a 4-dimensional eigenspace associated to the eigenvalue $\lambda = 5$.

5. Determine whether the following statements are true or false, and give justification or a counterexample.
   (a) (3 points) If $S = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\} \subset \mathbb{R}^n$ is an orthogonal set, then the matrix $A = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_n \end{bmatrix}$ is invertible.
   (b) (3 points) Suppose A is an $n \times n$ matrix and $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$. If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A corresponding to distinct eigenvalues then $\mathbf{v} \cdot \mathbf{w} = 0$.
   (c) (3 points) If U and V are subspaces of $\mathbb{R}^n$ and $U \subseteq V$ then $V^\perp \subseteq U^\perp$.
   (d) (3 points) If $T = \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$ is a linearly independent set, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\} \subset \mathbb{R}^n$ is also linearly independent.

6. Let A be a square matrix.

   (a) (5 points) Suppose A is diagonalizable and has characteristic polynomial

   $$p(x) = (x-1)(x-2)^3(x-3)^6.$$

   Determine $\operatorname{rank}(A - kI)$ for every possible value of the scalar k.

   (b) (4 points) Suppose A^2 has characteristic polynomial $p(x) = (x^2 - 10x + 9)^2$ and that A has 4 distinct eigenvalues. Determine the characteristic polynomial of A.