Problem Set 9

Due: Tuesday, November 22

Problem 1. Two graphs are *co-spectral* if they have the same spectrum (i.e. their adjacency matrices have the same eigenvalues with the same multiplicities). For a tree, the coefficient of λ^{n-2k} in the characteristic polynomial is $(-1)^k \mu_k(G)$, where $\mu_k(G)$ is the number of matchings of size k (you do not need to prove this). Use this to construct a pair of nonisomorphic co-spectral 8-vertex trees, both with characteristic polynomial $\lambda^8 - 7\lambda^6 + 9\lambda^4$. (Comment: as $n \to \infty$, almost no trees are uniquely determined by their spectra.)

Problem 2. The *odd girth* of a graph G is the length of a shortest odd cycle of G; a graph with no odd cycle has infinite odd girth. Show that co-spectral graphs have the same odd girth.

Problem 3. Determine the number of spanning trees in $K_{m,m}$.

Problem 4. Prove that G is bipartite if G is connected and $\lambda_{\max}(G) = -\lambda_{\min}(G)$.

Problem 5. Light bulbs $l_1, \ldots, l_n$ are controlled by switches $s_1, \ldots, s_n$. The ith switch changes the on/off status of the ith light and possibly others, but s_i changes the the status of l_j if and only if s_j changes the status of l_i. Initially all the lights are off. Prove that it is possible to turn all the lights on. (Hint: this uses vector spaces, not eigenvalues.)