Problem Set 8

Due: Tuesday, March 3 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. In the problems below, G denotes a finite group. For problems involving decomposing representations into irreducibles, it may be helpful to use the character tables in 19.1. Turn in Problems 1–10.

Problem 1. Let $\phi : Q_8 \to GL_4(\mathbb{C})$ be the representation determined by

$$i \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Determine the decomposition of ϕ into irreducible representations.

Problem 2. Let $\chi : G \to \mathbb{C}$ be a character. Show that $\chi(g) = \overline{\chi(g^{-1})}$ for every $g \in G$.

Problem 3. Let $\psi : G \to \mathbb{C}$ be the character of any 2-dimensional representation of a group G and let x be an element of order 2 in G. Prove that $\psi(x) = 2, 0$, or -2. Generalize this to n-dimensional representations.

Problem 4. Let $\chi : G \to \mathbb{C}$ be an irreducible character of G. Prove that for every element z in the center of G we have $\chi(z) = \epsilon\chi(1)$, where ϵ is some root of 1 in $\mathbb{C}$.

Problem 5. Let $\phi : G \to GL(V)$ be a representation and let $\chi : G \to \mathbb{C}^\times$ be a degree 1 representation. Prove that $\chi\phi : G \to GL(V)$ defined by $\chi\phi(g) = \chi(g)\phi(g)$ is a representation (note that multiplication of the linear transformation $\phi(g)$ by the complex number $\chi(G)$ is well defined). Show that $\chi\phi$ is irreducible if and only if ϕ is irreducible. Show that if ψ is the character afforded by ϕ then $\chi\psi$ is the character afforded by $\chi\phi$. Deduce that the product of any irreducible character with a character of degree 1 is also an irreducible character.

Problem 6. The action of $\mathcal{S}_4$ on $\{1, 2, 3, 4\}$ induces an action of $\mathcal{S}_4$ on subsets of $\{1, 2, 3, 4\}$ of size 2. For instance, if π is the cycle (132), then

$$\pi(\{1,2\}) = \{3,1\}, \pi(\{1,3\}) = \{3,2\}, \pi(\{1,4\}) = \{3,4\}, \pi(\{2,3\}) = \{1,2\}, \pi(\{2,4\}) = \{1,4\}, \pi(\{3,4\}) = \{2,4\}.$$

Let $M_{2,2}$ denote the $\mathbb{C}\mathcal{S}_4$-module corresponding to this action. Determine the decomposition of $M_{2,2}$ into irreducibles.

Problem 7. The group $\mathcal{S}_3$ acts on itself by conjugation. Formally, the action $\sigma : \mathcal{S}_3 \times \mathcal{S}_3 \to \mathcal{S}_3$ is given by $\sigma(g, x) = g^{-1}xg$. Let W be the corresponding $\mathbb{C}\mathcal{S}_3$-module. Determine the decomposition of W into irreducibles.

Problem 8. Determine the character table of D_{4n} (assume the field is $\mathbb{C}$).

Problem 9. Show that the character table (over $\mathbb{C}$) is an invertible matrix (you can use the fact that it is a square matrix). Use this to prove the *second orthogonality relations*: Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G. For any $x, y \in G$,

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

Here $C_G(x)$ denotes the centralizer of x in G.

Problem 10. Recall that for FG-modules V and W, $\mathrm{Hom}_F(V, W)$ is an FG-module via $(g \cdot \phi)(v) = g\phi(g^{-1}v)$ for every $\phi \in \mathrm{Hom}_F(V, W)$, $g \in G$, $v \in V$. Show that the character $\chi_{\mathrm{Hom}_F(V,W)}$ is given by

$$\chi_{\mathrm{Hom}_F(V,W)}(g) = \chi_V(g^{-1})\chi_W(g) \quad \text{for every } g \in G.$$

Hint: One possible route to proving this is to choose bases for V and W (say V has dimension n and W has dimension m); then $\mathrm{Hom}_F(V, W)$ can be identified with $m \times n$-matrices and has a basis $\{E_{ij}\}$, where E_{ij} denotes the matrix with 1 in the i, j-th spot and 0 elsewhere. Then the matrix corresponding to the action of g on $\mathrm{Hom}_F(V, W)$ is an $mn \times mn$-matrix that can be computed explicitly in terms of matrices corresponding to the action of g^{-1} on V, and the action of g on W.