Problem Set 8

Due: Tuesday, November 15

Problem 1. Let $A(x) = a_0+a_1x+a_2x^2+\cdots$ be the generating function for a sequence $a_0, a_1, a_2, \ldots$. Express in terms of A the generating functions for the following sequences:

a) $a_0 + a_1, a_1 + a_2, a_2 + a_3, \ldots$;

b) $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$;

c) $a_0, a_1b, a_2b^2, a_3b^3, \ldots,$ b is a constant;

d) $a_0, 0, a_2, 0, a_4, 0, a_60, a_8, 0, \ldots$

Problem 2. Solve the recurrence:

$$g_0 = 1,$$
$$g_n = g_{n-1} + 2g_{n-2} + \cdots + ng_0.$$

Problem 3. The adjacency matrix of the path with n vertices looks like

$$A_n := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Determine the eigenvalues of A_n.

Problem 4. Let M_n be the number of paths from $(0, 0)$ to $(n, 0)$ in an $n \times n$ grid using only steps $U = (1, 1)$, $F = (1, 0)$ and $D = (1, -1)$ (This is the same as the Catalan numbers except we allow a horizontal step F in addition to U and D). The beginning of the sequence is: $M_0 = 1, M_1 = 1, M_2 = 2, M_3 = 4, M_4 = 9$. Compute some further terms of this sequence. Find a recurrence relation and the generating function $M(x) := \sum_{n\geq 0} M_n x^n$ for this sequence.

Problem 5. The Chebyshev polynomial T_n is defined by the equality

$$\cos n\varphi = T_n(\cos\varphi).$$

Here are the first few Chebyshev polynomials:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Prove that

$$\sum_{n\geq 0} T_n(x)t^n = \frac{1 - tx}{1 - 2tx + t^2}.$$