Problem Set 7

Due: Tuesday, February 24 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. We define the *quaternion group* Q_8 by generators and relations:

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1,\ i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

There is a faithful representation $\phi : Q_8 \to GL_2(\mathbb{C})$ defined by

$$\phi(i) = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} \text{ and } \phi(j) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Write out the matrices $\phi(g)$ for every $g \in Q_8$ for this representation.

Problem 2. Let R be a ring. Let M, N be R-modules and $S \subseteq M$ a submodule of M. Let $\pi : M \to M/S$ be the natural projection. Let $\Theta : \mathrm{Hom}_R(M/S, N) \to \mathrm{Hom}_R(M, N)$ given by $\phi \mapsto \phi \circ \pi$. Show that Θ is injective and that the image of Θ consists of those $\alpha \in \mathrm{Hom}_R(M, N)$ such that $S \subseteq \ker(\alpha)$. This shows that "giving a map from M/S to N is the same as giving a map from M to N that sends S to 0".

Problem 3. Prove that the degree 1 representations of G are in bijective correspondence with the degree 1 representations of the abelian group G/G' where $G' := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$ is the commutator subgroup of G.

Problem 4. Prove that if $|G| > 1$ then every irreducible FG-module has dimension $< |G|$.

Problem 5. Let $V = \mathbb{C}\{e_1, e_2, e_3, e_4\}$ be the 4-dimensional $\mathbb{C}\mathcal{S}_4$-module and $\phi : \mathcal{S}_4 \to GL_4(\mathbb{C})$ be the corresponding representation, where $\mathcal{S}_4$ acts on the basis $\{e_i\}$ by permuting indices. Let $\pi : D_8 \to \mathcal{S}_4$ be the group homomorphism given by viewing $\mathcal{S}_4$ as the permutations of the vertices of a square (this realizes D_8 as the subgroup of $\mathcal{S}_4$ that preserves the edges of the square). Hence $\phi \circ \pi$ is a representation of D_8 and we can consider V as the corresponding $\mathbb{C}D_8$-module. Write V as a direct sum of irreducible $\mathbb{C}D_8$-modules.

Problem 6. Let $\phi : G \to GL_n(\mathbb{C})$ be a representation of the finite group G. Show that for every $g \in G$, $\phi(g)$ is diagonalizable and its eigenvalues are roots of unity.

Problem 7. Let G be a finite group. For any automorphism $\psi : G \to G$ and representation $\phi : G \to GL(V)$, we get a new representation $\phi \circ \psi : G \to GL(V)$. However, this new representation $\phi \circ \psi$ may be equivalent to ϕ. Find an example of a group G, an automorphism ψ, and a representation ϕ such that $\phi \circ \psi$ and ϕ are not equivalent.

Problem 8. We say that $n \times n$ matrices $A_1, \ldots, A_k$ are *simultaneously diagonalizable* if there is an invertible matrix P such that $P^{-1} A_i P$ are diagonal matrices for all i. Let $\{A_1, \ldots, A_k\} \subseteq GL_n(\mathbb{C})$ be a subgroup of commuting matrices. Show that these matrices are simultaneously diagonalizable using representation theory.

Problem 9. Let $\phi : G \to GL_n(\mathbb{C})$ be an irreducible representation of the finite group G. Show that if $g \in Z(G)$, then $\phi(g) = cI_n$ for some $c \in \mathbb{C}$.

Problem 10. Determine the list of 2-dimensional irreducible representations of the dihedral group D_{4n} of order $4n$.