Problem Set 7

Due: Tuesday, November 8

Problem 1. Prove combinatorially that the number of compositions of n with odd parts is the n-th Fibonacci number F_n by finding a simple bijection between such compositions and the compositions of $n-1$ with parts 1 and 2. (For example, for $n=5$ this requires finding a bijection between $\{311, 131, 11111, 113, 5\}$ and $\{112, 121, 211, 22, 1111\}$.)

Problem 2. How many permutations $w = a_1 a_2 \dots a_n \in \mathcal{S}_n$ have the property that for all $1 \leq i < n$, the numbers appearing in w between i and $i+1$ (whether i is to the left or right of $i+1$) are all less than i? (Prove that your formula is correct.) An example of such a permutation is 976412358.

Problem 3. Using the generating function for the Fibonacci numbers, prove the following identities (recall our indexing convention $F_1 = F_2 = 1, F_3 = 2, \dots$):

a) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$;

b) $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$;

c) $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$;

d) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Problem 4. An n-th root of unity is a complex number ζ satisfying $\zeta^n = 1$, i.e. $\zeta = e^{2\pi i k/n}$ for some $k \in \mathbb{Z}$. An n-th root of unity is *primitive* if $\zeta^k \neq 1$ for all $k = 1, \dots, n-1$. Let $f_n(z)$ be the function that has as its zeros all primitive n-th roots of unity. Prove that

$$f_n(z) = \prod_{k|n} (z^k - 1)^{\mu(\frac{n}{k})}.$$

Problem 5. Let T_n be the number of triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors. Find and prove a formula for T_n.