Problem Set 6

Due: Tuesday, February 17 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. Determine the Jordan canonical form of the $n \times n$ matrix A with 1's on the diagonal and 2's on the superdiagonal.

$$A := \begin{bmatrix} 1 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Problem 2. Prove that if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the $n \times n$ matrix A then $\lambda_1^k, \dots, \lambda_n^k$ are the eigenvalues of A^k for any $k \geq 0$.

Problem 3. Prove that any matrix A is similar to its transpose A^T.

Problem 4. Prove that an $n \times n$ matrix A with entries from $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over *any* field F?

Problem 5. Prove that there are no 3×3 matrices A over $\mathbb{Q}$ with $A^8 = I$ but $A^4 \neq I$.

Problem 6. Show that the following matrices are similar in $M_p(\mathbb{F}_p)$:

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Problem 7. Determine the Jordan canonical form for the $n \times n$ matrix over $\mathbb{F}_p$ whose entries are all equal to 1 (the answer depends on whether or not p divides n).

Problem 8. Let A be an $n \times n$ matrix over a field F. For $0 \leq i \leq n$, let d_i be the g.c.d. of the determinants of all the $i \times i$ minors of $xI - A$ (take the 0×0 minor to be 1). Prove that the ith element along the diagonal of the Smith Normal Form for A is d_i/d_{i-1}. This gives the invariant factors of A. (Hint: Show that these g.c.d.s do not change under the following row and column operations: (1) Add a multiple of one row to another row, (2) Swap two rows, (3) Rescale a row by an element of F; column operations are analogous.)

Problem 9. Determine the 1-dimensional representations of the dihedral group D_{4n} of order $4n$. (This means giving a list of 1-dimensional representations, showing that each pair of these is nonequivalent, and showing that any 1-dimensional representation belongs to the list; note that for 1×1 matrices, conjugation does nothing, so showing two 1-dimensional representations are nonequivalent only requires showing they are not equal.)

Problem 10. Let Z_n be the cyclic group of order n. Use rational canonical form to determine the irreducible $\mathbb{F}_p Z_4$-modules. Here $\mathbb{F}_p$ denotes the field with p elements where p is a prime. (Hint: You can use the fact that the group $\mathbb{F}_p^\times$ is the cyclic group of order $p-1$.)