Problem Set 5

Due: Tuesday, February 10 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. Let $A_1, \dots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, \dots, n$. Prove that

$$(A_1 \oplus \cdots \oplus A_n)/(B_1 \oplus \cdots \oplus B_n) \cong (A_1/B_1) \oplus \cdots \oplus (A_n/B_n).$$

Problem 2. Prove that two 3×3 matrices over a field F are similar if and only if they have the same characteristic and same minimal polynomials. Give an explicit counterexample to this assertion for 4×4 matrices.

Problem 3. Find the rational canonical forms of the following matrices over $\mathbb{Q}$:

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 4. Find all similarity classes of 3×3 matrices A over $\mathbb{F}_2$ satisfying $A^6 = I$.

Problem 5. Determine up to similarity all 2×2 rational matrices A (i.e., $A \in M_2(\mathbb{Q})$) such that $A^4 = I$ and $A^k \neq I$ for $k < 4$. Do the same if the matrix has entries from $\mathbb{C}$.

Problem 6. Let R be any commutative ring, let V be an R-module, and let $x_1, x_2, \dots, x_n \in V$. Suppose $A \in M_n(R)$ and

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0}.$$

Prove that $(\det A)x_i = 0$ for all $i \in \{1, 2, \dots, n\}$.

Problem 7. Determine representatives for the conjugacy classes for $GL_3(\mathbb{F}_2)$.

Problem 8. Find an integral domain R and an R-module M such that M is torsion-free and M is not a free module.

Problem 9. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is torsion-free but not free. Why does this not contradict the Structure Theorem proven in class?

Problem 10. Let M be a finitely generated module over a PID R. Show that any submodule of M is finitely generated. (Do not use the Structure Theorem since we needed this to prove the Structure Theorem.)