Problem Set 4

Due: Tuesday, February 3 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. For $a, b \in R$, where R is a commutative ring and a, b are nonzero, a *least common multiple* of a and b is an element c of R such that $a \mid c$, $b \mid c$, and if ($a \mid c'$ and $b \mid c'$) then $c \mid c'$. Prove that any two nonzero elements of a PID have a least common multiple.

Problem 2. Let R be an integral domain. Prove that if the following two conditions hold then R is a PID:

(i) for any $a, b \in R$, there is a $d \in R$ such that $(a, b) = (d)$, and

(ii) if $a_1, a_2, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

Problem 3. Let F be a field and $R = F[x, y]$. Every ideal of R is finitely generated (we have not proved this, but you can use it for this problem). For a finitely generated ideal I, let $s(I)$ be the smallest possible size of a generating set of I. Determine

$$\max\{s(I) \mid I \text{ is an ideal of } R\}.$$

(Define max of a set to be ∞ if the set is unbounded from above.)

Problem 4. Let $R = \mathbb{Z}$. Use the row and column operations discussed in the proof in class to bring the matrix $A = \begin{bmatrix} 2 & 4 & 4 \\ -4 & 8 & 8 \\ 3 & 2 & 4 \end{bmatrix}$ into the form $\begin{bmatrix} a_1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & \cdots & 0 \end{bmatrix}$ such that $a_1 \mid a_2 \mid \cdots \mid a_m$ and $m \leq 3$.

Problem 5. Let R be a ring. Show that R is a field if and only if every R-module has a basis.

Problem 6. Let M be a module over the integral domain R.

(a) Let $x \in M$ be a torsion element. Show that x is linearly dependent. Conclude that the rank of $\operatorname{Tor}(M)$ is 0, so that in particular any torsion R-module has rank 0.

(b) Show that the rank of M is the same as the rank of the (torsion free) quotient $M/\operatorname{Tor} M$.

Problem 7. Let M be a module over the integral domain R. Suppose that M has rank n and that $x_1, \ldots, x_n$ is any maximal set of linearly independent elements of M. Let $N = Rx_1 + \cdots + Rx_n$ be the submodule generated by $x_1, \ldots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module.

Problem 8. Let R be a commutative ring and let M be the free R-module R^n. Show that if the elements $x_1, \ldots, x_n \in M$ are linearly independent, then they form a basis of M.

Problem 9. Let R be a commutative ring and let M be the free R-module R^n. Show that if the elements $x_1, \ldots, x_n \in M$ generate M, then they form a basis of M.

Problem 10. Let R be a commutative ring and let $b_1, \ldots, b_n$ be a basis of R^n. Let $C = [c_{ij}]$ be an $n \times n$ matrix with coefficients in R, i.e., $C \in M_n(R)$. Suppose that $\det(C)$ is a unit in R.

(a) Show that C is a unit in $M_n(R)$.

(b) For $i = 1, \ldots, n$, let $d_i = \sum_j c_{ij} b_j$. Show that the elements $d_1, \ldots, d_n$ form a basis of R^n.