Problem Set 2
Due: Tuesday, September 20

Problem 1. Let $S$ be the set $\{1, 2, \ldots, mn\}$. Partition $S$ into $m$ sets $A_1, \ldots, A_m$ of size $n$ each. Also partition $S$ into $m$ sets $B_1, \ldots, B_m$ of size $n$ each. Show that the $A_i$ can be renumbered so that $A_i \cap B_i$ is non-empty for every $i$.

Problem 2. For a list of trails $T_1, \ldots, T_m$, let $L(T_1, \ldots, T_m)$ be the number of trails $T_i$ that are not closed. Let $\tau(G)$ be the smallest value of $L(T_1, \ldots, T_m)$ over all lists of trails $T_1, \ldots, T_m$ such that their edge sets partition $E(G)$ (i.e. each edge of $G$ appears in exactly one trail). For example, if $G$ is a cycle with an extra edge, then $\tau(G) = 1$. Determine a simple expression for $\tau(G)$ in terms of the vertex degrees of $G$ (and prove that this is correct).

Problem 3. For a spanning tree $T$ in a weighted graph, let $m(T)$ denote the maximum among the weights of the edges in $T$. Let $x$ denote the minimum of $m(T)$ over all spanning trees of a weighted graph $G$. Prove that if $T$ is a spanning tree in $G$ with minimum total weight, then $m(T) = x$ (in other words, $T$ also minimizes the maximum weight). Construct an example to show that the converse is false. (Comment: A tree that minimizes the maximum weight is called a bottleneck or minimax spanning tree.)

Problem 4. Let $T, T'$ be two spanning trees of a connected graph $G$. For $e \in E(T) - E(T')$, prove that there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of $G$.

Problem 5. Let $G$ be a bipartite graph with vertex sets $V_1$ and $V_2$. Let $A$ be the set of vertices of maximal degree. Show that there is a matching in $G$ that covers $A$. 