Problem Set 2

Due: Tuesday, September 20

Problem 1. Let S be the set $\{1, 2, \ldots, mn\}$. Partition S into m sets $A_1, \ldots, A_m$ of size n each. Also partition S into m sets $B_1, \ldots, B_m$ of size n each. Show that the A_i can be renumbered so that $A_i \cap B_i$ is non-empty for every i.

Problem 2. For a list of trails $T_1, \ldots, T_m$, let $L(T_1, \ldots, T_m)$ be the number of trails T_i that are not closed. Let $\tau(G)$ be the smallest value of $L(T_1, \ldots, T_m)$ over all lists of trails $T_1, \ldots, T_m$ such that their edge sets partition $E(G)$ (i.e. each edge of G appears in exactly one trail). For example, if G is a cycle with an extra edge, then $\tau(G) = 1$. Determine a simple expression for $\tau(G)$ in terms of the vertex degrees of G (and prove that this is correct).

Problem 3. For a spanning tree T in a weighted graph, let $m(T)$ denote the maximum among the weights of the edges in T. Let x denote the minimum of $m(T)$ over all spanning trees of a weighted graph G. Prove that if T is a spanning tree in G with minimum total weight, then $m(T) = x$ (in other words, T also minimizes the maximum weight). Construct an example to show that the converse is false. (Comment: A tree that minimizes the maximum weight is called a **bottleneck** or **minimax** spanning tree.)

Problem 4. Let T, T' be two spanning trees of a connected graph G. For $e \in E(T) - E(T')$, prove that there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

Problem 5. Let G be a bipartite graph with vertex sets V_1 and V_2. Let A be the set of vertices of maximal degree. Show that there is a matching in G that covers A.