

# Problem Set 1

Due: Tuesday, January 13 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problems A–C but do not turn these in. Turn in Problems 1–10.

**Problem A.** Let  $R$  be a ring. Show that  $(-1)^2 = 1$  in  $R$ .

**Problem B.** Decide which of the following are subrings of the ring of all functions from the closed interval  $[0, 1]$  to  $\mathbb{R}$ :

- (a) the set of all functions  $f(x)$  such that  $f(q) = 0$  for all  $q \in \mathbb{Q} \cap [0, 1]$
- (b) the set of all polynomial functions
- (c) the set of all functions which only have a finite number of zeros, together with the zero function
- (d) the set of all functions which have an infinite number of zeros
- (e) the set of all functions  $f$  such that  $\lim_{x \rightarrow 1^-} f(x) = 0$ .
- (f) the set of all rational linear combinations of the functions  $\sin(nx)$  and  $\cos(mx)$ , where  $m, n \in \{0, 1, 2, \dots\}$ .

**Problem C.** Decide which of the following are ideals of the ring  $\mathbb{Z} \times \mathbb{Z}$ :

- (a)  $\{(a, a) \mid a \in \mathbb{Z}\}$
- (b)  $\{(2a, 2b) \mid a, b \in \mathbb{Z}\}$
- (c)  $\{(2a, 0) \mid a \in \mathbb{Z}\}$
- (d)  $\{(a, -a) \mid a \in \mathbb{Z}\}$ .

**Problem 1.** An element  $x$  in a ring  $R$  is called *nilpotent* if  $x^m = 0$  for some  $m \in \mathbb{Z}^+$ . Let  $x$  be a nilpotent element of the commutative ring  $R$ .

- (a) Prove that  $x$  is either zero or a zero divisor.
- (b) Prove that  $rx$  is nilpotent for all  $r \in R$ .
- (c) Prove that  $1 + x$  is a unit in  $R$ .
- (d) Deduce that the sum of a nilpotent element and a unit is a unit.

**Problem 2.** Let  $R$  be a ring with  $1 \neq 0$ . A nonzero element  $a$  is called a *left zero divisor* in  $R$  if there is a nonzero element  $x \in R$  such that  $ax = 0$ . Symmetrically,  $b \neq 0$  is a *right zero divisor* if there is a nonzero  $y \in R$  such that  $yb = 0$  (so a zero divisor is an element which is neither a left or a right zero divisor). An element  $u \in R$  has a *left inverse* in  $R$  if there is some  $s \in R$  such that  $su = 1$ . Symmetrically,  $v$  has a *right inverse* if  $vt = 1$  for some  $t \in R$ .

- (a) Prove that  $u$  is a unit if and only if it has both a right and a left inverse (i.e.  $u$  must have a two-sided inverse).
- (b) Prove that if  $u$  has a right inverse then  $u$  is not a right zero divisor.
- (c) Prove that if  $u$  has more than one right inverse then  $u$  is a left zero divisor.
- (d) Prove that if  $R$  is a finite-dimensional algebra over a field then every element that has a right inverse is a unit (i.e., has a two-sided inverse).

**Problem 3.** Let  $\mathcal{K} = \{k_1, \dots, k_m\}$  be a conjugacy class in the finite group  $G$ .

- (a) Prove that the element  $K = k_1 + \dots + k_m$  is in the center of the group ring  $RG$ .

- (b) Let  $\mathcal{K}_1, \dots, \mathcal{K}_r$  be the conjugacy classes of  $G$  and for each  $\mathcal{K}_i$  let  $K_i$  be the element of  $RG$  that is the sum of the members of  $\mathcal{K}_i$ . Prove that an element  $\alpha \in RG$  is in the center of  $RG$  if and only if  $\alpha = a_1K_1 + a_2K_2 + \dots + a_rK_r$  for some  $a_1, a_2, \dots, a_r \in R$ .

**Problem 4.** Prove that the rings  $\mathbb{Z}[x]$  and  $\mathbb{Q}[x]$  are not isomorphic.

**Problem 5.** Decide which of the following are ideals of the ring  $\mathbb{Z}[x]$  (and justify your answer):

- the set of all polynomials whose constant term is a multiple of 3
- the set of all polynomials whose coefficient of  $x^2$  is a multiple of 3
- the set of all polynomials whose constant term, coefficient of  $x$ , and coefficient of  $x^2$  are zero
- $\mathbb{Z}[x^2]$  (i.e., the polynomials in which only even powers of  $x$  appear)
- the set of polynomials whose coefficients sum to zero
- the set of polynomials  $p(x)$  such that  $p'(0) = 0$ , where  $p'(x)$  is the usual first derivative of  $p(x)$  with respect to  $x$ .

**Problem 6.** Prove that every (two-sided) ideal of  $M_n(R)$  is equal to  $M_n(J)$  for some (two-sided) ideal  $J$  of  $R$ .

**Problem 7.** Let  $I$  and  $J$  be ideals of  $R$ .

- Prove that  $I + J$  is the smallest ideal of  $R$  containing both  $I$  and  $J$ .
- Prove that  $IJ$  is an ideal contained in  $I \cap J$ .
- Give an example where  $IJ \neq I \cap J$ .
- Prove that if  $R$  is commutative and if  $I + J = R$  then  $IJ = I \cap J$ .

**Problem 8.** For the following two rings, give an example of a prime ideal that is not maximal (and prove that your answer is correct):

- $\mathbb{Z}[x]$
- $F[x, y]$  for a field  $F$ .

**Problem 9.** Let  $\mathcal{S}_3$  denote the symmetric group on three letters. Determine all two-sided ideals of  $\mathbb{C}\mathcal{S}_3$ .

**Problem 10.** Let  $R$  be the ring of all continuous functions from the closed interval  $[0, 1]$  to  $\mathbb{R}$  and for each  $c \in [0, 1]$  let  $M_c = \{f \in R \mid f(c) = 0\}$  (recall that  $M_c$  was shown to be a maximal ideal of  $R$ ).

- Prove that if  $M$  is *any* maximal ideal of  $R$  then there is a real number  $c \in [0, 1]$  such that  $M = M_c$ .
- Prove that if  $b$  and  $c$  are distinct points in  $[0, 1]$  then  $M_b \neq M_c$ .
- Prove that  $M_c$  is not equal to the principal ideal generated by  $x - c$ .
- Prove that  $M_c$  is not a finitely generated ideal.