Problem Set 11

Do not turn in

Problem 1. Let G and H be subgraphs of some graph, possibly overlapping.

a) Prove that $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G \cap H;k)}$ when $G \cap H$ is a complete graph.

b) Show that the formula may fail when $G \cap H$ is not a complete graph.

Problem 2. Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n-1$.

Problem 3. Prove that $\chi(G; x+y) = \sum_{U \subseteq V(G)} \chi(G[U]; x)\chi(G[\overline{U}]; y)$.

Problem 4. Let D be an acyclic orientation of G, and let f be a map from $V(G)$ to $[k]$. We say that (D, f) is a *compatible pair* if $u \to v$ in D implies $f(u) \le f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. Prove that $\eta(G; k) = (-1)^{n(G)}\chi(G; k)$.

Problem 5. A set $B \subseteq E(G)$ is a *bicycle* if $B = [S, \overline{S}]$ is an edge cut and the subgraph corresponding to B has even vertex degrees. Prove that a connected graph G has no bicycles if and only if it has an odd number of spanning trees.