Problem Set 10

Optional problems, do not turn in.

Problem 1. Prove that if M is a cyclic R-module then $T(M) = S(M)$, i.e., the tensor algebra $T(M)$ is commutative.

Problem 2. Let F be a field in which $-1 \neq 1$ and let V be a vector space over F. Prove that $V \otimes_F V \cong S^2(V) \oplus \bigwedge^2(V)$.

Problem 3. Let G be a finite group and let V be a finite-dimensional $\mathbb{C}G$-module. Recall from the previous problem set that $V \otimes_{\mathbb{C}} V$ is a $\mathbb{C}G$-module.

(a) Show that $S^2(V)$ and $\bigwedge^2(V)$ are $\mathbb{C}G$-submodules of $V \otimes_{\mathbb{C}} V$.

(b) Prove that $\chi_{S^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$ for all $g \in G$.

(c) Prove that $\chi_{\bigwedge^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$ for all $g \in G$.

Problem 4. Let V be the 4-dimensional defining representation of $\mathcal{S}_4$ over $\mathbb{C}$. Determine the decomposition of $\bigwedge^k(V)$ into irreducibles for all k.

Problem 5. Let F be a field and let U, V, W be finite-dimensional F-vector spaces. Let $\phi : U \to V$ and $\psi : V \to W$ be linear maps. Recall that $T(\phi)$ and $\bigwedge(\phi)$ denote the induced maps $T(\phi) : T(U) \to T(V)$ and $\bigwedge(\phi) : \bigwedge(U) \to \bigwedge(V)$.

(a) Prove that $T(\psi) \circ T(\phi) = T(\psi \circ \phi)$.

(b) Prove that $\bigwedge(\psi) \circ \bigwedge(\phi) = \bigwedge(\psi \circ \phi)$.

(c) Deduce that if A and B are $n \times n$ matrices then $\det(A)\det(B) = \det(AB)$.