Problem Set 10
Due: Tuesday, December 6

Problem 1. Prove that a \(d\)-regular simple graph \(G\) has a decomposition into copies of \(K_{1,d}\) if and only if it is bipartite. (A decomposition means that the edge sets of the copies of the \(K_{1,d}\) form a partition of \(E(G)\).)

Problem 2. A one-distance set of \(\mathbb{R}^n\) is a set of points in \(\mathbb{R}^n\) such that the distance between any two of them is the same. Prove that a one-distance set of \(\mathbb{R}^n\) has at most \(n + 1\) points.

Problem 3. Determine the number of spanning trees of the \(k\)-cube \(B_k\). The \(k\)-cube is a graph with \(2^k\) vertices and can be defined as the \(k\)-fold cartesian product of the edge \(B_1 = K_2\). The cartesian product of graphs \(G\) and \(H\), written \(G \Box H\), is the graph with vertex set \(V(G) \times V(H)\) specified by putting \((u,v)\) adjacent to \((u',v')\) if and only if (1) \(u = u'\) and \(vv' \in E(H)\), or (2) \(v = v'\) and \(uu' \in E(G)\).

Problem 4. Let \(Q\) be an \(n \times n\) matrix such that the sum of the entries in any row is 0 and the sum of the entries in any column is zero. Let \(\mu_1, \ldots, \mu_n\) be the eigenvalues of \(Q\), labeled so that \(\mu_n = 0\). Show that any principal cofactor of \(Q\) is equal to \(\frac{1}{n} \mu_1 \cdots \mu_{n-1}\).

Problem 5. Let \(T\) be a tree. Prove that the independence number \(\alpha(T)\) is the number of nonnegative eigenvalues of \(T\). (Hint: See Theorem 8.6.20 in West.)