Problem Set 10

Due: Tuesday, December 6

Problem 1. Prove that a d-regular simple graph G has a decomposition into copies of $K_{1,d}$ if and only if it is bipartite. (A decomposition means that the edge sets of the copies of the $K_{1,d}$ form a partition of $E(G)$.)

Problem 2. A *one-distance set of* $\mathbb{R}^n$ is a set of points in $\mathbb{R}^n$ such that the distance between any two of them is the same. Prove that a one-distance set of $\mathbb{R}^n$ has at most $n + 1$ points.

Problem 3. Determine the number of spanning trees of the k-cube B_k. The k-cube is a graph with 2^k vertices and can be defined as the k-fold cartesian product of the edge $B_1 = K_2$. The *cartesian product* of graphs G and H, written $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

Problem 4. Let Q be an $n \times n$ matrix such that the sum of the entries in any row is 0 and the sum of the entries in any column is zero. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of Q, labeled so that $\mu_n = 0$. Show that any principal cofactor of Q is equal to $\frac{1}{n}\mu_1 \cdots \mu_{n-1}$.

Problem 5. Let T be a tree. Prove that the independence number $\alpha(T)$ is the number of nonnegative eigenvalues of T. (Hint: See Theorem 8.6.20 in West.)