COMBINATORICS OF MULTIGRADED POINCARÉ SERIES FOR MONOMIAL RINGS

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Abstract. Backelin proved that the multigraded Poincaré series for resolving a residue field over a polynomial ring modulo a monomial ideal is a rational function. The numerator is simple, but until the recent work of Berglund there was no combinatorial formula for the denominator. Berglund's formula gives the denominator in terms of ranks of reduced homology groups of lower intervals in a certain lattice. We now express this lattice as the intersection lattice $L_{\mathcal{A}(I)}$ of a subspace arrangement $\mathcal{A}(I)$, use Crapo's Closure Lemma to drastically simplify the denominator in some cases (such as monomial ideals generated in degree two), and relate Golodness to the Cohen-Macaulay property for associated posets. In addition, we introduce a new class of finite lattices called *complete lattices*, prove that all geometric lattices are complete and provide a simple criterion for Golodness of monomial ideals whose lcm-lattices are complete.

1. introduction

This paper uses topological combinatorics of posets to study Poincaré series for free resolutions of a residue field k over a polynomial ring $k[x_1, \ldots, x_n]/I$, where I is a monomial ideal. Backelin showed in [Ba] that the Poincaré series for such a resolution, i.e. the generating function for its multigraded Betti numbers, is a rational function. Recently, Berglund [Be] expressed the denominator of this rational function as a sum of polynomials with coefficients that are ranks of reduced homology groups of lower intervals in a certain lattice denoted K_I. Our work may be viewed as a follow-up to [Be]. We now show how to interpret this lattice as the intersection lattice $L_{\mathcal{A}(I)}$ of a diagonal subspace arrangement $\mathcal{A}(I)$ (also known as a hypergraph arrangement). In many cases, the topological structure of this intersection lattice has been studied before, allowing translation of results from combinatorics. We also provide general results for simplifying the denominator in further cases.

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Section 2 briefly provides background on free resolutions and tools from topological combinatorics. In Section 3, we prove that $K_I \cong L_{\mathcal{A}(I)}$; we also introduce a poset denoted M_I which equals $L_{\mathcal{A}(I)}$ in many important cases, which always contains $L_{\mathcal{A}(I)}$, and is quite useful in proofs. In Section 4, we provide a closure map f on M_I whose image is the lcm-lattice L_I of I. It is well-known that the rank of the $(i-2)$-nd reduced homology group for the lower interval $(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}})$ in L_I is the multigraded Betti number $\beta_{i,\mathbf{S}}$ for the ideal I; we observe that the lower intervals $(\hat{0}, u)$ for $u \in f^{-1}(\mathbf{x}^{\mathbf{S}})$ collectively determine the multigraded Betti number $\beta_{i,\mathbf{S}}$ for the minimal free resolution of the residue field k over the polynomial ring $k[x_1, \ldots, x_n]/I$. The closure map f thereby gives a combinatorial explanation for the relationship between the Poincaré series given by these two types of Betti numbers for a monomial ideal I.

When $L_{\mathcal{A}(I)}$ is a Cohen-Macaulay poset equalling M_I, we express the multigraded Betti number $\beta_{i,\boldsymbol{S}}$ for resolving k over $k[x_1, \ldots, x_n]/I$ as a single quantity, namely as $\mu_{L_I}(\hat{0}, \mathbf{x}^{\mathbf{S}})$ when i is top dimensional and as 0 otherwise (see Section 4). This dramatically simplifies the Poincaré series denominator in this case. When L_I has the added property of being Cohen-Macaulay, then we show that $k[x_1, \ldots, x_n]/I$ is Golod. The key observation is that Golodness is equivalent to the map f satisfying a refined version of Crapo's Closure Lemma. This viewpoint yields results regarding multigraded Betti numbers and Golodness for several classes of monomial ideals, including monomial ideals generated in degree two (see Section 5) and monomial ideals generated by the bases of a matroid (see Section 6). Finally, in Section 9 we introduce a class of lattices, called *complete* lattices, for which the Golod property is trivial to decide. We prove that this class is closed under taking direct products and contains all geometric lattices.

Any result in this paper which includes $L_{\mathcal{A}(I)} = M_I$ as a hypothesis can be reformulated to give a related result without this hypothesis, typically involving degree shifts; however, the statements become much more cumbersome, while proofs remain virtually unchanged, so all such variations are left to the interested reader.

2. Preliminaries

Throughout this paper k is a field. Let R be a commutative, finitely generated k-algebra. A free resolution of an R-module M is a complex of free R-modules

$$\ldots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

that is exact everywhere. We consider modules with an $\mathbb{N}^n$-grading. For the cases that interest us, the F_i have finite rank and we can write each F_i as $R^l = \bigoplus_{j=1}^{l} R(-\boldsymbol{\alpha}_j)$ (for some l, $\boldsymbol{\alpha}_j$ depending on i), where $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l \in \mathbb{N}^n$ and $R(-\boldsymbol{\alpha})$ is the free module of rank 1 generated in multidegree $\boldsymbol{\alpha}$. If the free resolution is minimal, $F_i = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} R(-\boldsymbol{\alpha})^{\beta^R_{i,\boldsymbol{\alpha}}(M)}$ and $\beta^R_{i,\boldsymbol{\alpha}}(M)$ is called the i^{th}*-multigraded Betti number* of M in degree $\boldsymbol{\alpha}$ (over R). One way to compute multigraded Betti numbers is via the relationship $\beta^R_{i,\boldsymbol{\alpha}}(M) = \dim_k \operatorname{Tor}^R_i(k, M)_{\boldsymbol{\alpha}}$ (see e.g. [MS]).

Let I be a monomial ideal in $P = k[\mathbf{x}] = k[x_1, \ldots, x_n]$. If $M = P/I$ and $R = k[\mathbf{x}]$, an explicit minimal free resolution is known for many nice classes of monomial ideals, for instance, generic ideals, Borel-fixed ideals, and ideals in 3 variables (see e.g. [MS]). We focus on the case $R = P/I$ where the (typically infinite) free resolutions are much less well understood. This paper considers such resolutions for M equal to the residue field k, a situation which already captures quite a bit of the structure regarding resolutions of more general finitely generated R-modules over the ring $R = P/I$. See for instance [AP] or [Av] for results in this direction.

The multigraded Poincaré series of k over R, denoted $\mathrm{P}_R(\mathbf{x}, z)$, is the formal power series

$$\sum_{i\geq 0,\ \boldsymbol{\alpha}\in\mathbb{N}^n} \dim_k \operatorname{Tor}^R_i(k,k)_{\boldsymbol{\alpha}}\ \boldsymbol{x}^{\boldsymbol{\alpha}} z^i$$

recording multigraded Betti numbers. An $\mathbb{N}^n$-graded algebra R always satisfies the so-called "Serre bound," namely the coefficientwise Poincaré series bound

$$\mathrm{P}_R(\mathbf{x}, z) \leq \frac{\prod_{i=1}^{n}(1 + x_i z)}{1 - z\sum_{i\geq 0,\boldsymbol{\alpha}\in\mathbb{N}^n} \beta^P_{i,\boldsymbol{\alpha}}(R)\boldsymbol{x}^{\boldsymbol{\alpha}} z^i}.$$

R is *Golod* when this bound is sharp. When $R = P/I$ for I a monomial ideal, this equality is achieved if and only if the posets L_I and $L_{\mathcal{A}(I)}$ to be introduced shortly have the same ranks of reduced homology groups on corresponding collections of lower intervals.

2.1. **The lcm-lattice and resolutions of monomial ideals.** The *lcm-lattice* of a monomial ideal I with minimal set of generators M is the set $L_I = \{m_S \mid S \subseteq M\}$ of all least common multiples $m_S := \operatorname{lcm}_{s\in S}(s)$ of subsets $S \subseteq M$, partially ordered by divisibility. It is a finite atomic lattice with the generators of I as atoms and the least common multiple operation as join. The following result is proven in [GPW] and also may be derived from results in [Ho].

Theorem 2.1. *For $i \geq 1$ and $m \in L_I$, the multigraded Betti numbers of P/I over P are determined by the simplicial homology of intervals in L_I as follows:*

$$\beta^P_{i,\boldsymbol{m}}(P/I) = \dim \tilde{H}_{i-2}((\hat{0}, m)_{L_I}).$$

From this, it follows that the coarsely graded Betti numbers are:

$$\beta^P_i(P/I) = \sum_{m \in L_I \setminus \hat{0}} \dim \tilde{H}_{i-2}((\hat{0}, m)_{L_I}).$$

2.2. **Poincaré series for free resolutions over monomial rings.** Backelin proved in [Ba] that the multigraded Poincaré series for resolving k over a polynomial ring modulo a monomial ideal, is given by

$$\mathrm{P}_R(\mathbf{x}, z) = \frac{\prod_{i=1}^n (1 + x_i z)}{b_R(\mathbf{x}, z)}$$

for some polynomial $b_R(\mathbf{x}, z) \in \mathbb{Z}[\mathbf{x}, z]$. In [CR], Charalambous and Reeves gave a formula for $b_R(\mathbf{x}, z)$ in the case when the Taylor resolution (cf., [Ta]) is minimal. In [Av2], Avramov proved that after fixing the characteristic of the ground field, the polynomial $b_R(1, z)$ depends only on the lcm-lattice and gcd-graph of the minimal set of generators. Recently, Berglund provided a formula for $b_R(\mathbf{x}, z)$, which we state next after introducing the necessary notation.

For S a finite set of monomials, let $G(S)$ be a graph whose vertices are the elements of S and whose edges are pairs of monomials having a non-trivial common factor. Call this the *gcd graph* of S. Let $c(S)$ denote the number of connected components in $G(S)$. Let m_S denote the least common multiple of all monomials in S. If S is a subset of a monomial set M with $S = S_1 \cup \ldots \cup S_r$ its decomposition into connected components, then the *saturation* of S in M is the set $\overline{S} = \overline{S_1} \cup \ldots \cup \overline{S_r}$, where $\overline{S_i} = \{m \in M \mid m \text{ divides } m_{S_i}\}$ Clearly $S \subseteq \overline{S}$, and S is called *saturated in M* if equality holds.

Define K_M to be the set of saturated subsets of M, partially ordered by containment. It is a lattice with $S \wedge S' := S \cap S'$ and $S \vee S' := \overline{S \cup S'}$. If I is a monomial ideal with minimal set of generators M, then define K_I to be K_M. Let $\hat{K}_I = K_I \setminus \{\emptyset\}$. The gcd-graph of I is by definition the gcd-graph of M.

If Q is a poset and $x, y \in Q$, then (x, y) denotes the open interval $\{z \in Q \mid x < z < y\}$. If Q is finite, then $\Delta(Q)$ denotes the simplicial complex of chains in Q, refered to as the *order complex* of Q. We write $\Delta_Q(x, y)$ for the order complex of the interval (x, y) in the poset Q, if Q is not clear from context. By definition the i-th reduced homology of Q with coefficients in k is given by $\widetilde{\mathrm{H}}_i(Q; k) = \widetilde{\mathrm{H}}_i(\Delta(Q); k)$. We will work over the field k throughout and from now on we omit the dependence on k. Denote by $\widetilde{\mathrm{H}}(Q)(z)$ the generating function $\sum_{i\geq -1}\dim_k \widetilde{\mathrm{H}}_i(Q)z^i$. The following is proven in [Be].

Theorem 2.2. *Let k be any field. Let I be an ideal in $P = k[x_1, \ldots, x_n]$ generated by monomials of degree at least 2. The denominator of the Poincaré series of $R = P/I$ is given by*

$$b_R(\mathbf{x}, z) = 1 + \sum_{S\in \hat{K}_I} m_S(-z)^{c(S)+2}\widetilde{\mathrm{H}}((\emptyset, S))(z). \tag{1}$$

2.3. **An equivalent expression.** We will show for I generated in degree two and higher that the lattice K_I can be interpreted as the intersection lattice $L_{\mathcal{A}(I)}$ of a subspace arrangement $\mathcal{A}(I)$ obtained from I as follows.

If $I \subseteq k[x_1, \ldots, x_n]$ is squarefree with minimal set of generators M, then $\mathcal{A}(I)$ is the collection $\{U_m \mid m \in M\}$ where U_m is the subspace $\left\{\mathbf{x} \in \mathbb{R}^n \mid x_{i_1} = \cdots = x_{i_q}\right\}$ if $m = x_{i_1}\cdots x_{i_q}$.

If I is not squarefree, then let $\mathcal{A}(I) = \mathcal{A}(I^{pol})$, where I^{pol} is a squarefree monomial ideal called the *polarization* of I, obtained as follows: In each generator of I replace each power x_i^d of a variable x_i by the product of d distinct variables $x_{i,1}\cdots x_{i,d}$. Of course, I and I^{pol} live in different polynomial rings.

The intersection lattice of $\mathcal{A}(I)$ is the set $L_{\mathcal{A}(I)}$ of all possible intersections of subspaces in $\mathcal{A}(I)$ partially ordered by reverse inclusion. Each atom $a \in L_{\mathcal{A}(I)}$ corresponds to a generator of I which we denote $m(a)$. For each $u \in L_{\mathcal{A}(I)}$, let $m(u) = \mathrm{lcm}_{a\leq u}(m(a))$.

Each subspace $u \in L_{\mathcal{A}(I)}$ also has naturally associated to it a partition $\pi(u)$ of $\{1, 2, \ldots, n\}$ by saying i, j are in the same block of $\pi(u)$ for each pair i, j such that u satisfies $x_i = x_j$ for every $\mathbf{x} \in u$. We say that a set partition block is *nontrivial* if it contains at least two elements.

Proposition 3.1 will provide a poset isomorphism $L_{\mathcal{A}(I)} \cong K_I$. This will translate the formula of Theorem 2.2 into:

$$b_R(\mathbf{x}, z) = 1 + \sum_{u\in L_{\mathcal{A}(I)}\setminus\{\hat{0}\}} m(u)\widetilde{\mathrm{H}}((\hat{0}, u))(z)(-z)^{2+|B(u)|}, \tag{2}$$

where $B(u)$ denotes the set of nontrivial blocks in the set partition associated to u. Connections between the topology of $\mathcal{A}(I)$ and $\mathrm{Tor}^R(k, k)$ have been studied in [PRW].

2.4. **Combinatorial preliminaries.** All posets considered in this paper are assumed to be finite. We often will speak of a poset having a topological property, by which we mean that its order complex (as defined in Section 2.2) has this property. See [Bj2] for further background on topological combinatorics (such as the crosscut theorem and the Quillen Fibre Lemma), and see [Ox] for background in matroid theory.

The Möbius function of P may often be computed using the formula $\mu_P(u,v) = \tilde{\chi}(\Delta(u,v))$. subposet of P consisting of The next lemma will play a key role in simplifying Poincaré series denominators. Recall that a *closure map* is a poset map $f : P \to P$ which is idempotent and which satisfies $f(u) \geq u$ for all $u \in P$.

Theorem 2.3 (Crapo's Closure Lemma)**.** *Let f be a closure map on P with $f(\hat{0}) = \hat{0}$. Let $Q = im(f)$. Then*

$$\mu_Q(\hat{0},\hat{1}) = \sum_{u \in f^{-1}(\hat{1})} \mu_P(\hat{0},u).$$

A poset is *graded* if all maximal chains have the same length. A poset is *Cohen-Macaulay* over a field k if it is graded and each interval has homology concentrated in top degree, when coefficients are taken in k. Intersection lattices of central hyperplane arrangements (i.e. arrangements in which every hyperplane contains the origin) are geometric lattices, hence are Cohen-Macaulay posets (see [Bj1] or [OT]). If a graded poset is shellable (e.g. has an EL-labelling), then it is Cohen-Macaulay over all fields (see [Bj1]). If a poset is not graded, it may still be nonpure shellable (as in [BjWa]), in which case each interval has the homotopy type of a wedge of (not necessarily equidimensional) spheres, and the poset's Betti numbers may still be recovered from its Möbius function (see [Wa1]).

3. Reinterpreting K_I as an intersection lattice

This section shows for I generated in degree two and higher that K_I, as defined in Section 2, is isomorphic to the intersection lattice $L_{\mathcal{A}(I)}$ of a diagonal subspace arrangement. See Section 2 for a description of how to associate a diagonal arrangement $\mathcal{A}(I)$ to I. This viewpoint has the benefit that many such intersection lattices have already been studied.

Proposition 3.1. *The map A from $L_{\mathcal{A}(I)}$ to the set of subsets of the minimal set of generators M of I defined by*

$$A(u) = \left\{ m(a) \in M \mid \hat{0} \prec a \leq u \right\},$$

i.e. the map sending each u to the set of generators $m(a)$ for atoms $a \leq u$, has image K_I and defines an isomorphism of posets $K_I \cong L_{\mathcal{A}(I)}$.

Proof. Suppose I is squarefree. Then A clearly gives a bijection on atoms. Since $L_{\mathcal{A}(I)}$ is atomic, each u is a join of atoms, and it is easy to check that extending a set S of atoms in K_I to a saturated set is exactly equivalent to including in $A(u)$ all atoms below $u = \vee_{a\in S} a$, not just those in S. Moreover, for an atomic lattice $u \le v$ holds if and only if $A(u) \subseteq A(v)$, implying A is not only a bijection on elements, but also preserves covering relations. For arbitrary I, it is easy to see that $L_I \cong L_{I^{pol}}$ and that the ideals I, I^{pol} have identical gcd graphs, hence $K_I \cong K_{I^{pol}}$. Since by definition $L_{\mathcal{A}(I)} = L_{\mathcal{A}(I^{pol})}$, we are done. $\square$

3.1. **An alternate expression for Poincaré series denominator.** In this section we relate $L_{\mathcal{A}(I)}$ to a slightly larger poset, denoted M_I, motivated by the fact that M_I will be useful in some proofs later.

As a meet-semilattice, $L_{\mathcal{A}(I)}$ is generated by the subspaces U_m, for $m \in M$, where M is the minimal set of generators for I. In view of the fact that $U_m \cap U_n = U_{\mathrm{lcm}(m,n)}$ if m and n have a common factor, any subspace $u \in L_{\mathcal{A}(I)}$ may be brought to the form $U_{m_1} \cap \ldots \cap U_{m_r}$, where $m_1, \ldots, m_r$ are pairwise relatively prime and the interval $(\hat{0}, m_i)$ in L_I has connected gcd-graph. So we may think of $L_{\mathcal{A}(I)}$ as collections of pairwise relatively prime elements m of L_I such that the interval $(\hat{0}, m)$ has connected gcd-graph, or is empty.

Definition 3.2. *The elements of M_I are the collections of pairwise relatively prime elements of L_I, referred to as blocks. Call M_I an lcm decomposition poset. Order elements of M_I by refinement, that is, set $u \le v$ whenever each block of u divides a block of v.*

Example 3.3. If $I = \langle x_1x_2, x_3x_4, x_5x_6, x_1x_3x_5\rangle$, then $\{x_1x_2x_3x_4, x_5x_6\} \in M_I \setminus L_{\mathcal{A}(I)}$; in the case $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5\rangle$, note that $\{x_1x_2x_4x_5\} \in M_I \setminus L_{\mathcal{A}(I)}$.

The following is immediate from the definition.

Proposition 3.4. *$L_{\mathcal{A}(I)} = M_I$ if and only if all nonempty lower intervals $(\hat{0}, u)$ in L_I have connected gcd graph.*

When $L_{\mathcal{A}(I)} \ne M_I$, we still have the following relationship on homology of lower intervals.

Proposition 3.5. *There is a poset map $f : M_I \to L_{\mathcal{A}(I)}$ such that for each $u \in L_{\mathcal{A}(I)}$, the induced map $\Delta_{M_I}(\hat{0}, u) \to \Delta_{L_{\mathcal{A}(I)}}(\hat{0}, u)$ is a deformation retract, implying $\Delta_{L_{\mathcal{A}(I)}}(\hat{0}, u) \simeq \Delta_{M_I}(\hat{0}, u)$. For each $u \in M_I \setminus L_{\mathcal{A}(I)}$, $\Delta_{M_I}(\hat{0}, u)$ is collapsible.*

Proof. Let $f : M_I \to L_{\mathcal{A}(I)}$ be the map

$$f(u) = \bigvee_{\substack{a \le u \\ a \in A(L_{\mathcal{A}(I)})}} a,$$

where $A(P)$ denotes the set of atoms in P. First notice that f is a closure map on M_I^* with $im(f) = L_{\mathcal{A}(I)}$ and with $f(\hat{0}) = \hat{0}$. If $u \in L_{\mathcal{A}(I)}$ then $f(u) = u$, so the open interval $(\hat{0}, u)$ in M_I is mapped onto the open interval $(\hat{0}, u)$ in $L_{\mathcal{A}(I)}$. Thus, by the remark just after Corollary 10.12 in [Bj2], f induces a simplicial map $\Delta_{M_I}(\hat{0}, u) \to \Delta_{L_{\mathcal{A}(I)}}(\hat{0}, u)$ which is a deformation retract.

Now consider $u \in M_I \setminus L_{\mathcal{A}(I)}$, which implies $f(u) < u$. The order complex $\Delta_{M_I}(\hat{0}, u)$ is homotopy equivalent to the crosscut complex for this interval. But since the join of the set of all atoms below u is $f(u)$, the crosscut complex is a simplex, hence is collapsible. □

Corollary 3.6. *$L_{\mathcal{A}(I)}$ may be replaced by M_I in the formula for $b_R(\mathbf{x}, z)$ in Section 2.3 to obtain an alternate denominator expression.*

Some of the classes of monomial ideals and hypergraph arrangements seemingly of widest interest satisfy $L_{\mathcal{A}(I)} = M_I$, for instance, stable and squarefree stable monomial ideals, as discussed in the next section.

4. A closure map on lcm partition posets

This section gives a closure map on M_I with image L_I, enabling coefficents in Berglund's denominator formula to be expressed as Möbius functions when $L_{\mathcal{A}(I)}$ is Cohen-Macaulay and $L_{\mathcal{A}(I)} = M_I$, then provides several classes of examples. This closure map also enables substantial simplification of the formula under related weaker conditions.

Proposition 4.1. *There is a closure map f on M_I with image L_I. Hence,*

$$\sum_{u \in f^{-1}(v)} \mu_{M_I}(\hat{0}, u) = \mu_{L_I}(\hat{0}, v).$$

Proof. Let $f(u)$ consist of a single block, namely the monomial obtained by multiplying together all blocks of u. It is immediate that $f(u) \geq u$ for all u and that f is idempotent, i.e. $f^2(u) = f(u)$. Hence, we have a closure map whose image is obviously isomorphic to L_I. Crapo's Closure Lemma then immediately yields the above Möbius function formula. □

This enables a simplification of Poincaré series denominator as follows. We think of L_I as the subset of single-block partitions in M_I.

Corollary 4.2. *Suppose $L_{\mathcal{A}(I)}$ is graded with each lower interval having homology concentrated in top degree. Suppose also that $L_{\mathcal{A}(I)} = M_I$, or equivalently that all lower intervals in L_I have connected gcd graph.*

Then the coefficient of $\mathbf{x}^{\mathbf{S}}$ *in the Poincaré series denominator is*

$$(-1)^{\mathrm{rk}(\mathbf{x}^{\mathbf{S}})+1}\mu_{L_I}(\hat{0},\mathbf{x}^{\mathbf{S}})z^{\mathrm{rk}(\mathbf{x}^{\mathbf{S}})+1}.$$

Proof. Let $|B(u)|$ denote the number of blocks in an element u of M_I. Note that if $f(u)=\boldsymbol{x}^{\boldsymbol{S}}$, then $\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}})-\mathrm{rk}(u)=|B(u)|-1$, since a saturated chain from u to $\boldsymbol{x}^{\boldsymbol{S}}$ is given by merging two blocks at a time. For Q a graded poset whose lower intervals have homology concentrated in top degree,

$$\mu_Q(\hat{0},u)=(-1)^{\mathrm{rk}(u)}\dim\tilde{H}_{\mathrm{rk}(u)-2}(\Delta(\hat{0},u)).$$

Thus, Crapo's Closure Lemma yields,

$$\begin{aligned}
\mu_{L_I}(\hat{0},\boldsymbol{x}^{\boldsymbol{S}}) &= \sum_{u\in f^{-1}(\boldsymbol{x}^{\boldsymbol{S}})}\mu_{M_I}(\hat{0},u)\\
&= \sum_{u\in f^{-1}(\boldsymbol{x}^{\boldsymbol{S}})}(-1)^{\mathrm{rk}(u)-2}\dim\tilde{H}_{\mathrm{rk}(u)-2}(\Delta(\hat{0},u))\\
&= (-1)^{\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}})-1}\sum_{u\in f^{-1}(\boldsymbol{x}^{\boldsymbol{S}})}(-1)^{|B(u)|}\dim\tilde{H}_{\mathrm{rk}(u)-2}(\Delta(\hat{0},u)),
\end{aligned}$$

noting that the final expression is the coefficient for $\boldsymbol{x}^{\boldsymbol{S}}z^{1+\mathrm{rk}_{L_{\mathcal{A}(I)}}(\boldsymbol{x}^{\boldsymbol{S}})}$ in $b_R(\mathbf{x},z)$ because the gcd graph is connected. □

This will allow simplification of denominator for monomial ideals generated in degree at most two, as well as monomial ideals related to matroids. Before turning to these and other examples, we make a few more general observations.

Proposition 4.3 (Refined Closure Lemma). *If* $L_{\mathcal{A}(I)}=M_I$, *then Golodness of* I *is equivalent to the following relationship on Betti numbers of poset order complexes:*

$$\dim\tilde{H}_j((\hat{0},\boldsymbol{x}^{\boldsymbol{S}})_{L_I})=\sum_{u\in f^{-1}(\boldsymbol{x}^{\boldsymbol{S}})}(-1)^{|B(u)|+1}\dim\tilde{H}_{j-|B(u)|+1}((\hat{0},u)_{L_{\mathcal{A}(I)}})$$

where $B(u)$ *denotes the set of blocks in* u.

Proof. Recall that I is Golod if and only if the Serre bound (cf. Section 2) yields coefficientwise equality. Then the equivalence is immediate from the interpretation of LHS as the Betti number $\beta^P_{j+2,\boldsymbol{S}}(I)$ in multidegree $\boldsymbol{x}^{\boldsymbol{S}}$ for resolving the monomial ideal I over $k[\boldsymbol{x}]$ together with the interpretation of the RHS as -1 times the coefficient of $\boldsymbol{x}^{\boldsymbol{S}}z^{j+3}$ in Berglund's formula for the Poincaré series denominator. □

The above refines Crapo's Closure Lemma since the Möbius function may be viewed as an alternating sum of ranks of poset reduced homology groups.

Corollary 4.4. *If $L_{\mathcal{A}(I)}$ and L_I are both graded with all lower intervals having homology concentrated in top degree and having connected gcd graph, then I is Golod.*

PROOF. Use

$$\mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}) = (-1)^{\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}})-2} \dim \tilde{H}_{\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}})}(\Delta(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}))$$

together with the proof of Corollary 4.2. □

Next we give several classes of monomial ideals to which Proposition 4.3 applies. Recall that a monomial ideal is *stable* if for every $m \in I$ and x_j the variable of largest index dividing m, $\frac{x_i}{x_j}m \in I$ for each $i < j$. A monomial ideal I is *squarefree stable* if its generators are squarefree and for every generator m, $\frac{x_i}{x_j}m \in I$ for x_j the variable of largest index dividing m and any $i < j$ such that x_i does not divide m.

Proposition 4.5. *If a monomial ideal I is stable, square-free stable, or has associated diagonal arrangement as in Theorem 4.6, then $L_{\mathcal{A}(I)} = M_I$.*

PROOF. If I is square-free stable, then consider any pair of atoms a_i, a_j in $L_{\mathcal{A}(I)}$ not connected by an edge in the gcd graph, and suppose without loss of generality that $\max(a_i) > \max(a_j)$, by which we mean the largest variables in the nontrivial blocks of a_i and a_j; then $\max(a_i)$ may be replaced by any element of a_j to obtain an atom $a \in L_{\mathcal{A}(I)}$ connected to both in the gcd graph, such that $a \vee a_i \vee a_j = a_i \vee a_j$ in L_I. This implies that all lower intervals $(\hat{0}, \vee_{i\in S} a_i)$ for S saturated with $|S| \geq 2$ have connected gcd graph, just as is needed to apply Proposition 3.4. A very similar argument yields the result for stable ideals and for monomial ideals meeting the conditions of Theorem 4.6, using the fact that these also satisfy suitable variations on the matroid exchange axiom. □

A *hypergraph* is a set of subsets of $[n]$ such that for each pair of subsets, one is not contained in the other. A hypergraph $\mathcal{H}$ naturally corresponds to the squarefree monomial ideal with a generator $\mathbf{x}^H$ for each $H \in \mathcal{H}$; let $I_{\mathcal{H}}$ denote this monomial ideal. The diagonal subspace arrangement $\mathcal{A}(I_{\mathcal{H}})$, as defined in Section 2, is also the *hypergraph arrangement* of $\mathcal{H}$. Its intersection lattice is denoted by $\Pi_{\mathcal{H}}$.

Theorem 4.6 (Kozlov)**.** *Fix a partition $\{1, \dots, n\} = E_1 \cup \cdots \cup E_r$ such that $\max E_i < \min E_{i+1}$ for $i = 1, \dots, r-1$. Let $\mathcal{H}$ be a hypergraph $\{H_1, \dots, H_l\}$ without singletons such that the following conditions are satisfied:*

(1) *$|H_i \cap E_j| \leq 1$ for any $1 \leq i \leq l$ and $1 \leq j \leq r$;*

(2) *for any* H_i *and* $x \notin H_i$ *there exists* j *such that* $H_i \cup H_j = H_i \cup \{x\}$, *i.e.* $x \in H_j$, $H_j \subseteq H_i \cup \{x\}$;
(3) *let* $C = H_{i_1} \cup \cdots \cup H_{i_d}$, *then there exists* j *and* s *such that*

$$H_j \cap E_m = \begin{cases} \min(C \cap E_m) & \textit{if } C \cap E_m \neq \emptyset \textit{ and } 1 \leq m \leq s; \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Then $\Pi_{\mathcal{H}}$ *is EL-shellable.*

In fact, a substantial subset of the monomial ideals $I_{\mathcal{H}}$ above are squarefree stable. By a result in [HRW], this implies that these are Golod.

Proposition 4.7. *If* $\mathcal{H}$ *meets the requirements of Theorem 4.6 with* $|E_i| = 1$ *for all* i, *then* $I_{\mathcal{H}}$ *is squarefree stable.*

Since this is somewhat of a digression, and the proof is straightforward, it is left to the reader.

4.1. **Relaxing the** $L_{\mathcal{A}(I)} = M_I$ **requirement.** Let g be the map sending $u \in L_{\mathcal{A}(I)}$ to the product of the monomials corresponding to the blocks of u. Then removing the assumption $L_{\mathcal{A}(I)} = M_I$ still yields:

Proposition 4.8. *Let* f *be the map on* $L_{\mathcal{A}(I)}$ *which sends* $u \in L_{\mathcal{A}(I)}$ *to the largest element of* $\{v \in L_{\mathcal{A}(I)} \mid g(v) = g(u)\}$. *Then* f *is a closure map whose image is isomorphic to* L_I. *Moreover, if* $L_{\mathcal{A}(I)}$ *is graded with all lower intervals having homology concentrated in top degree, then*

$$b_R(\mathbf{x}, z)|_{\boldsymbol{x}^{\boldsymbol{S}} z^t} = \sum_{\substack{u \in g^{-1}(\boldsymbol{x}^{\boldsymbol{S}}) \\ |B(u)| + \mathrm{rk}(u) = t}} \mu_{L_{\mathcal{A}(I)}}(\hat{0}, u).$$

This is useful, for instance, for monomial ideals generated in degree two, since then $L_{\mathcal{A}(I)}$ is a geometric lattice, hence Cohen-Macaulay.

Remark 4.9. *It would be interesting to better understand the relationship between Golodness of* I *and nonpure shellability of* $L_{\mathcal{A}(I)}$, *since quite often when the topology of* $L_{\mathcal{A}(I)}$ *is well-understood, it is by virtue of* $L_{\mathcal{A}(I)}$ *being nonpure shellable.*

Next we give a very simple application, before turning to more substantial examples in upcoming sections.

Proposition 4.10. *For* $I = \langle x_1, \ldots, x_n \rangle$, *both* L_I *and* M_I *are graded and shellable (hence Cohen-Macaulay).*

PROOF. L_I is the Boolean algebra, hence graded and shellable. M_I is the partial partition lattice $\Pi_{\leq n}$ which was introduced and proven to be supersolvable (hence graded and shellable) in [HHS]. $\square$

Corollary 4.11. *Berglund's formula holds for all monomial ideals, not just for those whose generators all have degree at least two.*

PROOF. It is shown in [HHS] that $\mu_{\pi_{\leq n}}(\hat{0}, u) = (-1)^{\mathrm{rk}(u)}$ if all blocks of u have size one and is 0 otherwise. Thus, $b_R(\mathbf{x}, z)|_{\mathbf{x}^{\mathbf{S}}} = z^{|\boldsymbol{S}|}$, so $\mathrm{P}_R(\mathbf{x}, z) = 1$, as needed. $\square$

5. Monomial ideals generated in degree two

Much is already known about monomial ideals generated in degree two, but our viewpoint does yield some new results. This case also serves as prototype for what one may hope to glean from Berglund's formula via combinatorics. For I generated in degree two, Fröberg has constructed a minimal free resolution for k over $R = k[x_1, \ldots, x_n]/I$, where the generators in homological degree i correspond to the monomials of degree i in the Koszul dual ring $R^!$ (see [Fr]). However, a closed formula counting these generators is not provided. Our approach easily yields the following:

Proposition 5.1. *If I is a monomial ideal generated in degree two with all lower intervals having connected gcd graph, then the coefficient of $\boldsymbol{x}^{\boldsymbol{S}}$ in the Poincaré series denominator is*

$$\mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}) z^{\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}}) - |B(\boldsymbol{x}^{\boldsymbol{S}})| + 2}.$$

PROOF. Notice that $L_{\mathcal{A}(I)}$ is a geometric lattice, namely the lattice of flats for the graphic matroid given by the graph G whose edges are exactly those $e_{i,j}$ such that $x_i x_j$ is a generator of I, hence $L_{\mathcal{A}(I)}$ is Cohen-Macaulay in this case and 4.2 applies. $\square$

Comparing with Fröberg's minimal free resolution in this case yields:

Corollary 5.2.

$$Hilb(R^!, z) = \frac{\prod_{i=1}^{n}(1 + x_i z)}{1 + \sum_{\boldsymbol{x}^{\boldsymbol{S}} \in L_I \setminus \{0\}} \mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}) \boldsymbol{x}^{\boldsymbol{S}} z^{\mathrm{rk}(\boldsymbol{x}^{\boldsymbol{S}}) - |B(\boldsymbol{x}^{\boldsymbol{S}})| + 2}}$$

For geometric lattices, $|\mu(\hat{0}, u)|$ is the number of NBC bases in the matroid whose ground set is the set of atoms a satisfying $a \leq u$; the sign of $\mu(\hat{0}, u)$ is $(-1)^{\mathrm{rk}(u)}$.

Question 5.3. *Determine the Möbius functions $\mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}})$ for I generated in degree two.*

See Section 7 for the case $I = \langle x_{i_1} \cdots x_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n \rangle$ with $k \geq 2$. Many other cases should also be manageable, since there is an extensive repertoire of techniques available for calculating Möbius functions.

Theorem 5.4. *If I is a monomial ideal generated in degree two, then $L_{\mathcal{A}(I)}$ is a geometric lattice, hence Cohen-Macaulay, and L_I is graded.*

Proof. We may reduce to the squarefree case by polarization. Since $L_{\mathcal{A}(I)}$ is the intersection lattice of a central hyperplane arrangement in this case, it is geometric, hence graded and is shellable. For each $u \in L_I$, let $m_1 \cdots m_r$ be its expression as a product of monomials also in L_I, chosen with r as small as possible so that each interval $(\hat{0}, m_i)$ has connected gcd graph. Then the grading for L_I is the function g given by $g(u) = \prod_{i=1}^{r}(\deg(m_i) - 1)$. □

Remark 5.5. *L_I is not always Cohen-Macaulay, as indicated by the example $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5 \rangle$.*

Corollary 5.6. *If one further requires that each lower interval in L_I have connected gcd graph, then I is Golod if and only if each lower interval in L_I has homology concentrated in top degree.*

Proof. This is immediate from the definitions, the result of [GPW] expressing multigraded Betti numbers $\beta^P_{i,\boldsymbol{\alpha}}(I)$ in terms of lower intervals in L_I, and Proposition 5.1. □

6. Monomial ideals generated by the bases of a matroid

Throughout this section, let $k[x_1, \dots, x_n]$ be a polynomial ring with variables indexed by the ground set of a matroid M. Let I be the monomial ideal generated by the bases of M. Notice that these monomial ideals are Alexander dual to the matroid ideals which were defined and studied in [NPS].

Proposition 6.1. *If I is generated by the bases of a matroid, then $L_I^* \cong F(Indep(M^*))$, where $F(Indep(M^*))$ denotes the face poset for the independence complex of the matroid dual to M.*

Proof. Taking the dual of L_I and relabelling the elements that are coatoms in L_I^* with the bases of the dual matroid, notice that L_I^* is the face poset for the independence complex of the dual matroid: the matroid exchange axiom implies L_I^* contains all possible faces, i.e. that $L_I \setminus \hat{0}$ is the filter over the generators of I. □

In the case of the uniform matroid, $L_{\mathcal{A}(I)}$ is the intersection lattice for the k-equal arrangement, an example to be discussed further in Section 7.

Corollary 6.2. *For I generated by the bases of a matroid, L_I is Cohen-Macaulay.*

Proof. $F(Indep(M^*))$ is graded and shellable, because its order complex is the first barycentric subdivision of a pure, shellable simplicial complex, so L_I is Cohen-Macaulay. □

Corollary 6.3. *For matroids in which each basis involves more than half the ground set, I is Golod.*

Proof. In this case, $L_{\mathcal{A}(I)} = L_I$ since the gcd graph is the complete graph, so it follows from shellability and gradedness of L_I that both $L_{\mathcal{A}(I)}$ and L_I are Cohen-Macaulay. The gcd graph for each lower interval is complete, therefore connected. Corollary 4.4 now applies. □

Question 6.4. *Is $L_{\mathcal{A}(I)}$ shellable for I generated by the bases of a matroid?*

More generally, it would be interesting to known how exactly shellability of $L_{\mathcal{A}(I)}$ is related to shellability of L_I for I any monomial ideal. It is not always the case that $L_{\mathcal{A}(I)}$ Cohen-Macaulay implies L_I Cohen-Macaulay, since for instance there are monomial ideals generated in degree two for which L_I is not Cohen-Macaulay. However, when I is the Stanley-Reisner ideal of Δ, it is easy to show

$$\Delta^* \simeq \Delta(L_I \setminus \{\hat{0}, \hat{1}\}),$$

applying the Quillen Fibre Lemma to the poset map $f : f(\Delta^*) \to L_I$ with $f(u) = \vee_{a \le u} a$. It is known that Δ^* is Cohen-Macaulay if and only if I has a linear resolution (see [ER]), and Δ^* is sequentially Cohen-Macaulay iff I has a componentwise linear resolution (see [HRW]).

7. Explicit denominator calculations in the case of fatpoints and the k-equal arrangement

If $I = \langle x_{i_1} \cdots x_{i_k} | 1 \le i_1 < \cdots < i_k \le n \rangle$, then $L_{\mathcal{A}(I)}$ is the intersection lattice for the k-equal arrangement, namely the arrangement generated by all subspaces $x_{i_1} = \cdots = x_{i_k}$ for a fixed k. The k-equal arrangement is known to be nonpure shellable, and its homology is well understood (see [BjWe]). Recall that for P any graded poset and S any subset of the set of ranks appearing in P, then the *rank-selected subposet* P^S is the subposet of P consisting of those $u \in P$ with $\mathrm{rk(u)} \in \mathrm{S}$. It is easy to see for $I = \langle x_{i_1} \cdots x_{i_k} | 1 \le i_1 < \cdots < i_k \le n \rangle$ that L_I is the rank-selected Boolean algebra $B_n^{0,k,k+1,k+2,\dots,n}$. See [PRW] for a connection between the multilinear part of $\mathrm{Tor}^{k[x_1,\dots,x_n]/(x_1,\dots,x_n)^r}(k,k)$ and the r-equal arrangement which predated Berglund's denominator formula from [Be].

Theorem 7.1. *If* $I = \langle x_{i_1} \cdots x_{i_k} | 1 \le i_1 < \cdots < i_k \le n \rangle$*, then the coefficient for* $\boldsymbol{x}^{\boldsymbol{S}} z^t$ *in Berglund's denominator formula equals* $(-1)^{|S|-k}\binom{|S|-1}{k-1}$ *when* $t = |S| - k + 2$ *and is* 0 *otherwise.*

Proof. The rank-selected Boolean algebra is graded and shellable, hence Cohen-Macaulay, so that $\mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}) = (-1)^{|S|-1} \operatorname{rk}(\tilde{H}_{top}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}}))$, while $\operatorname{rk}(\tilde{H}_i(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}})) = 0$ otherwise. This ideal is known to be Golod by virtue of being squarefree stable, implying the coefficient for $\boldsymbol{x}^{\boldsymbol{S}} z^t$ equals $-\beta^P_{t-1,\boldsymbol{S}}(P/I)$. The evaluation of $\mu_{L_I}(\hat{0}, \boldsymbol{x}^{\boldsymbol{S}})$ counts descending chains in a shelling for the rank-selected Boolean algebra, which are indexed by all k-subsets U of S such that the largest element of S is in U, so there are $\binom{|S|-1}{k-1}$ such sets. □

Next we deduce a combinatorial corollary. Given a set partition u, let *nontriv*(u) be the set of elements appearing in blocks of size larger than one.

Theorem 7.2. *Summing over partitions where each nontrivial block has size at least* k *for some fixed* k*, i.e. elements of the* k*-equal partition lattice* $\Pi_{n,k}$*,*

$$\sum_{\substack{u \in \Pi_{n,k} \\ nontriv(u) = [n]}} \mu_{\Pi_{n,k}}(\hat{0}, u) = (-1)^{n-k+1}\binom{n-1}{k-1}.$$

Proof. Consider the closure map f with $f(u) = nontriv(u)$. Notice that $im(f) = B_n^{k,k+1,\ldots,n-2,n-1}$, i.e. all subsets of $[n]$ of size at least k. Crapo's Closure Lemma then gives the relationship

$$\mu_{B_n^{k,k+1,\ldots,n-2,n-1}}(\hat{0}, \hat{1}) = \sum_{u \in f^{-1}(\hat{1})} \mu_{\Pi_{n,k}}(\hat{0}, u).$$

Now apply the Möbius function computation from the proof of Theorem 7.1. □

8. Realizations of lattices

In this section we show that any finite lattice can be realized as an lcm-lattice of some set of monomials. Furthermore, each lattice admits a 'minimal realization', see Definition 8.2.

Define the lcm-lattice of any set of monomials M to be the set $L_M = \{m_S \mid S \subseteq M\}$ of least common multiples of subsets of M partially ordered by divisibility. An isomorphism of a lattice L with the lcm-lattice of some set of monomials M such that M maps to the irreducible elements of L is called a *realization* of L. We also call this monomial set M a realization of L.

The next proposition says that the gcd graph of any realization of L contains the edges

$$\{(x,y) \mid x, y \not\leq c, \text{ for some coirreducible } c \in L\}.$$

Proposition 8.1. *Let M be a monomial set. If m and n are elements of L_M satisfying* $\gcd(m,n) = 1$*, then for all coirreducible elements $c \in L_M$ either $m \mid c$ or $n \mid c$.*

Proof. Let $\underline{2}$ be the set $\{0,1\}$ with its usual partial order. For any finite join-semilattice L, there is an order-reversing bijection between L and the set L^* of all morphisms of join-semilattices $L \to \underline{2}$, given by $x \mapsto f_x$, where $f_x \colon L \to \underline{2}$ is defined by $f_x(y) = 0 \Leftrightarrow y \leq x$.

Let X be the variables used in M. For each $x \in X$ and each $n \geq 1$, consider the function $\alpha_{x^n} \colon L_M \to \underline{2}$ defined by

$$\alpha_{x^n}(w) = \begin{cases} 0 & x^n \nmid w \\ 1 & x^n \mid w \end{cases}$$

α_{x^n} is an element of $(L_M)^*$. We claim that the set $\{\alpha_{x^n} \mid x \in X,\ n \geq 1\}$ generates $(L_M)^*$ as a join-semilattice. Indeed, if $f \in (L_M)^*$, then setting $v = \vee f^{-1}(0)$ one checks that

$$f = \bigvee_{x^n \nmid v} \alpha_{x^n}.$$

Now let c be a coirreducible element of L_M. Then f_c is irreducible in $(L_M)^*$. Therefore $f_c = \alpha_{x^n}$ for some $x \in X$ and some $n \geq 1$, that is, $w \mid c$ if and only if $x^n \nmid w$ for $w \in L_M$. If $\gcd(m,n) = 1$, then either $x^n \nmid m$ or $x^n \nmid n$, i.e., $m \mid c$ or $n \mid c$. ☐

Definition 8.2. *Let L be a finite lattice and let I (resp. C) be its set of irreducible (resp. coirreducible) elements. The* minimal realization *of L is the monomial set $M = \{m_a \mid a \in I\}$, where for each $z \in L$, m_z is the squarefree monomial in the variables $\{x_c\}_{c \in C}$ defined by*

$$m_z = \prod_{\substack{c \in C \\ z \nmid c}} x_c.$$

The next proposition justifies the term 'minimal realization'. The minimal realization of a geometric lattice yields the monomial set given by Peeva in Construction 2.3 of [Pe].

Proposition 8.3. *Let L be a finite lattice. The map $z \mapsto m_z$ is an isomorphism $L \to L_M$, where M is the minimal realization of L. Furthermore the graph structure induced on L via this isomorphism is the minimal possible, i.e., $x, y \in L$ are connected by an edge if and only if $x, y \not\leq c$ for some coirreducible $c \in L$.*

Proof. That we have an isomorphism of lattices follows from the fact that $x \leq y$ in L if and only if $C_y \subseteq C_x$, where C_x denotes the set of coirreducible elements above x. Also, the graph structure on L is the minimal one allowed by Proposition 8.1. □

Corollary 8.4. *Any finite lattice L is the lcm-lattice of some set of monomials. L is the lcm-lattice of a monomial ideal if and only if it is atomic.*

9. Complete monomial sets and Golodness for geometric monomial ideals

This section examines when the morphism of join-semilattices $K_M \to L_M$, $S \mapsto m_S$, associated to any monomial set M is an isomorphism. A finite lattice whose minimal realization M has this property will be called a *complete lattice*. We will show that the class of complete lattices is closed under direct products and that all geometric lattices are complete. The main feature of this class is that if the lcm-lattice L_I of a monomial ideal I is complete, then it is trivial to decide Golodness of I.

Let M_m denote the set $\{n \in M \mid n \text{ divides } m\}$ if m is a monomial and M is a set of monomials. L_M embeds into K_M as a meet-semilattice by mapping $x \in L_M$ to $\overline{\{x\}} = M_x$. The map $K_M \to L_M$ sending S to m_S is a map of join-semilattices and a retraction onto L_M, because $m_{M_x} = x$. Thus K_M is isomorphic to L_M if and only if the equality $M_{m_S} = S$ holds for every saturated subset S of M.

Definition 9.1. *A monomial set M is called* complete *if $K_M \cong L_M$.*

For instance, monomial sets with complete gcd graph are complete.

Proposition 9.2. *M is complete if and only if for all $x, y \in L_M$ with $\gcd(x, y) = 1$ and for all $m \in M$, $m \mid xy$ implies $m \mid x$ or $m \mid y$.*

Proof. Assume M complete and suppose $x, y \in L_M$ and $\gcd(x, y) = 1$. Let $S = M_x \cup M_y$. S is saturated in M because the saturated sets M_x and M_y are the connected components of S. Note that $m_S = xy$, so by completeness $M_x \cup M_y = M_{xy}$, which is exactly what is required.

Conversely, if $M_{xy} = M_x \cup M_y$ whenever $\gcd(x, y) = 1$ and $x, y \in L_M$, then for $S \in K_M$, decompose S into connected components as $S = S_1 \cup \ldots \cup S_r$. Since $S_i = M_{m_{S_i}}$, it follows that $S = M_{m_{S_1}} \cup \ldots \cup M_{m_{S_r}} = M_{m_{S_1} \ldots m_{S_r}} = M_{m_S}$. □

Let M be the minimal realization of a lattice L and let N be any realization of L. Then by Propositions 8.1 and 8.3, the induced lattice isomorphism $f \colon L_M \to L_N$ gives a morphism of gcd graphs, i.e., the graph for L_M is obtained from the graph for L_N by removing some edges. Then there is a commutative diagram

$$\begin{tikzcd} K_M \arrow[r, two heads, "\bar{f}"] \arrow[d, two heads, "m_-"] & K_N \arrow[d, two heads, "m_-"] \\ L_M \arrow[r, "\cong", "f"'] & L_N \end{tikzcd}$$

where $\bar{f}(S) = \overline{f(S)}$. Therefore, if M is complete, i.e., if $K_M \to L_M$ is an isomorphism, then so is N. In other words, if the minimal realization of a lattice L is complete, then all realizations of L are complete. This leads us to call the lattice L complete if its minimal realization is a complete monomial set, since we have now proven the following:

Theorem 9.3. *The following are equivalent for a finite lattice L:*

- *L is complete.*
- *The minimal realization of L is complete.*
- *Every realization of L is complete.*
- *For any $x, y \in L$ such that $L_{\geq x} \cup L_{\geq y}$ contains all coirreducible elements of L, if $a \in L$ is irreducible, then $a \leq x \vee y$ only if $a \leq x$ or $a \leq y$.*

If M and N are sets of monomials in the variables X and Y, respectively, then $M \bigoplus N$ is the monomial set $M \cup N$ in the variables $X \sqcup Y$. The graph underlying $M \bigoplus N$ is the disjoint union of the graphs of M and N. Clearly, $L_{M \bigoplus N} \cong L_M \times L_N$ and $K_{M \bigoplus N} \cong K_M \times K_N$. Therefore $M \bigoplus N$ is complete if M and N are complete. One can also verify that if M and N are the minimal realizations of the lattices L and K, then $M \bigoplus N$ is the minimal realization of $L \times K$. Consequently, direct products of complete lattices are complete. A lattice is *indecomposable* if it is not isomorphic to a direct product of smaller lattices.

Theorem 9.4 ([Gr], Theorems IV.3.5 and IV.3.6)**.** *Every geometric lattice is isomorphic to a direct product of indecomposable geometric lattices. A geometric lattice L is indecomposable if and only if for any two atoms $a, b \in L$, there is a coatom $c \in L$ such that $a \not\leq c$ and $b \not\leq c$.*

Theorem 9.4 easily implies that all indecomposable geometric lattices are complete, as shown next.

Proposition 9.5. *The graph underlying the minimal realization of a geometric lattice L is a disjoint union of complete graphs, the components being in one-to-one correspondence with the factors of the decomposition of L as a direct product of indecomposable lattices.*

Proof. A geometric lattice is coatomic, so the coirreducibles of L are exactly the coatoms. Thus in the minimal realization $f\colon L_M \xrightarrow{\cong} L$, two monomials $m, n \in M$ have a common factor if and only if there is a coatom c of L such that $f(m), f(n) \not\leq c$, implying indecomposable geometric lattices have complete gcd-graph, hence are complete. By the discussion following Theorem 9.3, this implies all geometric lattices have gcd graphs which are collections of cliques, since all geometric lattices are direct products of indeomposable geometric lattices. □

Since complete gcd graph implies completeness, and since being complete is closed under direct product, the above implies:

Corollary 9.6. *Geometric lattices are complete.*

Thus, if I is a monomial ideal with L_I geometric, then $K_I \cong L_I$. On the other hand, there are monomial ideals I with $L_{\mathcal{A}(I)}$ geometric but $L_{\mathcal{A}(I)} \ncong L_I$, e.g. some monomial ideals generated in degree two.

Remark 9.7. *Not all complete lattices are geometric; consider for instance $M = \{x^2y, xz, yz\}$. Not all shellable lattices are complete, as exhibited by $M = \{x^2, xy, y^2\}$, though geometric implies shellable.*

9.1. **The Golod property for complete monomial sets.** Our starting point is the following well-known fact (see e.g. [CR] or [Jo]). Let $P = k[x_1, \ldots, x_n]$.

Proposition 9.8. *If each pair m_i, m_j of minimal generators for a monomial ideal $I \subseteq P$ have a common factor, then P/I is Golod.*

Proof. In this case, $L_I = L_{\mathcal{A}(I)}$, and the result follows from the fact that L_I determines multigraded Betti numbers of a monomial ideal (by [GPW]) while $L_{\mathcal{A}(I)}$ determines Poincaré series denominator for resolving k over P/I. □

The converse of Proposition 9.8 does not hold, as exhibited by $I = (x^2, xy, y^2)$. In [CR] it is proved that if L_I is a boolean lattice then the converse holds, i.e., P/I is Golod if and only if the gcd-graph of I is complete. Boolean lattices are geometric and hence complete, and we have the following generalization of the cited result.

Proposition 9.9. *If I is a monomial ideal whose minimal set of generators M is complete, then the converse of Proposition 9.8 holds, i.e., P/I Golod implies the graph underlying M is complete.*

Proof. Suppose P/I is Golod and that M is complete. According to [Jo] Lemma 8.4, P/I Golod implies that if $m, n \in M$ were relatively prime then there would be a $w \in M \setminus \{m, n\}$ such that

$w \mid \mathrm{lcm}(m,n) = mn$. But this would contradict Proposition 9.2. Hence no two monomials in M are relatively prime.

□

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References

[AHH] A. Aramova, J. Herzog, and T. Hibi, *Weakly stable ideals*, Osaka J. Math. **34** (1997), 745–755.

[AH] A. Aramova and J. Herzog, *Square-free lexsegment ideals*, Math. Z. **228** (1998), 353–378.

[Av] L. Avramov, Infinite free resolutions. *Six lectures on commutative algebra (Bellaterra, 1996)*, 1–118, Prog. Math., **166**, Birkhäuser, Basel, 1998.

[Av2] L. Avramov, *Homotopy Lie algebras and Poincaré series of algebras with monomial relations*, Homology Homotopy Appl. **4** (2002), no.2, 17–27.

[AP] L. Avramov and I. Peeva, *Finite regularity and Koszul algebras*, Amer. J. Math., **123** (2001), no. 2, 275–281.

[Ba] J. Backelin, *Les anneaux locaux a' relations monomiales ont des se'ries de Poincaré-Betti rationelles*, C.R. Aced. Sc. Paris (1982), 607–610.

[Bat] E. Batzies, *Discrete Morse theory for cellular resolutions*, Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften, Marburg/Lahn 2002.

[BW] E. Batzies and V. Welker, *Discrete Morse theory for cellular resolutions*, J. Reine Angew. Math., **543** (2002), 147–168.

[Be] A. Berglund, *Poincaré series of monomial rings*, J. Algebra, **295, No. 1** (2006), 211–230.

[Bj1] A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260, No. 1** (1980), 159–183.

[Bj2] *Topological methods*, Handbook of combinatorics, Vol. 1, 2, 1819-1872, Elsevier, Amsterdam, 1995.

[Bj3] *Homology and shellability of matroids and geometric lattices*, Matroid applications (N. White, ed.), Encyclopedia Math. Appl. **40**, Cambridge University Press, Cambridge, 1992, 226–283.

[Bj4] A. Björner, *Subspace arrangements*, First Euopean Congress of Mathematics, Vol. I (Paris, 1992), Progr. Math. **119**, 321–370, Birkhäuser 1994

[BjWa] A. Björner and M. Wachs, *Shellable nonpure complexes and posets I*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1299–1327.

[BjWe] A. Björner and V. Welker, *Homology of the "k-equal" manifolds and related partition lattices*, Adv. Math. **110** (1995), no. 2, 277–306.

[CR] H. Charalambous and A. Reeves, *Poincaré series and resolutions of the residue field over monomial rings*, Comm. Algebra **23** (1995), 2389–2399.

[ER] J. Eagon and V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Alg. **130** (1998), 265–275.

[Fr] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), 29–39.

[GPW] V. Gasharov, I. Peeva and V. Welker, *The lcm-lattice in monomial resolutions*, Math. Res. Lett. **6** (1999), no. 5-6, 521–532.

[Gr] G. Grätzer, General Lattice Theory, *Pure and Applied Mathematics*, **75**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, xiii + 381 pages.

[HHS] P. Hanlon, P. Hersh and J. Shareshian, *A $GL_n(q)$ analogue of the partition lattice*, preprint 2005.

[Ha] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[HRW] J. Herzog, V. Reiner and V. Welker, *Componentwise linear ideals and Golod rings*, Michigan Math. J. **46**, (1999), 211–223.

[Ho] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26, Dekker, New York, 1977.

[Jo] M. Jöllenbeck, *On the multi-graded Hilbert and Poincaré-Betti series and the Golod property of monomial ideals*, J. Pure Appl. Algebra **207**, (2006), 261–298.

[Ko] D. Kozlov, *A class of hypergraph arrangements with shellable intersection lattice*, J. Combin. Theory, Ser. A, **86** (1999), no. 1, 169–176.

[MS] E. Miller and B. Sturmfels, *Combinatorial Commutative algebra*, Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005. xiv+417 pp.

[NPS] I. Novik, A. Postnikov and B. Sturmfels, *Syzygies of oriented matroids*, Duke Math. J., **111** (2002), no. 2, 287–317.

[OT] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, (Springer-Verlag, Berlin), 1992.

[Ox] J. Oxley, *Matroid theory*, Oxford University Press, New York, 1992.

[Pe] I. Peeva, *Resolutions and lattices*. The Roos festschrift volume, 2. Homology Homotopy Appl. **4** (2002), no. 2, part 2, 427–437.

[PRW] I. Peeva, V. Reiner and V. Welker, *Cohomology of real diagonal subspace arrangements via resolutions*, Compositio Math. **117** (1999), no.1, 99–115

[Qu] D. Quillen, *Homotopy properties of the poset of non-trivial p-subgroups of a group*, Adv. Math., **28** (1978), 101–128.

[Ro] G.-C. Rota, *On the foundations of Combinatorial Theory I. Theory of Möbius functions*, Z. Wahrsch. Verw. Gebiete, **2** (1964), 340–368.

[Ta] D. Taylor, *Ideals generated by monomials in an R-sequence*, Ph. D. Thesis, University of Chicago, 1966.

[Wa1] M. Wachs, *Whitney homology of semipure shellable posets*, J. Algebraic Combin. **9** (1999), no. 2, 173–207.