Math 201 – Midterm 1

Winter 2015
Solutions

Name: ______________________________

Question	Points	Score
1	10	
2	8	
3	8	
4	8	
5	12	
6	6	
Total:	52	

1. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 4 & 12 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ -8 \end{bmatrix}$.

(a) (3 points) Find an echelon form (EF) of the augmented matrix $[A\,\mathbf{b}]$.

Solution: Start with the augmented matrix $[A\,\mathbf{b}]$ and row-reduce as follows:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 4 \\ -4 & 4 & 12 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 4 \\ 0 & -4 & 16 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (2 points) Find the reduced row echelon form (RREF) of the augmented matrix $[A\,\mathbf{b}]$.

Solution: Continue row-reducing from the echelon form of $[A\,\mathbf{b}]$ above:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 4 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) (2 points) Find the parametric description of the solution set of the system $A\mathbf{x} = \mathbf{b}$.

Solution: The RREF found above has pivot columns 1 and 2 and x_3 is free. Hence the parametric description of the solution set is

$$\begin{cases} x_1 & = 4 + 7x_3 \\ x_2 & = 2 + 4x_3 \\ x_3 & \text{is free.} \end{cases}$$

(d) (2 points) Do the columns of A span $\mathbb{R}^3$? Justify your answer.

Solution: By a theorem in the book, the columns of A span $\mathbb{R}^3$ if and only if there is a pivot in every row of the RREF of A. The RREF of A is $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$, which does not have a pivot in every row, so the columns of A do not span $\mathbb{R}^3$.

(e) (1 point) Write down one solution to the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 4 \\ -4x_1 + 4x_2 + 12x_3 &= -8 \end{aligned}$$

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Solution: In part (c) we determined all the solutions to this system. We can obtain one by, for instance, setting $x_3 = 0$ and obtaining the solution $x_1 = 4$, $x_2 = 2$, $x_3 = 0$.

2. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ -1 & -2 & -3 \end{bmatrix}$.

(a) (4 points) Find the inverse of A.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

Hence $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) (2 points) Determine the solution set of the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Solution: By a theorem in the book, the system has the unique solution

$$\mathbf{x} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Hence the solution set is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

(c) (2 points) Is there a $\mathbf{b} \in \mathbb{R}^3$ such that the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions?

Solution: By a theorem in the book, if A is invertible then $A\mathbf{x} = \mathbf{b}$ has the unique solution $A^{-1}\mathbf{b}$, so the answer is no.

3. Let

$$C = \begin{pmatrix} 1 & 2 & 1 & 21 \\ 2 & 6 & 0 & 20 \\ 0 & 0 & 0 & 3 \\ 2 & 4 & 3 & 19 \end{pmatrix}.$$

(a) (4 points) Compute the determinant of C.

Solution: Expanding along the second-to-last row yields:

$$\det(C) = \begin{vmatrix} 1 & 2 & 1 & 21 \\ 2 & 6 & 0 & 20 \\ 0 & 0 & 0 & 3 \\ 2 & 4 & 3 & 19 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 6 & 0 \\ 2 & 4 & 3 \end{vmatrix}$$

Row-reducing this 3×3 matrix, keeping track of scalar multiplications and row swaps, we obtain:

$$-3 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 6 & 0 \\ 2 & 4 & 3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{vmatrix} = -6,$$

where the last equality is because the determinant of an upper-triangular matrix is the product of its diagonal entries.

(b) (2 points) Compute the determinant of C^{-1}.

Solution: Using the multiplicativity of the determinant,

$$1 = \det(I) = \det(CC^{-1}) = \det(C)\det(C^{-1}),$$

hence $\det(C^{-1}) = \frac{1}{\det(C)}$. Thus $\det(C^{-1}) = -\frac{1}{6}$.

(c) (2 points) Compute the determinant of the matrix $D = \begin{bmatrix} 1 & 2 & 1 & 21 \\ 4 & 12 & 0 & 40 \\ 0 & 0 & 0 & 3 \\ 2 & 4 & 3 & 19 \end{bmatrix}$.

Solution: Since D is obtained from C by the row operation $R_2 \to 2R_2$, by the theorem that says how determinants change under row operations,

$$\det(D) = 2\det(C) = 2(-6) = -12.$$

4. For this question, no justification is required (justification is required on all other questions on this midterm). Circle your final answer clearly. For parts (c)–(h), determine whether the statement is true or false.

(a) (1 point) Write down a vector $\mathbf{v} \in \mathbb{R}^2$ such that the set $\left\{ \mathbf{v}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

Solution: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) (1 point) Let A be a 4×2 matrix. Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution to the system $A\mathbf{x} = \mathbf{0}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a solution to the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$. Write down a solution to the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ that is not equal to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Solution: Since a solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ exists, the solutions of this system are translates of solutions to the system $A\mathbf{x} = \mathbf{0}$. Hence $\mathbf{v}$ is a solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ if and only if $\mathbf{v} - \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So $\begin{bmatrix} 2 \\ 4 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution to this nonhomogeneous system for any $c \in \mathbb{R}$. One possibility is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(c) (1 point) True/False: Let A be a square matrix. If A^2 is invertible, then A is invertible.

Solution: True. Since A^2 is invertible, $\det(A^2) \neq 0$. Thus, $\det(A)^2 = \det(A^2) \neq 0 \implies \det(A) \neq 0$, hence A is invertible.

(d) (1 point) True/False: Let A be a square matrix. If A is invertible, then A^2 is invertible.

Solution: True. Since A is invertible, $\det(A) \neq 0$. Thus, $\det(A^2) = \det(A)^2 \neq 0$, hence A^2 is invertible.

(e) (1 point) True/False: If the columns of a 4×7 matrix A span $\mathbb{R}^4$, then A has 7 pivots.

Solution: False. Since there is at most one pivot in each row, a 4×7 matrix cannot have more than 4 pivots.

(f) (1 point) True/False: If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^3$, then the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is always linearly dependent.

Solution: True. Any set of n vectors in $\mathbb{R}^m$ is linearly dependent if $n > m$.

(g) (1 point) True/False: Let A be a square $n \times n$ matrix. If the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^n$.

Solution: False. If A is the zero matrix, then $A\mathbf{x} = \mathbf{0}$ has solution set $\mathbb{R}^n$, but $A\mathbf{x} = \mathbf{b}$ has no solutions for any $\mathbf{b} \neq \mathbf{0}$.

(h) (1 point) True/False: Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation. The set

$$\left\{ T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right\}$$

is linearly dependent.

Solution: True. By linearity of T, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, thus the set is linearly dependent.

5. TRUE/FALSE. Determine whether the following statements are true or false, and give justification.

   (a) (3 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set then $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   Solution: FALSE. Counterexample: Take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ but $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent since $2 \cdot \mathbf{v}_1 - \mathbf{v}_2 + 0 \cdot \mathbf{v}_3 = 0$.

   (b) (3 points) Let A be a 2×4 matrix. if the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then the system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbb{R}^2$.

   Solution: FALSE. Counterexample: Let A be the zero matrix. Then $A\mathbf{x} = \mathbf{0}$ has solution set $\mathbb{R}^2$, but the system $A\mathbf{x} = \mathbf{b}$ has no solutions for any $\mathbf{b} \neq \mathbf{0}$.

   (c) (3 points) Suppose A is a 3×4 matrix whose columns span $\mathbb{R}^3$. There exists a 4×3 matrix B such that $AB = I_3$.

   Solution: TRUE. Since the columns of A span $\mathbb{R}^3$, there exist $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbb{R}^4$ such that $A\mathbf{b}_i = \mathbf{e}_i$ for $i = 1, 2, 3$ ($\mathbf{e}_i$ denotes the ith column of I_3). Letting $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, we obtain $AB = I_3$.

   (d) (3 points) Let A, B be 3×3 matrices. Suppose that A is invertible and A and B have different RREFs. Then B is not invertible.

   Solution: TRUE. By the Invertible Matrix Theorem, a matrix is invertible if and only if its RREF is the identity matrix. Hence A invertible $\implies$ the RREF of A is I_3 $\implies$ the RREF of B is not I_3 $\implies$ B is not invertible.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that:

$$T\left(\begin{bmatrix}1\\1\\3\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\ T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\ T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

(a) (3 points) Prove that $T\left(\begin{bmatrix}6\\4\\7\end{bmatrix}\right)=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

Solution:
First we solve

$$c_1\begin{bmatrix}1\\1\\3\end{bmatrix}+c_2\begin{bmatrix}1\\0\\2\end{bmatrix}+c_3\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}6\\4\\7\end{bmatrix}$$

We form the augmented matrix:

$$\begin{bmatrix}1&1&1&6\\1&0&1&4\\3&2&0&7\end{bmatrix}\longrightarrow \text{after row reduction}\longrightarrow\begin{bmatrix}1&0&0&1\\0&1&0&2\\0&0&1&3\end{bmatrix}$$

So $c_1=1$, $c_2=2$, $c_3=3$. Thus, $T\left(\begin{bmatrix}6\\4\\7\end{bmatrix}\right)=T\left(1\begin{bmatrix}1\\1\\3\end{bmatrix}+2\begin{bmatrix}1\\0\\2\end{bmatrix}+3\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=$

(since T is a linear function)

$$T\left(\begin{bmatrix}1\\1\\3\end{bmatrix}\right)+2T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)+3T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}+2\begin{bmatrix}0\\1\\0\end{bmatrix}+3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1+0+0\\0+2+0\\0+0+3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

(b) (3 points) Is T onto? Justify your answer.

Solution: Yes. Note that $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix}$. Hence by the linearity of T,

$$T\left(c_1\begin{bmatrix}1\\1\\3\end{bmatrix}+c_2\begin{bmatrix}1\\0\\2\end{bmatrix}+c_3\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=c_1T\left(\begin{bmatrix}1\\1\\3\end{bmatrix}\right)+c_2T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)+c_3T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}.$$

Since this holds for any $c_1, c_2, c_3 \in \mathbb{R}$, Range(T) $= \mathbb{R}^3$, hence T is onto.