The Joy of Sets

Autumn 2011

The study of modern mathematics requires a basic familiarity with the notions and notation of set theory. For a rigorous treatment of set theory, you may wish to take Math 582, *Introduction to Set Theory*.

What is a set?

A colony of beavers, an unkindness of ravens, a murder of crows, a team of oxen, . . . each is an example of a *set* of things. Rather than define what a set is, we assume you have the "ordinary, human, intuitive (and frequently erroneous) understanding"[1] of what a set is.

[1] Paul Halmos, *Naive Set Theory*, Springer–Verlag, NY 1974.

Sets have *elements*, often called *members*. The elements of a set may be flies, beavers, words, numbers, sets, vectors, . . . If x is some object and S is a set, we write $x \in S$ if x is an element of S and $x \notin S$ if x is not a member of S. For us, the MOST IMPORTANT PROPERTY a set S has is this: if x is an object, then either $x \in S$ or $x \notin S$, but not both.

The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are well-known to you, though you may not know their names. The set of *natural numbers* is denoted by $\mathbb{N}$, and its elements are the numbers 1, 2, 3, 4, . . . Note that if $n, m \in \mathbb{N}$, then $n + m \in \mathbb{N}$; that is, $\mathbb{N}$ is *closed* under addition. However, $\mathbb{N}$ is not closed under subtraction. For example, $(4 - 5) \notin \mathbb{N}$. To overcome this inconvenience we consider $\mathbb{Z}$, the set of *integers*, which has as its elements the numbers 0, ± 1, ± 2, ± 3, . . . While $\mathbb{Z}$ is closed under addition, subtraction, and multiplication, it is not closed under division. For example, $(-23)/57 \notin \mathbb{Z}$. To surmount this difficulty, we form the set of *rational numbers*, $\mathbb{Q}$. Intuitively, $\mathbb{Q}$ is the set of all numbers that can be expressed as a fraction n/m with $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. While closed under multiplication, division, addition, and subtraction, $\mathbb{Q}$ is missing important numbers like $\sqrt{2}$. There are MANY ways to overcome this inconvenience; the most common approach is to introduce $\mathbb{R}$, the set of *real numbers*. $\mathbb{R}$ is usually depicted as a line that extends forever in both directions.

Math 582, *Introduction to Set Theory*, provides a rigorous treatment of $\mathbb{N}$.

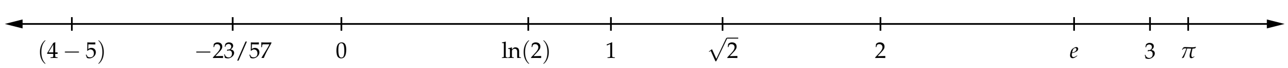
The symbol $\mathbb{Z}$ is derived from *Zahlen*, the German word for numbers.

NEVER divide by zero.

In Math 412, *Introduction to Modern Algebra*, $\mathbb{Q}$ is rigorously defined.

Approximately 1.41, $\sqrt{2}$ is the ratio of a square's diagonal to one of its sides. It was probably the first known example of a number that does not belong to $\mathbb{Q}$.

Introductory analysis courses, including Math 351 and Math 451, provide in-depth treatments of $\mathbb{R}$.



A way to specify a *finite* set is by listing all of its elements; this is sometimes called the *roster method*. The *cardinality* of a finite set is the number of elements that the set contains. For example, the sets

$$\{\pi, \sqrt{2}, 32, -5.4\} \quad \text{and} \quad \{\pi, -2, e, \{\pi, \sqrt{2}, 32, -5.4\}\}$$

both have cardinality four. The cardinality of a set A is denoted $|A|$.

Approximately 3.14, π is the ratio of a circle's circumference to its diameter.

Approximately 2.72, e is $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

The most common way to specify a set is by using *set-builder notation*. For example, the set of *primes* could be written

The first four primes are: 2, 3, 5, and 7. In particular, 1 is NOT a prime number.

$$\{n \in \mathbb{N} \mid n \text{ has exactly two distinct positive divisors}\},$$

the open interval $(\ln(2), 1)$ could be written

$$\{x \in \mathbb{R} \mid 2 < e^x < e\},$$

and the set of non-negative integers, $\mathbb{Z}_{\geq 0}$, could be written

$$\{m \in \mathbb{Z} \mid m \geq 0\}.$$

Russell's paradox provides a *non-example* of a set. Consider

$$\{S \text{ is a set} \mid S \notin S\}.$$

Call this candidate for set-hood T. As you should verify, we have both $T \in T$ and $T \notin T$. Thus, T does not have the MOST IMPORTANT PROPERTY, and so is not a set.

Approximately .69, $\ln(2)$ is $\displaystyle\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)}$.

For $a, b \in \mathbb{R}$ with $a \leq b$ we define

$$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\},$$
$$(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\},$$
$$[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\},$$
$$(a,b) := \{x \in \mathbb{R} \mid a < x < b\}, \text{ and}$$
$$[a,\infty) := \{x \in \mathbb{R} \mid x \geq a\}.$$

The sets (a,∞), $(-\infty,a)$, and $(-\infty,a]$ are defined similarly.

PRACTICE: Test your understanding of set notation using Doug Ensley's material at `www.math.lsa.umich.edu/courses/101/sets.html`

Set relations: Equality

> One can't do mathematics for more than ten minutes without grappling, in some way or other, with the slippery notion of *equality*. Slippery, because the way in which objects are presented to us hardly ever, perhaps never, immediately tells us — without further commentary — when two of them are to be considered equal.[2]

[2] Barry Mazur, *When is one thing equal to some other thing?*, Proof and other dilemmas, 2008.

Definition 1. *Two sets are defined to be equal when they have precisely the same elements. When the sets A and B are equal, we write $A = B$.*

The notations "=" and ":=" do NOT mean the same thing. The latter means: this is the definition of the object on the left.

That is, the sets A and B are equal if every element of A is an element of B, and every element of B is an element of A. For example, thanks to Lagrange's four-square theorem (1770),[3] we have

$$\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \text{ can be written as the sum of four squares}\}.$$

[3] When you encounter a new mathematical statement, work examples:

$$\begin{aligned} 0 &= 0^2 + 0^2 + 0^2 + 0^2 \\ 1 &= 1^2 + 0^2 + 0^2 + 0^2 \\ 2 &= 1^2 + 1^2 + 0^2 + 0^2 \\ 3 &= 1^2 + 1^2 + 1^2 + 0^2 \\ 4 &= 1^2 + 1^2 + 1^2 + 1^2 \\ &= 2^2 + 0^2 + 0^2 + 0^2 \\ 5 &= 2^2 + 1^2 + 0^2 + 0^2 \end{aligned}$$

Also try to formulate new questions based on your understanding of the statement. For example, you could ask: which numbers can, like 4, be written as a sum of four squares in more than one way?

The next example shows that order and inefficiency do not matter.

$$\begin{aligned} &\{T,O,M,M,A,R,V,O,L,O,R,I,D,D,L,E\} \\ &\quad = \{I,A,M,L,O,R,D,V,O,L,D,E,M,O,R,T\} \\ &\quad = \{A,D,E,I,L,M,O,R,T,V\}. \end{aligned}$$

Since two sets are the same provided that they have precisely the same elements, there is exactly one set with cardinality zero; it is called the *empty set* or *null set* and is denoted $\emptyset$. BEWARE: The set $\emptyset$ has zero elements, but the set $\{\emptyset\}$ has cardinality one.

Set relations: Subset

Definition 2. *If A and B are sets, then we say that A is a* subset *of B (or A is* contained in *B, or B* contains *A), and write $A \subset B$ or $A \subseteq B$, provided that every element of A is an element of B.*

WARNING: Some people write "$A \subset B$" to mean "$A \subseteq B$, but $A \neq B$." We will write "$A \subsetneq B$" for this.

For example, $\mathbb{N} \subset \mathbb{Z} \subseteq \mathbb{Q} \subset \mathbb{R}$; to emphasize that each inclusion is *proper*, we could write $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$. We also have $1 \in \{1, \sqrt{2}\} \subset (\sqrt{2}/2, \sqrt{2}] \subset [\ln(2), e) \subset [\ln(2), e]$ and the obvious[4] inclusion

$$\{n \in \mathbb{N} \mid n \text{ is even and the sum of two primes}\} \subset \{2m + 2 \mid m \in \mathbb{N}\}.$$

[4] If you can demonstrate the reverse inclusion, you will have proved the Goldbach conjecture, one of the oldest unsolved problems in mathematics.

$$\begin{aligned} 4 &= 2+2 \\ 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 7+3 \\ &= 5+5 \\ &\vdots \end{aligned}$$

Note that for any set A we have $\emptyset \subset A \subset A$.

Unreasonably Useful Result. *Suppose that X and Y are sets.*

$$X = Y \textit{ if and only if } X \subset Y \textit{ and } Y \subset X.$$

Proof. By Definition 1, to say that X and Y are equal means that every element of X is an element of Y AND every element of Y is an element of X. In other words, by Definition 2, to say $X = Y$ means that $X \subset Y$ AND $Y \subset X$. □

The symbol □ is called a tombstone or halmos, after former Michigan mathematics professor Paul Halmos. It means: my proof is complete, stop reading. It has replaced the initialism Q.E.D. which stands for *quod erat demonstrandum*; a phrase that means *that which was to be demonstrated*.

Venn diagrams

Representing sets using *Venn Diagrams* can be a useful tool for visualizing the relationships among them. In a Venn diagram a larger figure, often a rectangle, is used to denote a set of objects called the *universe* (for example the universe could be $\mathbb{R}$) and smaller figures, usually circles, within the diagram represent subsets of the universe — points inside a circle are elements of the corresponding subset.

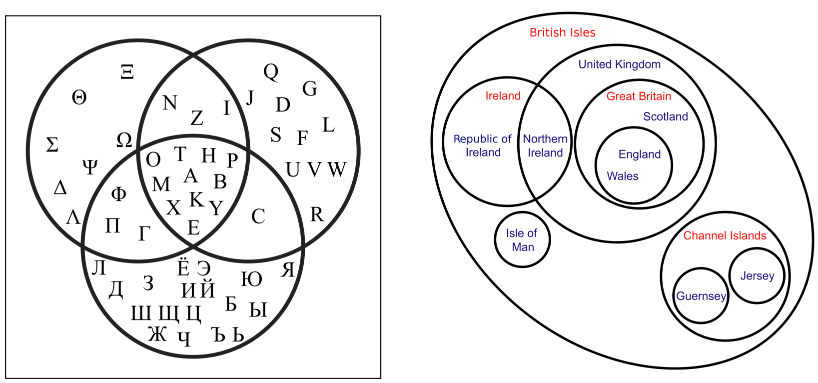


Figure 1: The left Venn diagram illustrates relationships among upper case letters in the Greek, Latin, and Russian alphabets. The universe consists of all upper case letters in these alphabets, and each language is represented by one of the circles. The Venn diagram on the right describes the geographical areas (red) and political entities (blue) that make up the British Isles. The elements of the universe are, with the exception of the United Kingdom, labeled in blue. The remaining words describe the rule for membership in their respective circles.

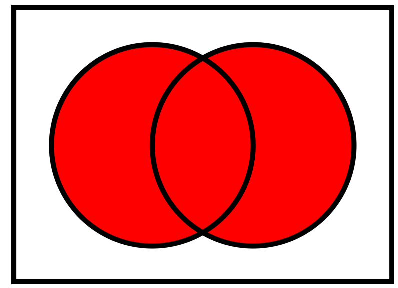
CAUTION. Because many statements about sets are intuitive and/or obvious, figuring out how to prove them can be difficult. While Venn diagrams are excellent tools for illustrating many of these statements, the diagrams are not substitutes for their proofs.

Set operations: Complement, union, and intersection

In the Venn diagrams illustrating the definitions of this section, the set A is represented by the circle to the left, the set B is represented

PRACTICE: Use Doug Ensley's materials to gain basic familiarity with set operations at `www.math.lsa.umich.edu/courses/101/venn2.html` and `www.math.lsa.umich.edu/courses/101/venn3.html` by the circle to the right, and the box represents a universe that contains both A and B.

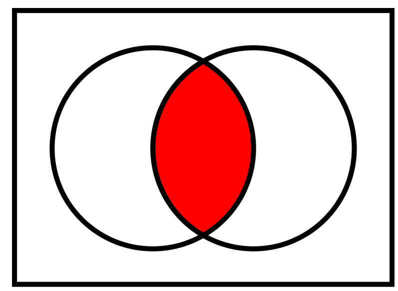
Definition 3. *The* union *of sets A and B, written $A \cup B$, is the set*



$$\{☺ \mid (☺ \in A) \text{ or } (☺ \in B)\}.$$

In other words, for an object to be an element of the union of two sets, it need only be a member of one or the other of the two sets. For example, the union of the sets $\{\varepsilon, \delta, \alpha\}$ and $\{\delta, \beta, \rho, \phi\}$ is the set $\{\alpha, \beta, \delta, \varepsilon, \rho, \phi\}$, the union of $\mathbb{Z}$ and $\mathbb{Q}$ is $\mathbb{Q}$, and $[\ln(2), \sqrt{2}] \cup (\sqrt{2}/2, e]$ is $[\ln(2), e]$. Note that $S \cup \emptyset = S$ for all sets S.

Definition 4. *The* intersection *of sets A and B, written $A \cap B$, is the set*

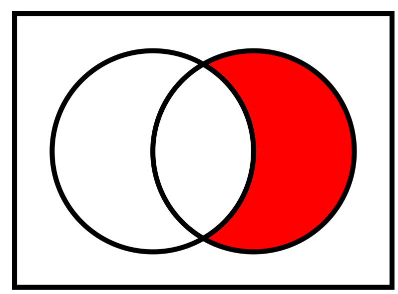


$$\{☹ \mid (☹ \in A) \text{ and } (☹ \in B)\}.$$

Thus, for an object to be a member of the intersection of two sets, it must be an element of both of the sets. For example, the intersection of the sets $\{\varepsilon, \delta, \alpha\}$ and $\{\delta, \beta, \rho, \phi\}$ is the *singleton* $\{\delta\}$, the intersection of $\mathbb{Z}$ and $\mathbb{Q}$ is $\mathbb{Z}$, and $[\ln(2), \sqrt{2}] \cap (\sqrt{2}/2, e]$ is $(\sqrt{2}/2, \sqrt{2}]$. Note that $T \cap \emptyset = \emptyset$ for all sets T.

Remark 5. *For S and T sets, $S \cap T \subset S \subset S \cup T$ and $S \cap T \subset T \subset S \cup T$.*

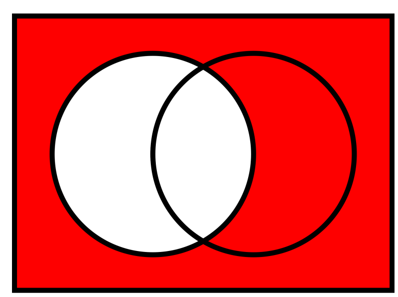
Definition 6. *Suppose A and B are sets. The* difference *of B and A, denoted $B \setminus A$ or $B - A$, is the set*



$$\{b \in B \mid b \notin A\}.$$

Note that, like subtraction, the difference operator is not symmetric. For example, $\{\varepsilon, \delta, \alpha\} \setminus \{\delta, \beta, \rho, \phi\}$ is $\{\alpha, \varepsilon\}$ while $\{\delta, \beta, \rho, \phi\} \setminus \{\varepsilon, \delta, \alpha\}$ is $\{\beta, \rho, \phi\}$. As another example, we have $[\ln(2), e] \setminus (\sqrt{2}/2, \sqrt{2}]$ is $[\ln(2), \sqrt{2}/2] \cup (\sqrt{2}, e]$ and $(\sqrt{2}/2, \sqrt{2}] \setminus [\ln(2), e] = \emptyset$.

Definition 7. *Let U denote a set that contains A and B as subsets. The* complement of A (with respect to U), *often written A^c, $A^{\complement}$, $\bar{A}$, or A', is the set $U \setminus A$.*



WARNING: It is common practice to suppress reference to the set U occurring in the definition of complement. Relying on the reader to implicitly identify the set U can cause confusion, but context often clarifies. For example, if asked to find $[-1, \pi)^{\complement}$, then from context the set U is $\mathbb{R}$ and $[-1, \pi)^{\complement} = (-\infty, -1) \cup [\pi, \infty)$.

Note that $A^{\complement} \cup A$ is U, and $A^{\complement} \cap A = \emptyset$. Two sets with empty intersection are said to be *disjoint*.

DeMorgan's Laws relate the set operations. You should use the definition of equality to verify[5] them. They say

$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement} \quad \text{and} \quad (A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}.$$

[5] Mathematics is not a spectator sport. In order to understand math, you need to do math; now is a good time to start.