Math 217 – Midterm 1

Fall 2012

Time: 120 mins.

1. Answer all questions in the spaces provided.
2. If you run out of room for an answer, continue on the back of the page.
3. No calculators, notes, or other outside assistance allowed.
4. Remember to justify your answers by showing all of your work, and by citing Theorems from lecture or from the text where appropriate.

Name: ______________________ Section: __________

Question	Points	Score
1	16	
2	10	
3	10	
4	16	
5	12	
6	8	
Total:	72	

1. Write complete, precise definitions for each of the following (italicized) terms.
   (a) (4 points) A *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$.
   (b) (4 points) The *span* of the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
   (c) (4 points) The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly dependent*.
   (d) (4 points) The function $f : X \to Y$ is *invertible*.

2. Consider the following system of three linear equations in four unknowns:

$$\begin{cases} x_1 - 2x_2 + x_3 + 4x_4 = -7 \\ 2x_1 - 4x_2 + 3x_3 + 9x_4 = -15 \\ -x_1 + 2x_2 - 3x_4 = 6 \end{cases}$$

(a) (4 points) Find the reduced row echelon form (RREF) of the augmented matrix of the system.

(b) (3 points) Express the solution set of the system in parametric form.

(c) (3 points) Let A be the coefficient matrix of the system. Is it possible to find a vector $\mathbf{b} \in \mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent? Briefly justify your answer.

3. Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 1 & 2 \\ 5 & k & 4 \end{bmatrix}$ and let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation induced by A.

   (a) (5 points) For what values of k is A invertible?

   (b) (5 points) For what values of k does T_A preserve volume?

4. Determine whether the following statements are true or false, and give justification or a counterexample.

   (a) (4 points) If A, B, and P are $n \times n$ matrices such that P is invertible and $B = PAP^{-1}$, then $\det(A) = \det(B)$.

   (b) (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the clockwise rotation about the origin through an angle of π, and let A be the standard matrix of T. Then $\det(A) = 0$.

   (c) (4 points) For any $m \times n$ matrix A and for any vector $\mathbf{b} \in \mathbb{R}^m$, the linear system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions if and only if the corresponding homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

   (d) (4 points) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the vector space V. If each of the sets $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is also linearly independent.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\3\\1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\2\end{bmatrix}\right) = \begin{bmatrix}0\\-2\\4\end{bmatrix}$$

(a) (3 points) Find $T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

(b) (3 points) Find the standard matrix of T.

(c) (2 points) Is T one-to-one?

(d) (2 points) Is T onto?

(e) (2 points) Find a linearly independent set of vectors that spans the range of T.

6. Determine whether the following statements are true or false, and give justification or a counterexample.
   (a) (4 points) For any square matrix A and scalar $c \in \mathbb{R}$, $c \cdot \det(A) = \det(cA)$.
   (b) (4 points) If an $n \times n$ matrix A can be written as a power $A = E^k$ of an elementary matrix E, then the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for all $\mathbf{b} \in \mathbb{R}^n$.